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Matching with preferences: Theory and Practice

Suppose that a CS department assigns a adviser to each new graduate student. Each student has his/her own preference for professors and each professor does for students. The assignment should be fair to both students and professors, where the theory of stable matchings plays a key role. In this course, stable matchings, or more generally matchings under preferences, are studied mainly from a theoretical point of view but also on its wide range of applications to the real world. The Nobel Prize 2012 for Economics was awarded to Alvin E. Roth and Lloyd S. Shapley. Shapley, with David Gale, first gave the notion of stable matchings with the Gale-Shapley Algorithm and Roth has had a lot of contribution from both angles of Computer Science and Economics. The topic is strongly related to the main research area of the instructor.

1. Stable matchings, Introduction
1.1 Maximum bipartite matchings
1.2 Maximum weighted bipartite matchings
1.3 Men-Women matchings; Preference lists
1.4 Blocking pairs
1.5 Removing blocking pairs

2. Stable marriage problem
2.1 Gale-Shapley algorithm; its novelty
2.2 Correctness and time complexity
2.3 Men/Women optimal stable matchings
2.4 Average-case complexity

3. Mathematical structures of stable matchings
3.1 Finding stable matchings with features
3.1 Describing all stable matchings
3.2 Rotations

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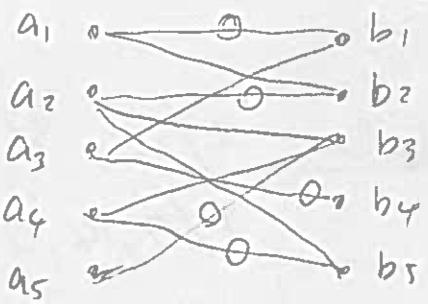
@ Bipartite matching

Instance: $A=\{a_1,\dots,a_n\}$, $B=\{b_1,\dots,b_n\}$.
$W \subseteq A \times B$

Question: Is Their a (perfect) matching?



$W=\{(a_1,b_1),(a_1,b_2),\dots\}$

matching: $\{(a_1,b_1),(a_2,b_2),(a_3,b_4),(a_4,b_5),(a_5,b_3)\}$

Namely $M \subseteq W$ is a matching

$\Updownarrow$

$\|M\| = n$ and

(a,b) & $(a',b') \in M$
$\Rightarrow a \neq a'$ $b \neq b'$

$W_1 = W - \{(a_4,b_5)\}$ $\Rightarrow$ W_1 does NOT have a matching

6.1 Stable roommate as stable marriage
6.2 How to obtain stable roommates

7. Real-world applications
7.1 Examples in US, UK, and others
7.2 Student assignment
7.3 Kidney exchange

8. Other models for matching with preferences
8.1 Weak and strong stabilities
8.2 Pareto optimality
8.3 Popular matchings
8.4 Online stable matchings

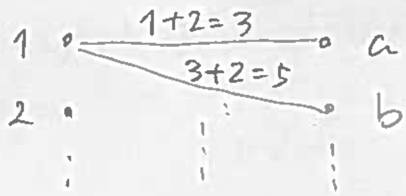
9. Game theoretical aspects of stable matchings

⓪ What is a preference list?

Men 1, 2, 3, 4, 5
Women a, b, c, d, e
He/she wants a "good" partner

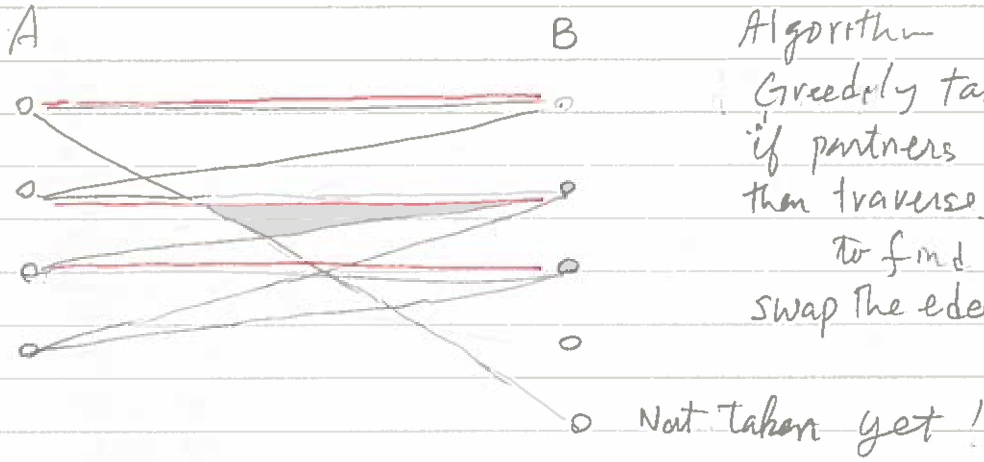
1: a, c, b, d, e
2: c, a, e, b, d
⋮

a: 2, 1, 3, 4, 5
b: 2, 1, 4, 5, 3
⋮



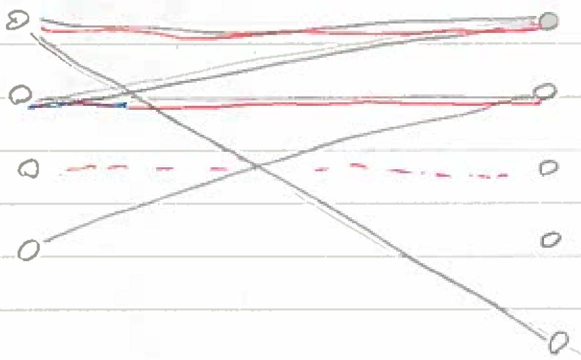
1.1 (Unweighted) bipartite matchings

How to obtain one.



Algorithm

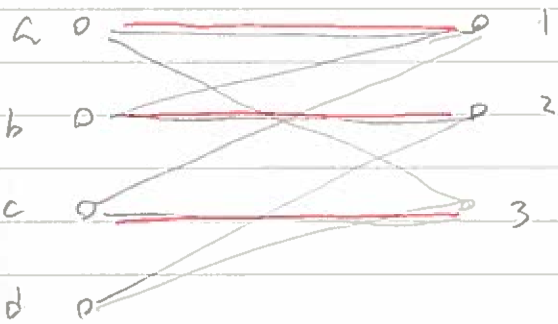
Greedely take a partner
if partners are taken
then traverse the graph
to find a free vertex
swap the ededges of the augmenting path



Augmenting path

Increased by 1

If we cannot find free vertices, then



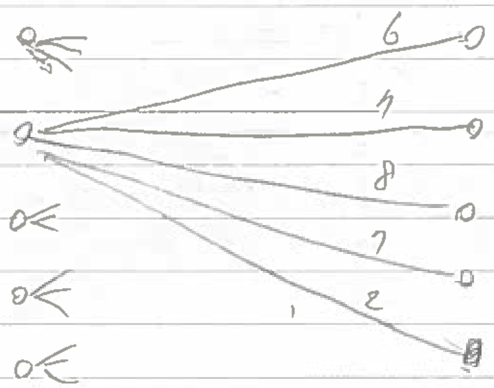
Neighbors of $\{a,b,c,d\}$ are $\{1,2,3\}$

$\Downarrow$

Obviously no matching

Th[1.1] A bipartite matching can be obtained in poly time.

1.2 Weighted bipartite matchings



We want a maximum matching.

	u_1	u_2	u_3	u_4	u_5
v_1	1	2	3	4	5 ← w_{15}
v_2	6	7	8	7	2
v_3	1	3	4	4	5
v_4	3	6	2	8	7
v_5	4	1	3	5	4

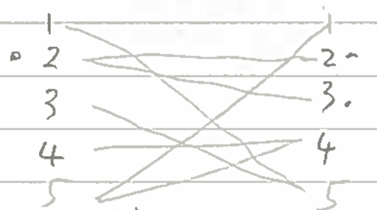
+1

		0	0	0	0	0
	5	4	3	2	1	0
	8	2	1	0	1	6
−1	5	4	2	1	1	0
	8	5	2	6	0	1
	5	1	4	2	0	1

$4 = 5 + 0 - 1$

	0	0	1	1	1
4	3	2	2	1	0
7	1	0	0	1	6
4	3	1	1	1	0
7	4	1	6	0	1
4	0	3	2	0	1

$4 = 4 + 1 - 1$



+1 +1

		0	0	1	1	1
−1	4	3	2	2	1	0
	7	1	0	0	1	6
	4	3	1	1	1	0
−1	7	4	1	6	0	1
	4	0	4	2	0	1

+1

Introduce variables $u_1 \sim u_5$ and $v_1 \sim v_5$, $w_{11} \sim w_{55}$

Maintain their values s.t.

① $\forall i, j$, $v_i + u_j \geq w_{ij}$

② matching value, calculated as $u_1 + \cdots + u_5 + v_1 + \cdots + v_5 - w_{ij}$ for a matching, does not change

$u_1 \sim u_5 = 0$

(1) Take a max for v_i in each row

and $w'_{ij} = v_i - w'_{ij}$

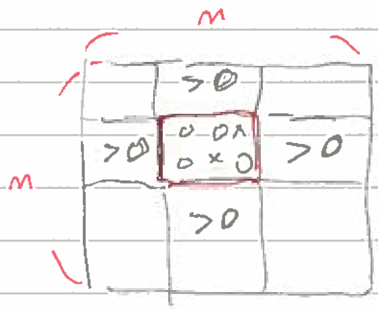
$0 = 31 - 5 = 26$

(2) Non-zero colums (or rows)

$0 = 29 - 3 = 26$

↳ if all 0's, we are done

No edges from $\{1, 3, 4, 5\}$ to $\{2, 3\}$
$\{2\}$ to $\{1, 4, 5\}$

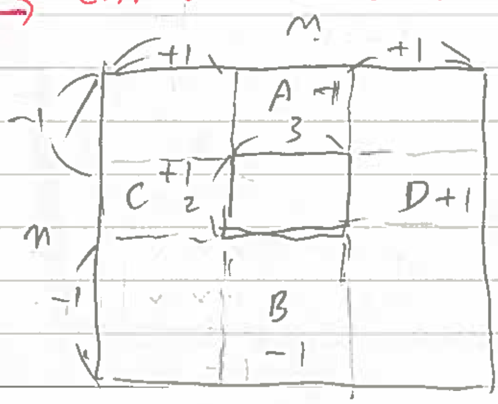


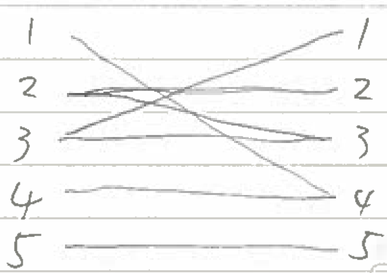
$$\begin{array}{c|ccccc} & 1 & 0 & 1 & 2 & 2 \\ \hline 3 & 3 & 1 & 1 & 1 & 0 \\ 7 & 2 & 0 & 0 & 2 & 7 \\ 3 & 3 & 0 & 0 & 1 & 0 \\ 6 & 4 & 0 & 5 & 0 & 1 \\ 3 & 0 & 3 & 1 & 0 & 1 \end{array}$$

$= 28 - 2 = 26$

$= \underline{28}$ <u>max</u>

correction of a mistake





$A+B > C+D$

$\boxed{\begin{array}{l} A \,\&\, B \; -1 \\ C \,\&\, D \; +1 \end{array}}$

<u>Th 1.2</u> A weighted maximum matching can be obtained in poly time

<u>Proof</u> (of sketch) Use the above algorithm
\# of 0's increases

Total value decreases strictly
Not the number of 0's

<u>Hungarian Method</u>

1.3 Men Women matchings with preferences

1	1	a	c	b	d	e		a	2	1	3	4	5
1	2	c	a	e	b	d		b	2	1	4	5	3
5	3	b	a	c	d	e		c	1	2	3	5	4
2	4	c	b	d	e	a		d	3	5	4	2	1
2	5	c	d	b	e	a		e	4	3	1	2	5

1+2=3

5

amount of regret (or favor)

Sum of weights =
$1+1+5+2+2+2+3+2+2+2=22$

1.4 Blocking pairs for a matching M

(3, d) : not matched in M &
each of 3 and "d" likes each other more than its current partner in M

Obviously not good since they have a strong motivation of escaping from the current partners.

New matching : No blocking pairs
↳ called a stable matching

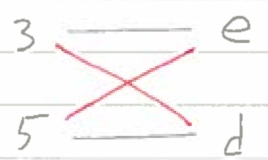
@ Man 3 has a complaint
d is too bad, b is 4's 2nd, ...
We can persuade: You say b is better but b does not like you as much as her current partner!

1.6 Removing blocking pairs

(3, d): blocking pair

⇓

what about just swapping partners



$\boxed{\text{conjectured by Knuth in 70's}}$

Good for this example, but could not prove for a long time

Finally disproved in [Tamura 93]

1	a	c	b	d	a	2	4	1	3
2	b	d	c	a	b	3	1	2	4
3	c	a	d	b	c	4	2	3	1
4	d	b	a	c	d	1	3	4	2

1 ⇔ 4

1	a	c	b	d	a	2	4	1	3
2	b	d	c	a	b	3	1	2	4
3	c	a	d	b	c	4	2	3	1
4	d	b	a	c	d	1	3	4	2

1 ⇔ 3

1	a	c	b	d	a	2	4	1	3
2	b	d	c	a	b	3	1	2	4
3	c	a	d	b	c	4	2	3	1
4	d	b	a	c	d	1	3	4	2

⇓

stable matching

2 Stable marriage problem

2.1 Gale Shapley algorithm

Obtaining a stable matching : Not Trivial at all

[Gale, Shapley 62]   David Gale : 1921–2008
Lloyd Shapley 1923–2016

Nobel Prize 2012

1	a	c	b	d	e
2	c	a	e	b	d
3	~~b~~	~~a~~	~~c~~	d	e
4	~~c~~	b	d	e	a
5	~~c~~	~~d~~	~~b~~	e	a

a	2	1	~~3~~	4	5
b	2	1	4	~~5~~	~~3~~
c	1	2	~~3~~	~~5~~	~~4~~
d	3	~~5~~	4	2	1
e	4	3	1	2	5

Algorithm GS
Input: preference lists
Output A stable matching

all men and women are single

While ∃ a single man whose list is not empty
select an arbitrary such man m
m proposes to his current best woman w
if w is single or m is better than her current partner m'
Then w accepts the proposal and engaged to m, rejects m',
m' deletes w' and becomes single
else w rejects the proposal
(keeps ~~single or~~ engaged to m')
and m deletes w (still single)
Output the current engagements

2.2 Correctness and Complexity

<u>Lemma 2.1</u> A woman is engaged to at most one man anytime. Similarly for a man

<u>Lemma 2.2</u> Once a woman gets engaged, she will never become single

<u>Lemma 2.3</u> A woman's partner never gets worse

<u>Th. 2.4</u> GS always outputs a stable matching in time $O(n^2)$ steps

<u>Proof</u> (1) GS always output a (complete) matching M

Suppose not. $\Rightarrow \exists$ a single man m and a single woman w by Lem. 2.1

$\Rightarrow$ By Lem. 2.2, w has been single all the time

$\Rightarrow$ m's list is empty, so m must have proposed to w, a contradiction

(2) M is stable

Suppose $\exists$ a blocking pair, (m, w)

$\Rightarrow m: \cdots w \cdots w' \cdots \qquad w: \cdots m \cdots m' \cdots$

$\Rightarrow$ m must have proposed to w

$\Rightarrow$ w is engaged to a man better than m then or sometime later

$\Rightarrow$ A contradiction to Lem. 2.3

(3) $O(n^2)$ steps since each man scans his list at most once, and at each propose there is some progress in women's lists

② Any instance has a sable matching

NO. 3

DATE

(~~always outputs a stable matching~~)

~~Th 2.4~~ ~~GS is correct, and runs in time at most $O(n^2)$ steps (n men & n women)~~

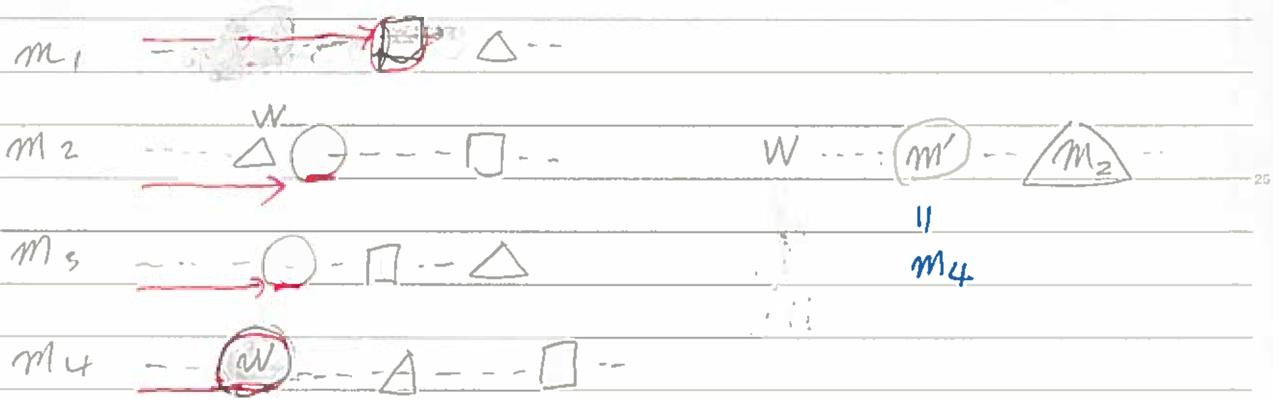
~~Proof~~ ~~Correctness : done~~

Time complexity : ~~at most one scan~~

Th. 2.5 GS does not depend on its execution sequence (always outputs The same matching)

Th. 2.6 Suppose that M is a matching provided by GS (for some execution sequence). Then there is no stable matching M′ such that ~~for some man m $M'(m) > M(m)$ (his partner is better in M′ than in M)~~

Proof. Suppose for contradiction That for some sequence of execution E, ∃ M and M′



□ : M   △ : M′   At The moment E first overtakes ~~...~~ (deletes) some △.

⟹ (m_4, w) is a blocking pair for △

Proof of Th 2.5 Suppose that M_1 and M_2 are both possible as outputs of GS (due to different sequences). Then M_1 and M_2 are both stable, so contradicts to Th. 2.6 (by just setting $M_1 = M$, $M_2 = M'$)

Output of GS : Men-optimal stable matching.

Remark : There are many different stable matchings in general for a single instance

1	a	b		a	2	1
2	b	a		b	1	2

1	a	b		a	2	1
2	b	a		b	1	2

$\Rightarrow 2^{n/2}$ different stable matchings

2.3 Men/Women optimal stable matchings

Women proposing GS $\Rightarrow$ Women optimal stable matching.

Men-optimal stable matching can be as bad as the following

1	a	b	c	d	e		a	2	3	4	5	1
2	b	c	d	a	e		b	3	4	5	1	2
3	c	d	a	b	e		c	4	5	1	2	3
4	d	a	b	c	e		d	5	1	2	3	4
5	a	b	c	d	e		e	1	2	3	4	5

2.4 Worst-Case and Average-Case complexity

Worst-Case ⇒ roughly n^2 proposals

propose ↓

1	c g a h ---		a	1	2	3	4	5	6	7	8	9
2	h e b a ---		b	1	2	3	4	5	6	7	8	9
3	a e c d ---		c	1	2	3	4	5	6	7	8	9
4	d f i c ---		d	1	2	3	4	5	6	7	8	9
5	~~h~~ ~~e~~ ~~a~~ g ---		e	1	2	3	4	5	6	7	8	9
6	d a c e ---		f	1	2	3	4	5	6	7	8	9
7	e f a b ---		g	1	2	3	4	5	6	7	8	9
8	a c g e b ---		h	1	2	3	4	5	6	7	8	9
9	a h c e h i ---		i	1	2	3	4	5	6	7	8	9

random

No big differences if women's lists are also random

Each man moves on his list until he gets to a single woman (cannot take an engaged woman)

each propose succeeds with

successful unless he proposes c

1 : prob n/n

2 : prob $n-1/n$

3 : prob $n-2/n$

⋮

$n-1$: prob $2/n$

n : " $1/n$

Expected # of proposals

= E[1's proposals + 2's proposals + ⋯ + n's proposals]

= E[1's proposals] + E[2's proposals] + ⋯ + E[n's proposals]

$= \frac{n}{n} + \frac{n}{n-1} + \cdots + \frac{n}{2} + \frac{n}{1}$

$= n \log_e n$

@ success prob = p
⇒ Expected # of tries = $1/p$

@ In many practical cases, GS is quick

3. Mathematical Structures of stable matchings

3.1 Many s.m.'s

I_0:

1	c	d	e	a	b	a	4	1	3	2	5
2	a	b	c	d	e	b	5	1	2	3	4
3	a	b	d	c	e	c	3	2	1	5	4
4	b	e	d	a	c	d	2	3	4	5	1
5	d	a	e	b	c	e	3	1	5	2	4

$M_1 = c\,b\,a\,e\,d$

$M_2 = c\,b\,a\,d\,e$

$M_3 = e\,c\,a\,d\,b$ $\qquad$ $M_4 = c\,b\,d\,a\,e$

$M_5 = e\,c\,d\,a\,b$

$M_6 = e\,d\,c\,a\,b$

$M_1 \succeq M_2 \quad M_2 \succeq M_3$
$M_2 \succeq M_4, \ \cdots$

$M_2 = M_3 \wedge M_4$
$M_5 = M_3 \vee M_4$

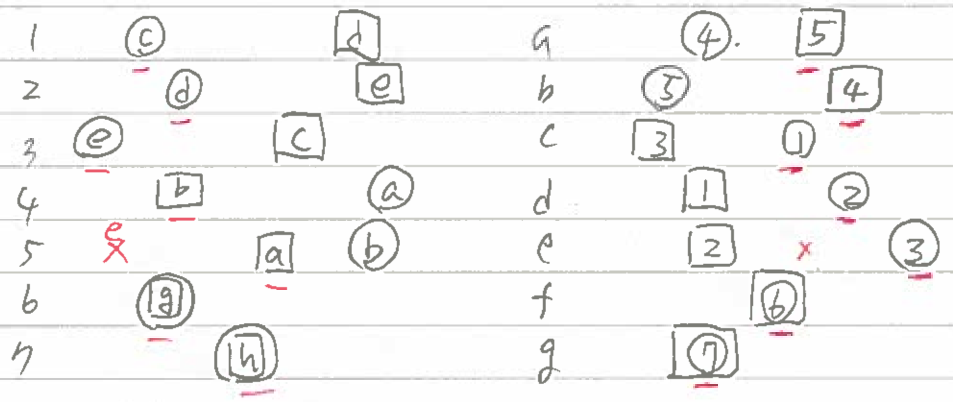
Def $M \succeq M' \overset{\text{def}}{\Longleftrightarrow}$ For $\forall$ man m, $M(m)$ is better (or equal to) than $M'(m)$

Def $M'' = \underset{\text{meet}}{M \wedge M'} \overset{\text{def}}{\Longleftrightarrow}$ For $\forall$ man m,
$M''(m) = \text{better}(M(m), M'(m))$

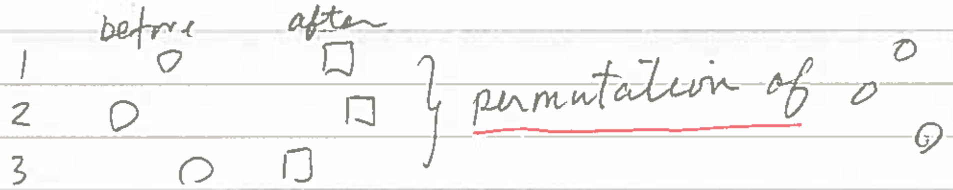
$M'' = \underset{\text{join}}{M \vee M'} \Longleftrightarrow M''(m) = \text{worse}(M(m), M'(m))$

$M, M' \in \mathfrak{M}(I)$: set of all stable matchings for an instance I

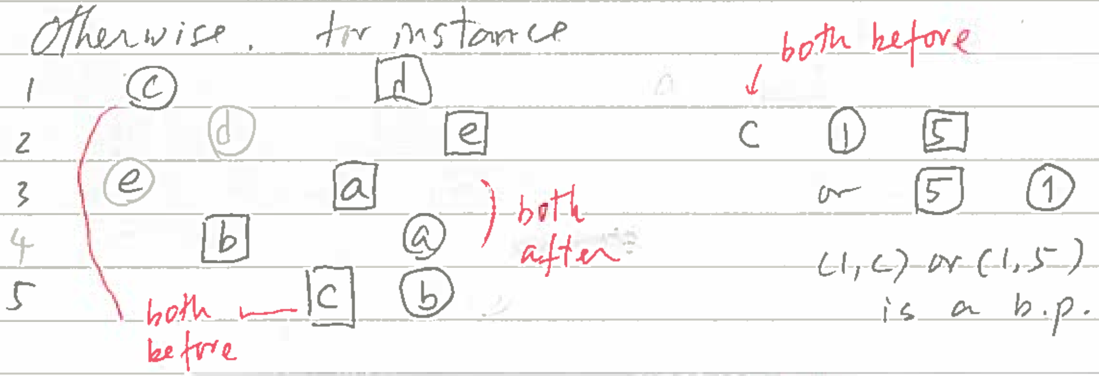
<u>Lemma 3.1</u> M: ○ and M': □ look like



(1)



(1') Otherwise, for instance



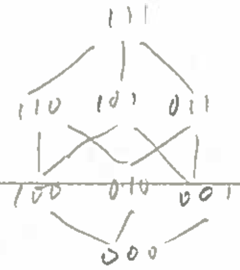
(2)

(i) otherwise $\exists$ an obvious b.p.

<u>Th 3.2</u> $M, M' \in \mathfrak{M}(I) \Rightarrow M \wedge M', M \vee M' \in \mathfrak{M}(I)$

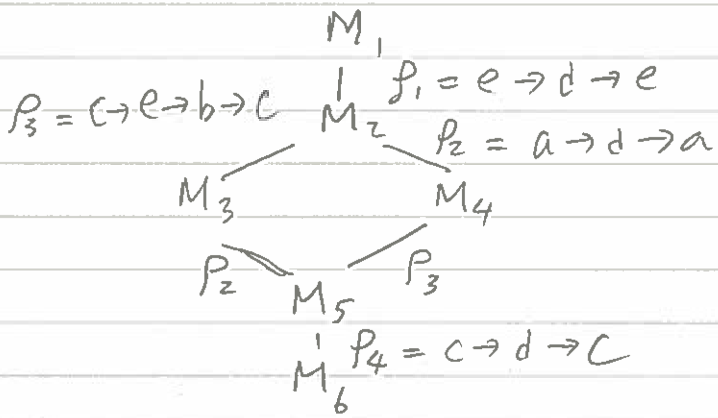
Proof a matching because of (1)

stable (1') $(5, x)$ is a b.p $\Rightarrow$ it is a b.p for □ or ○



3.2 Structure of $\mathcal{M}(I)$

- It's a POSET in terms of $\succeq$
- Closed under $\wedge$ and $\vee$ $\Rightarrow$ it's a lattice.



$\boxed{\rho_2 \text{ and } \rho_3 \text{ are disjoint}}$

Q How can we obtain each stable matchings in $\mathcal{M}(I)$?

M_1 : men-propose G.S

M_6 : women-propose G.S.

Def A rotation : A nice tool for this purpose

1	c	d	e	a	b	a	4	1	3	2	5
2	a	b	c	d	e	b	5	1	2	3	4
3	a	b	d	c	e	c	3	2	1	5	4
4	b	e	d	a	c	d	2	3	4	5	1
5	d	a	e	b	c	e	3	1	5	2	4

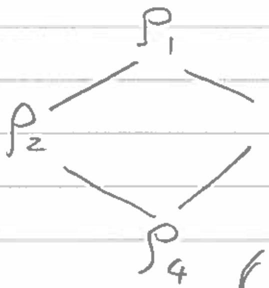
rotation ρ_1 : $(4,e) \to (4,d)$
$(5,d) \to (5,e)$
$(e \to d \to e)$

} permutation of

what about starting from $(1,c)$?

$M_1 \to M_2$ application (elimination) of rotation ρ_1

◎ Rotation POSET

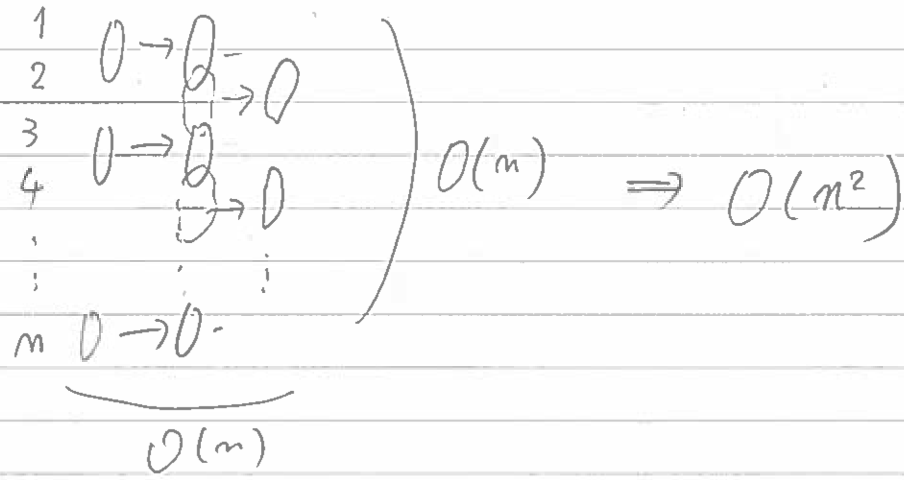


(ρ_3 can be eliminated only after ρ_1 is)

(only after ρ_2 and ρ_3 are eliminated)

Th. 3.3 All rotations of $\mathcal{M}(I)$ constitutes a POSET of size $O(n^2)$. They can be obtained in time $O(n^2)$

Proof



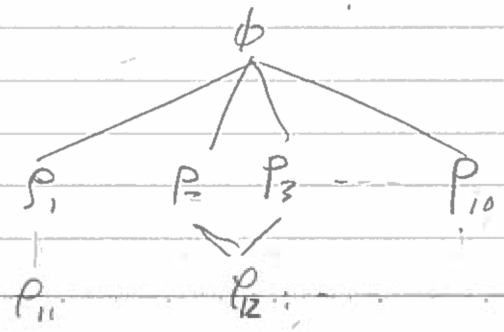
1	a	b	---
2	b	a	---
3	c	d	---
4	d	c	---
5			
6			
⋮			

a	2	1
b	1	2
c	4	3
d	3	4
e		
f		

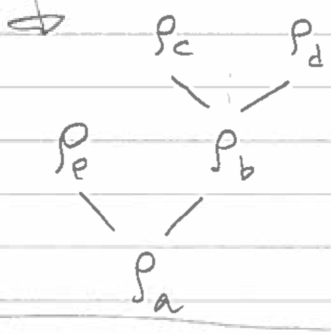
$\rho_1 = a \to b \to a$

$\rho_2 = c \to d \to c$

⋮



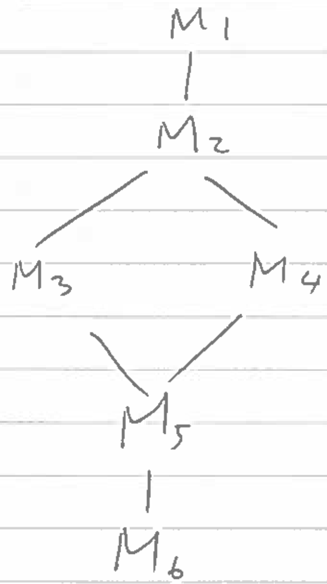
Th 3.4 Each matching in $\mathcal{M}(I)$ has a one-to-one correspondence to a closed subset of the rotation POSET of $\mathcal{M}(I)$

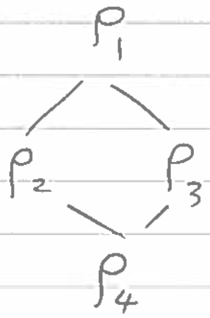


If a subset contains ρ_a then it also contains all rotations above ρ_a

ancesters decendents

Proof





$\phi \Leftrightarrow M_1$

$\{\rho_1\} \longleftrightarrow M_2$

$\{\rho_1, \rho_2\} \longleftrightarrow M_3$

$\{\rho_1, \rho_3\} \longleftrightarrow M_4$

$\{\rho_1, \rho_2, \rho_3\} \longleftrightarrow M_5$

$\{\rho_1, \rho_2, \rho_3, \rho_4\} \longleftrightarrow M_6$

$$\emptyset$$

ρ_1 ρ_2 ρ_3 --- $\rho_{n/2}$

$\emptyset \longleftrightarrow$

$\{\rho_1\} \longleftrightarrow$

$\{\rho_1, \rho_2\} \longleftrightarrow$

$\vdots$

exponentially many

3.4 "Good" stable matchings

1	c	e	a	d	b	a	$\boxed{4}$	1	3	2	5
2	a	b	c	d	e	b	5	1	2	3	4
3	a	$\boxed{d}$	c	e	b	c	2	3	1	5	4
4	b	e	d	$\boxed{a}$	c	d	$\boxed{3}$	2	4	5	1
5	d	a	e	b	c	e	3	1	2	5	4

$\rho_1 : e \to d \to e$ (−1) ; amount of goodness

$\rho_2 : c \to e \to b \to c$ (+3)

$\rho_3 : a \to d \to a$ (+2)

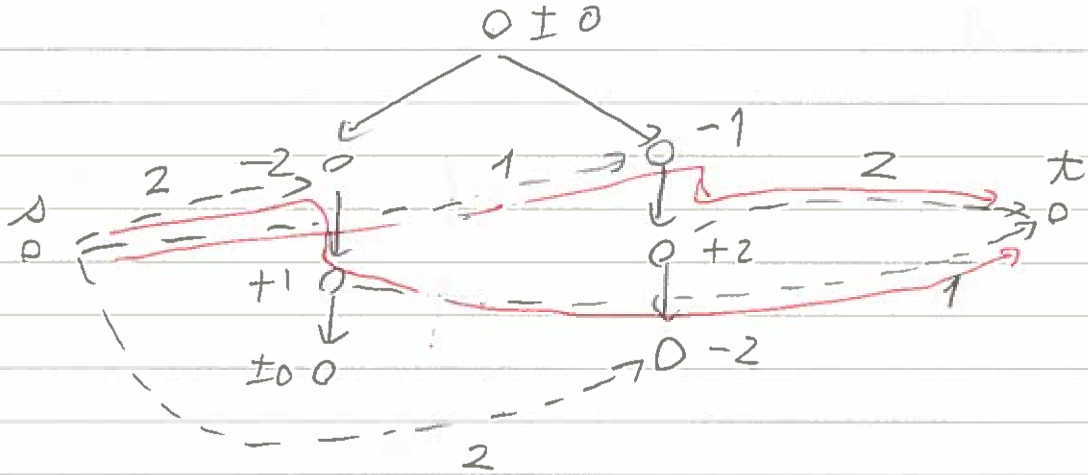
Elimination of $\{\rho_1\}$, $\{\rho_1, \rho_2\}$, $\{\rho_1, \rho_3\}$, $\{\rho_1, \rho_2, \rho_3\}$

-1 $\quad +1$ $\quad +2$ $\quad +4$

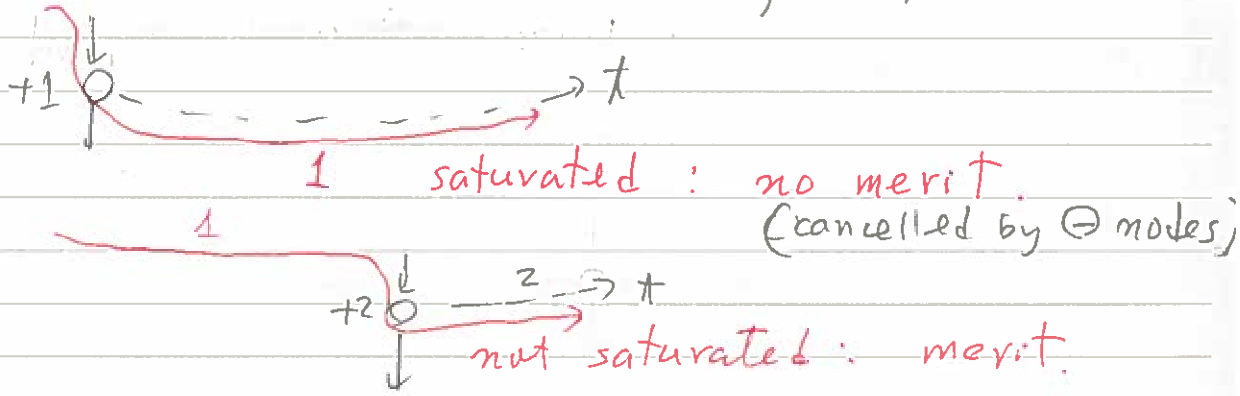
best

In general however,

We need to take a closed subset



We want to take + nodes as many as possible,



So, Take all + nodes that are not saturated and their predecessors (to make the set closed)

<u>Th. 3.5</u> Min-regret stable matching (in $\mathcal{M}(I)$) can be obtained in a poly time.

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⑥ Sex-equal stable matching

D= (man regret − woman regret)

M_1 $7-18=-11$

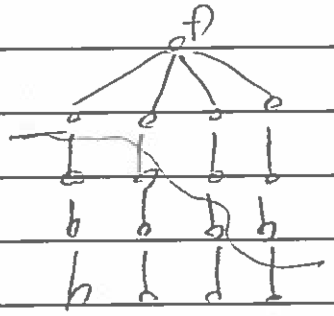
M_2 -5

$14-11=3$ M_3 M_4 $13-12=1$

M_5 $17-8=9$

M_6 $19-6=13$

We want a stable matching with this value 0 (or a very small)



ϕ $(D=-11)$

ρ_1 $+6$ $(=+3-(-3))$

$+8$ ρ_2 ρ_3 $+6$

ρ_4 $+4$

Our problem: Obtain a closed subset having the total value of D.

Th 3.5 Sex-equal stable matching is NP-hard

We can design approximation algorithms using a rotation POSET