Low-Distortion Embeddings of Metric Spaces

Yoshio Okamoto

Dept. Information and Computer Sciences, Toyohashi University of Technology

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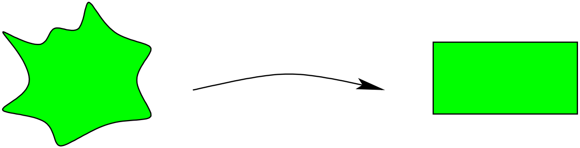
NHC Autumn School on Computational Geometry and Integer Programming

We want to do

Transform one metric space into another



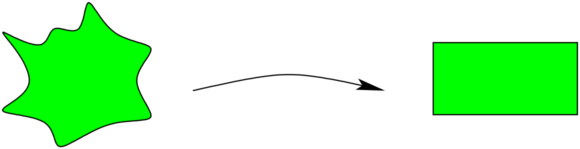
Algorithmic applications

- Geometric approximation algorithms
  Closest pairs, Nearest neighbors, ...
- Combinatorial approximation algorithms
  Sparsest cuts, Multi-commodity flows, Bandwidths, ...
  (cf. J. Lee's NHC Workshop Talk)
- Inapproximability (integrality gap of SDP relaxation)
  Vertex covers, Unique game conjectures, ...
- On-line algorithms
  Metrical task systems, ...
- Streaming algorithms
- ...

How we want to transform the space

Criteria

- efficiently
- into a space as low-dimensional as possible
- into a space as simple as possible
- without much distortion



Contents

- Introduction (10 min)
- Embedding into ℓ_∞ (10 min)
- Embedding into ℓ_2 (30 min)
- Lower bound for ℓ_2 (30 min)
- Remarks (10 min)

References

Books

- "Lectures on Discrete Geometry" by Matoušek, Springer, 2002
- "Geometry of Cuts and Metrics" by Deza and Laurent, Springer, 1997

Surveys

- "Low-distortion embeddings of finite metric spaces" by Indyk and Matoušek, in "Handbook on Discrete and Computational Geometry," 2004
- "Algorithmic applications of low-distortion embeddings" by Indyk, FOCS 2001
- "Finite metric spaces—combinatorics, geometry and algorithms" by Linial, ICM 2002

Def.: metric space

A pair (X, μ) of a set X and a map $\mu\colon X \times X \to \mathbb{R}_+$ is called a metric space if

- $\mu(x, y) = 0 \Leftrightarrow x = y$,
- $\mu(x, y) = \mu(y, x)$ for all $x, y \in X$,
- $\mu(x, y) + \mu(y, z) \geq \mu(x, z)$ for all $x, y, z \in X$.

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Def.: finite metric space

A finite metric space is a metric space (X, μ) with X finite.

- Note: This is *different* from a discrete metric space.

By a matrix

$$\begin{array}{c|ccccc|} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_1 & 0 & 2 & 3 & 4 & 3 \\ x_2 & 2 & 0 & 4 & 2 & 1 \\ x_3 & 3 & 4 & 0 & 2 & 5 \\ x_4 & 4 & 2 & 2 & 0 & 3 \\ x_5 & 3 & 1 & 5 & 3 & 0 \\ \hline \end{array}$$

The i, j-component represents $\mu(x_i, x_j)$.

Def.: normed spaces

A pair $(X, \|\cdot\|)$ of a vector space X on $\mathbb{R}$ and a map $x \in X \mapsto \|x\| \in \mathbb{R}_+$ is called a normed space if

- $\|x\| = 0 \Leftrightarrow x = 0$,
- $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{R}$, $x \in X$,
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

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Observation (or Exercise)

Given a normed space $(X, \|\cdot\|)$, we define $\mu \colon X \times X \to \mathbb{R}_+$ as

$$\mu(x, y) = \|x - y\|.$$

Then, (X, μ) is a metric space.

$\therefore$ we may think of normed spaces as metric spaces.

Def.: ℓ_p-norms

Define $\|x\|_p \in \mathbb{R}_+$ for every $x \in \mathbb{R}^d$ as

$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p}.$$

Then $(\mathbb{R}^d, \|\cdot\|_p)$ is a normed space, denoted by ℓ_p^d.
This norm is called the ℓ_p-norm.

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FANs (frequently asked norms)

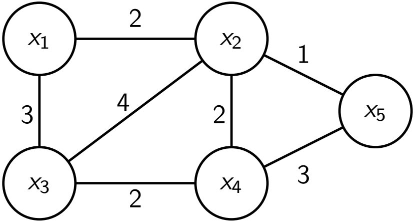
- $p = 1$: $\sum_{i=1}^n |x_i|$ (the Manhattan norm)
- $p = 2$: $\sqrt{\sum_{i=1}^n |x_i|^2}$ (the Euclidean norm)
- $p = \infty$: $\max_{i=1}^n |x_i|$ (the maximum norm)

$G = (V, E)$ a finite undirected graph, connected
$w\colon E \to \mathbb{R}_+$ an edge-weight function

Def.: shortest-path metrics

A pair (V, μ) is a shortest-path metric on G if
$\mu(u, v)$ is the shortest-path distance between u & v on G w.r.t w.

	x_1	x_2	x_3	x_4	x_5
x_1	0	2	3	4	3
x_2	2	0	4	2	1
x_3	3	4	0	2	5
x_4	4	2	2	0	3
x_5	3	1	5	3	0



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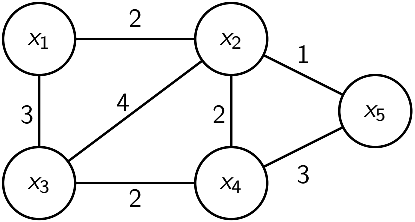


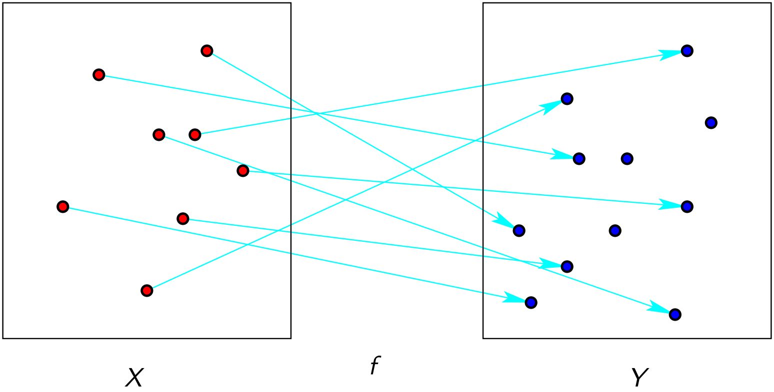
Remark (or Exercise)

Every finite metric space is a shortest-path metric on a graph.

(X, μ), (Y, ν) two metric spaces

Def.: embeddings

An embedding of (X, μ) into (Y, ν) is a map $f : X \to Y$.

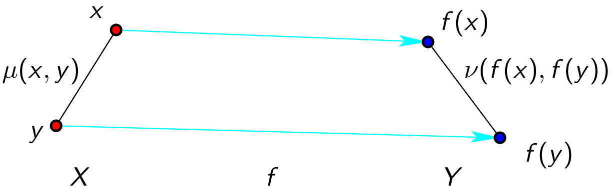


(X, μ), (Y, ν) two metric spaces, $D \geq 1$

Def.: D-embeddings

A *D-embedding* is an embedding $f : X \to Y$ s.t.
$\exists\ r > 0, \forall\ x, y \in X$

$$r \cdot \mu(x, y) \leq \nu(f(x), f(y)) \leq D \cdot r \cdot \mu(x, y).$$



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Def.: distortion

The *distortion* of an embedding f is $\inf\{D \colon f \text{ is a } D\text{-embedding}\}$.

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Def.: distortion

The distortion of an embedding f is $\inf\{D \colon f \text{ is a } D\text{-embedding}\}$.

Def.: isometry

An embedding is an isometry if its distortion is 1.

Def.: embedding into ℓ_p

An embedding into ℓ_p of a metric space (X, μ) is an embedding $(X, \mu) \to \ell_p^d$ for some finite d.

Note:

- We may also define the normed space ℓ_p, but it will be a bit subtle. We need to work around the infinity and convergence issues. Thus, we use ℓ_p just as a notational convenience.

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1. Every n-point metric space can be isometrically embedded into ℓ_∞.

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1. Every n-point metric space can be isometrically embedded into ℓ_∞.
2. Every n-point metric space can be embedded into ℓ_2 with distortion $O(\log n)$.
3. There exists an n-point metric space that requires the distortion of $\Omega(\log n)$ when embedded into ℓ_2.

Contents

Isometric embeddability into ℓ_∞

(X, μ) a finite metric space

Theorem

(X, μ) can be embedded into ℓ_∞ with distortion 1.

(X, μ) a finite metric space

Theorem

(X, μ) can be embedded into ℓ_∞ with distortion 1.

Proof Outline:

- Explicitly construct a particular embedding
- Prove that it is an isometry

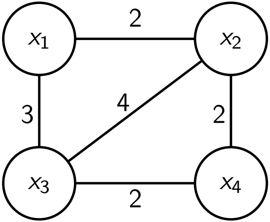
(X, μ) an n-point metric space; $X = \{x_1, \ldots, x_n\}$

Construction

Define a map $f : X \to \ell_\infty^n$ as

$$f(x)_i = \mu(x, x_i)$$

for every $x \in X$ and $i \in \{1, \ldots, n\}$



$$\begin{aligned} f(x_1) &= (0, 2, 3, 4) \\ f(x_2) &= (2, 0, 4, 2) \\ f(x_3) &= (3, 4, 0, 2) \\ f(x_4) &= (4, 2, 2, 0) \end{aligned}$$

Claim

The constructed f is an isometry.

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$$\|f(x_i) - f(x_j)\|_\infty \quad = \quad \max\{|f(x_i)_k - f(x_j)_k| \colon k \in \{1, \ldots, n\}\}$$

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$$\begin{aligned}\|f(x_i) - f(x_j)\|_\infty &= \max\{|f(x_i)_k - f(x_j)_k| \colon k \in \{1, \dots, n\}\} \\ &\geq |f(x_i)_j - f(x_j)_j|\end{aligned}$$

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$$\begin{aligned}\|f(x_i) - f(x_j)\|_\infty &= \max\{|f(x_i)_k - f(x_j)_k| : k \in \{1, \dots, n\}\} \\ &\geq |f(x_i)_j - f(x_j)_j| \\ &= |\mu(x_i, x_j) - \mu(x_j, x_j)|\end{aligned}$$

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$$\|f(x_i) - f(x_j)\|_\infty = |f(x_i)_k - f(x_j)_k| \quad \text{(for some } k\text{)}$$

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(X, μ) an n-point metric space

Theorem (Bourgain '85)

(X, μ) can be embedded into ℓ_2 with distortion $O(\log n)$.

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Proof Outline:

- Explicitly construct a particular embedding
  Some probability involved
- Prove that the distortion is $O(\log n)$

(X, μ) an n-point metric space; $q = \lfloor \log_2 n \rfloor + 1$

Construction

- Construct $A \subseteq X$ at random as follows:

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  - For each $j \in \{1, \ldots, q\}$ construct $A_j \subseteq X$ by sampling every point of X independently with prob. $1/2^j$

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  - Set $A = A_j$
- Define a map $f \colon X \to \ell_2^{2^n}$ as

$$f(x)_S = \sqrt{\Pr[S = A]}\, \mu(x, S)$$

for every $x \in X$ and $S \subseteq X$

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Notation

$\mu(x, S) = \min\{\mu(x, y) \colon y \in S\}$

(X, μ) an n-point metric space; $q = \lfloor \log_2 n \rfloor + 1$

Claim 1

The constructed f satisfies

$$\|f(x) - f(y)\|_2 \leq \mu(x, y)$$

for every $x, y \in X$

Proof: exercise

(X, μ) an n-point metric space; $q = \lfloor \log_2 n \rfloor + 1$

Claim 2

The constructed f satisfies

$$\mu(x, y) \leq 32q \|f(x) - f(y)\|_2$$

for every $x, y \in X$

(X, μ) an n-point metric space; $q = \lfloor \log_2 n \rfloor + 1$

Claim 2

The constructed f satisfies

$$\mu(x, y) \leq 32q \|f(x) - f(y)\|_2$$

for every $x, y \in X$

Note

The exact coefficient $32q$ is not important (it could be easily improved). It is only important that the coefficient is $O(\log n)$.

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Proof:

- Let's first look at some calculation

Notation

$p_S = \Pr[S = A]$

$$\|f(x) - f(y)\|_2 = \sqrt{\sum_{S \subseteq X} |f(x)_S - f(y)_S|^2}$$

Some calculation

Notation

$p_S = \Pr[S = A]$

$$
\begin{aligned}
\|f(x) - f(y)\|_2 &= \sqrt{\sum_{S \subseteq X} |f(x)_S - f(y)_S|^2} \\
&= \sqrt{\sum_{S \subseteq X} |\sqrt{p_S}\mu(x, S) - \sqrt{p_S}\mu(y, S)|^2}
\end{aligned}
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&= \sqrt{\sum_{S \subseteq X} p_S|\mu(x,S) - \mu(y,S)|^2}\sqrt{\sum_{S \subseteq X} p_S}
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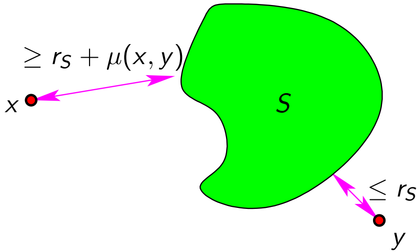
&= \sqrt{\sum_{S \subseteq X} p_S|\mu(x, S) - \mu(y, S)|^2}\sqrt{\sum_{S \subseteq X} p_S} \\
&\geq \sum_{S \subseteq X} p_S|\mu(x, S) - \mu(y, S)|
\end{aligned}
$$

Claim 2'

$$\sum_{S \subseteq X} p_S |\mu(x,S) - \mu(y,S)| \geq \frac{\mu(x,y)}{32q}$$

Proof idea:

- Try to show for "many" sets $S \subseteq X$ there exists r_S s.t. $\mu(x,S) \geq r_S + \mu(x,y)$ and $\mu(y,S) \leq r_S$



Suppose..

$\forall\ S \subseteq X\ \exists\ r_S > 0$ s.t. $\mu(x, S) \geq r_S + \mu(x, y)$ and $\mu(y, S) \leq r_S$

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Then,

$$\sum_{S \subseteq X} p_S |\mu(x,S) - \mu(y,S)| \ \geq \ \sum_{S \subseteq X} p_S((r_S + \mu(x,y)) - r_S)$$

Suppose..

$\forall\ S \subseteq X\ \exists\ r_S > 0$ s.t. $\mu(x,S) \geq r_S + \mu(x,y)$ and $\mu(y,S) \leq r_S$

Then,

$$\begin{aligned}\sum_{S\subseteq X} p_S|\mu(x,S) - \mu(y,S)| &\geq \sum_{S\subseteq X} p_S((r_S + \mu(x,y)) - r_S) \\ &= \sum_{S\subseteq X} p_S\mu(x,y) = \mu(x,y)\end{aligned}$$

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Thus, we have no expansion!!

Suppose..

$\forall\ S \subseteq X\ \exists\ r_S > 0$ s.t. $\mu(x, S) \geq r_S + \mu(x, y)$ and $\mu(y, S) \leq r_S$

Then,

$$\begin{aligned}\sum_{S\subseteq X} p_S|\mu(x,S) - \mu(y,S)| &\geq \sum_{S\subseteq X} p_S((r_S + \mu(x,y)) - r_S) \\ &= \sum_{S\subseteq X} p_S\mu(x,y) = \mu(x,y)\end{aligned}$$

Thus, we have no expansion!! But, ...

- Not all S satisfy this assumption
- However, if "many" sets satisfy the assumption, the result holds!

Claim 2'

$$\sum_{S \subseteq X} p_S |\mu(x,S) - \mu(y,S)| \geq \frac{\mu(x,y)}{32q}$$

Claim modified

Claim 2'

$$\sum_{S \subseteq X} p_S |\mu(x,S) - \mu(y,S)| \geq \frac{\mu(x,y)}{32q}$$

Proof idea:

- Try to show for "many" sets $S \subseteq X$
  there exists r_S s.t. $\mu(x,S) \geq r_S + \mu(x,y)$ and $\mu(y,S) \leq r_S$

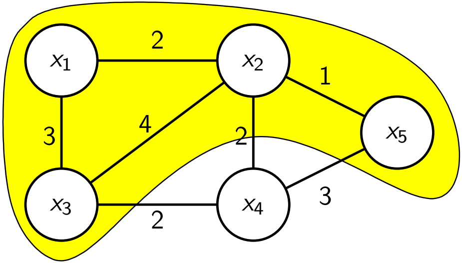
To this end

- Instead of counting, we look at probabilities...
- Quantize $|\mu(x,S) - \mu(y,S)|$ with j
- Throw out some sets S from the summation
- Bound p_S

Definition (a ball of radius r centered at c)

$B(c, r) = \{z \in X : \mu(c, z) \leq r\}$ (closed)
$B^{\circ}(c, r) = \{z \in X : \mu(c, z) < r\}$ (open)

$B(x_1, 3)$

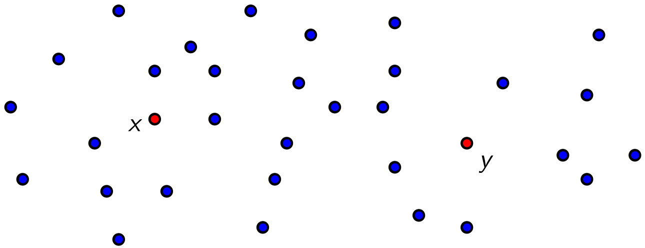


Fix $x, y \in X$

Notation

- For every $j \in \{0, 1, \ldots, q\}$

$$\tilde{r}_j = \min\{r \colon |B(x,r)| \geq 2^j \text{ and } |B(y,r)| \geq 2^j\}$$



Fix $x, y \in X$

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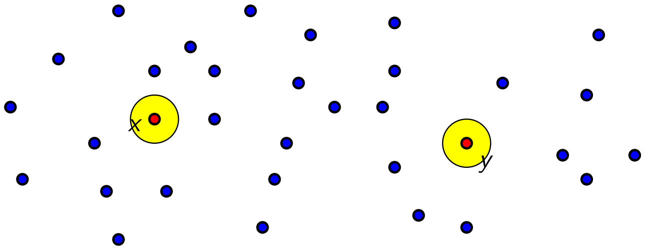

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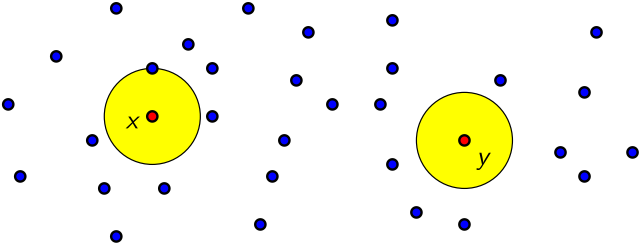

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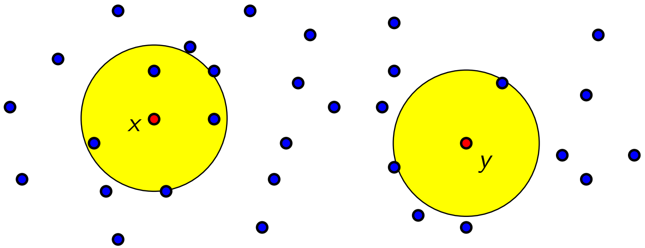

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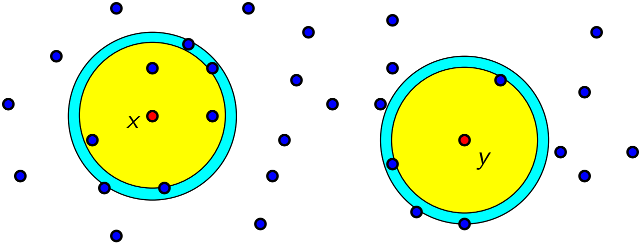

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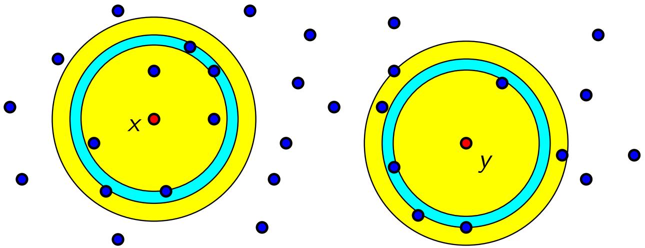

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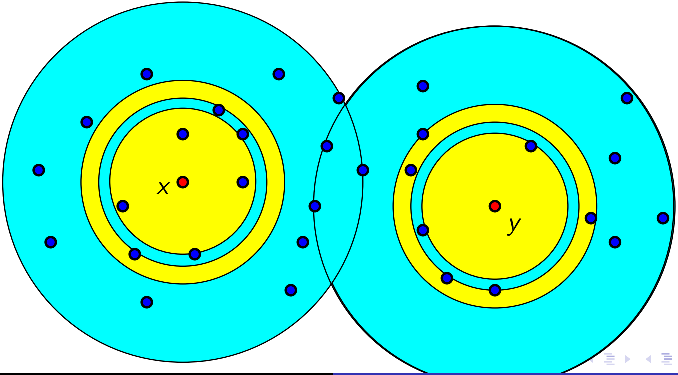


Fix $x, y \in X$

Notation

- For every $j \in \{0, 1, \ldots, q\}$

$$\tilde{r}_j = \min\{r \colon |B(x,r)| \geq 2^j \text{ and } |B(y,r)| \geq 2^j\}$$

Fix $x, y \in X$

Notation

- For every $j \in \{0, 1, \ldots, q\}$

$$\tilde{r}_j = \min\{r \colon |B(x, r)| \geq 2^j \text{ and } |B(y, r)| \geq 2^j\}$$

- Let i be an index satisfying

$$\tilde{r}_0 \leq \tilde{r}_1 \leq \cdots \leq \tilde{r}_{i-1} \leq \frac{1}{2}\mu(x, y) \leq \tilde{r}_i \leq \cdots \leq \tilde{r}_q$$

Fix $x, y \in X$

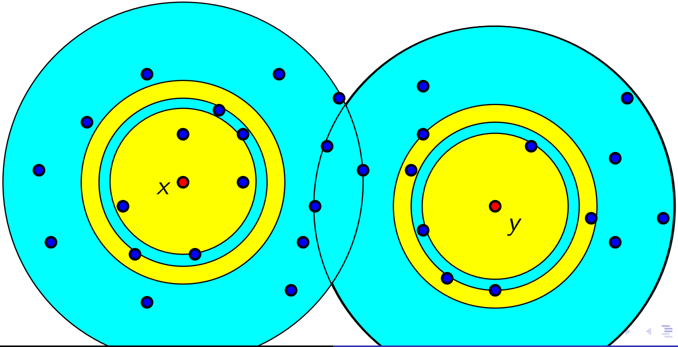
Notation

- For every $j \in \{0, 1, \dots, q\}$
$$\tilde{r}_j = \min\{r\colon |B(x, r)| \geq 2^j \text{ and } |B(y, r)| \geq 2^j\}$$
- Let i be an index satisfying
$$\tilde{r}_0 \leq \tilde{r}_1 \leq \dots \leq \tilde{r}_{i-1} \leq \frac{1}{2}\mu(x, y) \leq \tilde{r}_i \leq \dots \leq \tilde{r}_q$$
- For every $j \in \{0, 1, \dots, i\}$
$$r_j = \begin{cases} \tilde{r}_j & (j \in \{0, 1, \dots, i-1\}) \\ \frac{1}{2}\mu(x, y) & (j = i) \end{cases}$$

$j \in \{1, \ldots, i\}$ fixed

Observations

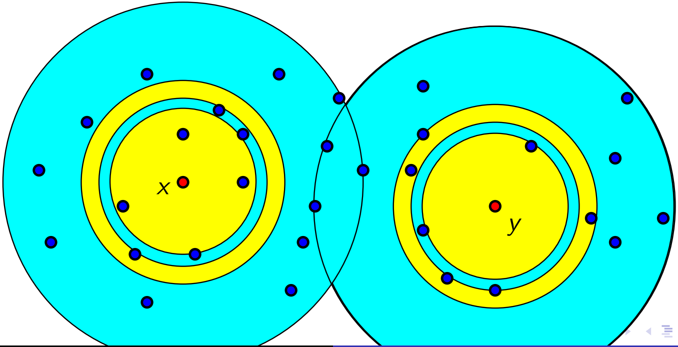
- $|B^\circ(x, r_j)| < 2^j$ or $|B^\circ(y, r_j)| < 2^j$ (from the def of r_j)



$j \in \{1, \ldots, i\}$ fixed

Observations

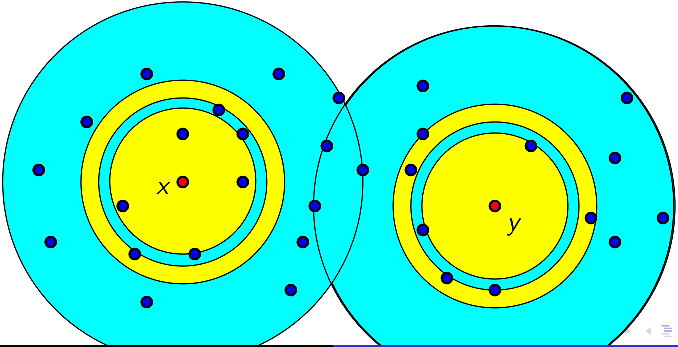
- $|B^\circ(x, r_j)| < 2^j$ or $|B^\circ(y, r_j)| < 2^j$ (from the def of r_j)
- WLOG $|B^\circ(x, r_j)| < 2^j$



$j \in \{1, \ldots, i\}$ fixed

Observations

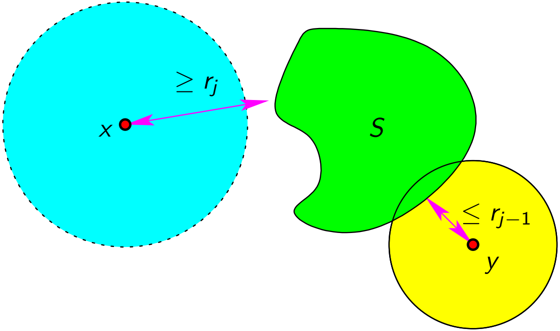
- $|B^\circ(x, r_j)| < 2^j$ or $|B^\circ(y, r_j)| < 2^j$ (from the def of r_j)
- WLOG $|B^\circ(x, r_j)| < 2^j$
- $|B(y, r_{j-1})| \geq 2^{j-1}$ (from the def of r_{j-1})



$j \in \{1, \ldots, i\}$ fixed

Important observation

$B^\circ(x, r_j) \cap S = \emptyset$ and $B(y, r_{j-1}) \cap S \neq \emptyset$
$\Rightarrow |\mu(x, S) - \mu(y, S)| \geq r_j - r_{j-1}$



Throw out some sets S from the summation (1)

$$\sum_{S \subseteq X} \Pr[S = A] |\mu(x, S) - \mu(y, S)|$$

Throw out some sets S from the summation (1)

$$\sum_{S\subseteq X} \Pr[S = A]|\mu(x,S) - \mu(y,S)|$$

$$= \sum_{S\subseteq X}\sum_{j=1}^{q} \Pr[S = A \mid j \text{ chosen}] \Pr[j \text{ chosen}]|\mu(x,S) - \mu(y,S)|$$

Throw out some sets S from the summation (1)

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$$= \sum_{S \subseteq X} \sum_{j=1}^{q} \Pr[S = A \mid j \text{ chosen}] \frac{1}{q} |\mu(x, S) - \mu(y, S)|$$

$$
\begin{aligned}
&\sum_{S\subseteq X} \Pr[S = A]|\mu(x,S) - \mu(y,S)| \\
&= \sum_{S\subseteq X}\sum_{j=1}^{q} \Pr[S = A \mid j \text{ chosen}]\Pr[j \text{ chosen}]|\mu(x,S) - \mu(y,S)| \\
&= \sum_{S\subseteq X}\sum_{j=1}^{q} \Pr[S = A \mid j \text{ chosen}]\frac{1}{q}|\mu(x,S) - \mu(y,S)| \\
&= \frac{1}{q}\sum_{j=1}^{q}\sum_{S\subseteq X} \Pr[S = A \mid j \text{ chosen}]|\mu(x,S) - \mu(y,S)|
\end{aligned}
$$

Throw out some sets S from the summation (1)

$$
\begin{aligned}
&\sum_{S\subseteq X} \Pr[S = A]|\mu(x,S) - \mu(y,S)| \\
&= \sum_{S\subseteq X}\sum_{j=1}^{q} \Pr[S = A \mid j \text{ chosen}]\Pr[j \text{ chosen}]|\mu(x,S) - \mu(y,S)| \\
&= \sum_{S\subseteq X}\sum_{j=1}^{q} \Pr[S = A \mid j \text{ chosen}]\frac{1}{q}|\mu(x,S) - \mu(y,S)| \\
&= \frac{1}{q}\sum_{j=1}^{q}\sum_{S\subseteq X} \Pr[S = A \mid j \text{ chosen}]|\mu(x,S) - \mu(y,S)| \\
&\geq \frac{1}{q}\sum_{j=1}^{i}\sum_{S\subseteq X} \Pr[S = A \mid j \text{ chosen}]|\mu(x,S) - \mu(y,S)|
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{S \subseteq X} \Pr[S = A] |\mu(x, S) - \mu(y, S)| \\
&= \sum_{S \subseteq X} \sum_{j=1}^{q} \Pr[S = A \mid j \text{ chosen}] \Pr[j \text{ chosen}] |\mu(x, S) - \mu(y, S)| \\
&= \sum_{S \subseteq X} \sum_{j=1}^{q} \Pr[S = A \mid j \text{ chosen}] \frac{1}{q} |\mu(x, S) - \mu(y, S)| \\
&= \frac{1}{q} \sum_{j=1}^{q} \sum_{S \subseteq X} \Pr[S = A \mid j \text{ chosen}] |\mu(x, S) - \mu(y, S)| \\
&\geq \frac{1}{q} \sum_{j=1}^{i} \sum_{S \subseteq X} \Pr[S = A \mid j \text{ chosen}] |\mu(x, S) - \mu(y, S)| \\
&\geq \frac{1}{q} \sum_{j=1}^{i} \sum_{\substack{S \subseteq X, \\ B^\circ(x, r_j) \cap S = \emptyset, \\ B(y, r_{j-1}) \cap S \neq \emptyset}} \Pr[S = A \mid j \text{ chosen}] |\mu(x, S) - \mu(y, S)|
\end{aligned}
$$

$$\sum_{S \subseteq X} \Pr[S = A] |\mu(x, S) - \mu(y, S)|$$

$$\geq \frac{1}{q} \sum_{j=1}^{i} \sum_{\substack{S \subseteq X, \\ B^\circ(x, r_j) \cap S = \emptyset, \\ B(y, r_{j-1}) \cap S \neq \emptyset}} \Pr[S = A \mid j \text{ chosen}] |\mu(x, S) - \mu(y, S)|$$

$$\sum_{S \subseteq X} \Pr[S = A] |\mu(x, S) - \mu(y, S)|$$

$$\geq \frac{1}{q} \sum_{j=1}^{i} \sum_{\substack{S \subseteq X, \\ B^\circ(x, r_j) \cap S = \emptyset, \\ B(y, r_{j-1}) \cap S \neq \emptyset}} \Pr[S = A \mid j \text{ chosen}] |\mu(x, S) - \mu(y, S)|$$

$$\geq \frac{1}{q} \sum_{j=1}^{i} \sum_{\substack{S \subseteq X, \\ B^\circ(x, r_j) \cap S = \emptyset, \\ B(y, r_{j-1}) \cap S \neq \emptyset}} \Pr[S = A \mid j \text{ chosen}] (r_j - r_{j-1})$$

Throw out some sets S from the summation (2)

$$
\begin{aligned}
&\sum_{S \subseteq X} \Pr[S = A] |\mu(x, S) - \mu(y, S)| \\
&\geq \frac{1}{q} \sum_{j=1}^{i} \sum_{\substack{S \subseteq X, \\ B^\circ(x, r_j) \cap S = \emptyset, \\ B(y, r_{j-1}) \cap S \neq \emptyset}} \Pr[S = A \mid j \text{ chosen}] |\mu(x, S) - \mu(y, S)| \\
&\geq \frac{1}{q} \sum_{j=1}^{i} \sum_{\substack{S \subseteq X, \\ B^\circ(x, r_j) \cap S = \emptyset, \\ B(y, r_{j-1}) \cap S \neq \emptyset}} \Pr[S = A \mid j \text{ chosen}] (r_j - r_{j-1}) \\
&= \frac{1}{q} \sum_{j=1}^{i} (r_j - r_{j-1}) \underbrace{\sum_{\substack{S \subseteq X, \\ B^\circ(x, r_j) \cap S = \emptyset, \\ B(y, r_{j-1}) \cap S \neq \emptyset}} \Pr[S = A \mid j \text{ chosen}]}_{\text{need to bound}}
\end{aligned}
$$

Bound the probability (1)

$$\sum_{\substack{S \subseteq X, \\ B^\circ(x, r_j) \cap S = \emptyset, \\ B(y, r_{j-1}) \cap S \neq \emptyset}} \underbrace{\Pr[S = A \mid j \text{ chosen}]}_{= \Pr[S = A_j]}$$

Bound the probability (1)

$$\sum_{\substack{S \subseteq X,\\ B^\circ(x,r_j)\cap S=\emptyset,\\ B(y,r_{j-1})\cap S\neq\emptyset}} \underbrace{\Pr[S = A \mid j \text{ chosen}]}_{=\Pr[S=A_j]}$$

$$= \Pr[B^\circ(x, r_j) \cap A_j = \emptyset \text{ and } B(y, r_{j-1}) \cap A_j \neq \emptyset]$$

Bound the probability (1)

$$\sum_{\substack{S\subseteq X,\\ B^\circ(x,r_j)\cap S=\emptyset,\\ B(y,r_{j-1})\cap S\neq\emptyset}} \underbrace{\Pr[S=A \mid j \text{ chosen}]}_{=\Pr[S=A_j]}$$

$$= \Pr[B^\circ(x,r_j)\cap A_j=\emptyset \text{ and } B(y,r_{j-1})\cap A_j\neq\emptyset]$$

$$= \Pr[B^\circ(x,r_j)\cap A_j=\emptyset]\cdot \Pr[B(y,r_{j-1})\cap A_j\neq\emptyset]$$

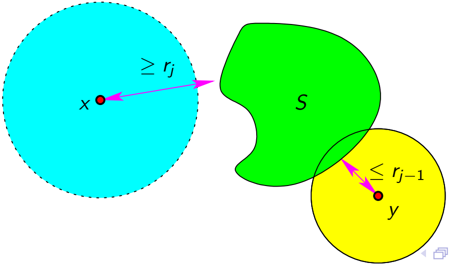
$$\sum_{\substack{S \subseteq X, \\ B^\circ(x, r_j) \cap S = \emptyset, \\ B(y, r_{j-1}) \cap S \neq \emptyset}} \underbrace{\Pr[S = A \mid j \text{ chosen}]}_{= \Pr[S = A_j]}$$

$$= \Pr[B^\circ(x, r_j) \cap A_j = \emptyset \text{ and } B(y, r_{j-1}) \cap A_j \neq \emptyset]$$

$$= \Pr[B^\circ(x, r_j) \cap A_j = \emptyset] \cdot \Pr[B(y, r_{j-1}) \cap A_j \neq \emptyset]$$

Note

$j \leq i$ implies $B^\circ(x, r_j) \cap B(y, r_{j-1}) = \emptyset$,
$\therefore$ the events $B^\circ(x, r_j) \cap S = \emptyset$ and $B(y, r_{j-1}) \cap S \neq \emptyset$ independent.



Bound the probability (2)

$$\Pr[B^\circ(x, r_j) \cap A_j = \emptyset]$$

Bound the probability (2)

$$\Pr[B^\circ(x, r_j) \cap A_j = \emptyset] \quad = \quad \Pr[z \notin A_j \text{ for all } z \in B^\circ(x, r_j)]$$

$$\begin{aligned}\Pr[B^\circ(x, r_j) \cap A_j = \emptyset] &= \Pr[z \notin A_j \text{ for all } z \in B^\circ(x, r_j)] \\ &= \left(1 - \frac{1}{2^j}\right)^{|B^\circ(x,r_j)|}\end{aligned}$$

Reminder (construction of A_j)

For each $j \in \{1, \dots, q\}$ construct $A_j \subseteq X$ by sampling every point of X independently with prob. $1/2^j$

$$\begin{aligned} \Pr[B^\circ(x, r_j) \cap A_j = \emptyset] &= \Pr[z \notin A_j \text{ for all } z \in B^\circ(x, r_j)] \\ &= \left(1 - \frac{1}{2^j}\right)^{|B^\circ(x,r_j)|} > \left(1 - \frac{1}{2^j}\right)^{2^j} \end{aligned}$$

Reminder (construction of A_j)

For each $j \in \{1, \ldots, q\}$ construct $A_j \subseteq X$ by sampling every point of X independently with prob. $1/2^j$

Reminder (assumptions and a fact)

$|B^\circ(x, r_j)| < 2^j$,

$$\begin{aligned}\Pr[B^\circ(x,r_j)\cap A_j=\emptyset] &= \Pr[z\notin A_j \text{ for all } z\in B^\circ(x,r_j)]\\ &= \left(1-\frac{1}{2^j}\right)^{|B^\circ(x,r_j)|} > \left(1-\frac{1}{2^j}\right)^{2^j}\\ &\geq \left(1-\frac{1}{2^1}\right)^{2^1}\end{aligned}$$

Reminder (construction of A_j)

For each $j\in\{1,\dots,q\}$ construct $A_j\subseteq X$ by sampling every point of X independently with prob. $1/2^j$

Reminder (assumptions and a fact)

$|B^\circ(x,r_j)|<2^j$, $\quad j\geq 1$ and $(1-1/2^j)^{2^j}$ monotonically increasing

$$
\begin{aligned}
\Pr[B^\circ(x,r_j)\cap A_j=\emptyset] &= \Pr[z\notin A_j \text{ for all } z\in B^\circ(x,r_j)] \\
&= \left(1-\frac{1}{2^j}\right)^{|B^\circ(x,r_j)|} > \left(1-\frac{1}{2^j}\right)^{2^j} \\
&\geq \left(1-\frac{1}{2^1}\right)^{2^1} = \frac{1}{4}
\end{aligned}
$$

Reminder (construction of A_j)

For each $j\in\{1,\ldots,q\}$ construct $A_j\subseteq X$ by sampling every point of X independently with prob. $1/2^j$

Reminder (assumptions and a fact)

$|B^\circ(x,r_j)|<2^j$, $\quad j\geq 1$ and $(1-1/2^j)^{2^j}$ monotonically increasing

$$\Pr[B(y, r_{j-1}) \cap A_j \neq \emptyset]$$

Bound the probability (3)

$$\Pr[B(y, r_{j-1}) \cap A_j \neq \emptyset] \quad = \quad 1 - \Pr[B(y, r_{j-1}) \cap A_j = \emptyset]$$

Bound the probability (3)

$$\begin{aligned}\Pr[B(y, r_{j-1}) \cap A_j \neq \emptyset] &= 1 - \Pr[B(y, r_{j-1}) \cap A_j = \emptyset] \\ &= 1 - \left(1 - \frac{1}{2^j}\right)^{|B(y, r_{j-1})|}\end{aligned}$$

Bound the probability (3)

$$\begin{aligned}
\Pr[B(y, r_{j-1}) \cap A_j \neq \emptyset] &= 1 - \Pr[B(y, r_{j-1}) \cap A_j = \emptyset] \\
&= 1 - \left(1 - \frac{1}{2^j}\right)^{|B(y, r_{j-1})|} \\
&\geq 1 - \left(1 - \frac{1}{2^j}\right)^{2^{j-1}}
\end{aligned}$$

Reminder (assumption and a well-known fact)

$|B(y, r_{j-1})| \geq 2^{j-1}$,

Bound the probability (3)

$$
\begin{aligned}
\Pr[B(y, r_{j-1}) \cap A_j \neq \emptyset] &= 1 - \Pr[B(y, r_{j-1}) \cap A_j = \emptyset] \\
&= 1 - \left(1 - \frac{1}{2^j}\right)^{|B(y, r_{j-1})|} \\
&\geq 1 - \left(1 - \frac{1}{2^j}\right)^{2^{j-1}} \\
&\geq 1 - \exp\left(\frac{1}{2^j} 2^{j-1}\right)
\end{aligned}
$$

Reminder (assumption and a well-known fact)

$|B(y, r_{j-1})| \geq 2^{j-1}$, $\quad 1 + x \leq \exp(x)$ for all $x \in \mathbb{R}$

Bound the probability (3)

$$
\begin{aligned}
\Pr[B(y, r_{j-1}) \cap A_j \neq \emptyset] &= 1 - \Pr[B(y, r_{j-1}) \cap A_j = \emptyset] \\
&= 1 - \left(1 - \frac{1}{2^j}\right)^{|B(y, r_{j-1})|} \\
&\geq 1 - \left(1 - \frac{1}{2^j}\right)^{2^{j-1}} \\
&\geq 1 - \exp\left(\frac{1}{2^j} 2^{j-1}\right) \\
&= 1 - \frac{1}{\sqrt{e}}
\end{aligned}
$$

Reminder (assumption and a well-known fact)

$|B(y, r_{j-1})| \geq 2^{j-1}$, $\quad 1 + x \leq \exp(x)$ for all $x \in \mathbb{R}$

$$
\begin{aligned}
\Pr[B(y, r_{j-1}) \cap A_j \neq \emptyset] &= 1 - \Pr[B(y, r_{j-1}) \cap A_j = \emptyset] \\
&= 1 - \left(1 - \frac{1}{2^j}\right)^{|B(y, r_{j-1})|} \\
&\geq 1 - \left(1 - \frac{1}{2^j}\right)^{2^{j-1}} \\
&\geq 1 - \exp\left(\frac{1}{2^j} 2^{j-1}\right) \\
&= 1 - \frac{1}{\sqrt{e}} \geq \frac{1}{4}
\end{aligned}
$$

Reminder (assumption and a well-known fact)

$|B(y, r_{j-1})| \geq 2^{j-1}$, $\quad 1 + x \leq \exp(x)$ for all $x \in \mathbb{R}$

$$\sum_{S \subseteq X} p_S |\mu(x,S) - \mu(y,S)|$$

$$\geq \frac{1}{q} \sum_{j=1}^{i} (r_j - r_{j-1}) \Pr[B^\circ(x, r_j) \cap A_j = \emptyset] \cdot \Pr[B(y, r_{j-1}) \cap A_j \neq \emptyset]$$

Summing up...

$$
\begin{aligned}
&\sum_{S\subseteq X} p_S|\mu(x,S)-\mu(y,S)| \\
&\quad\geq \frac{1}{q}\sum_{j=1}^{i}(r_j-r_{j-1})\,\Pr[B^\circ(x,r_j)\cap A_j=\emptyset]\cdot\Pr[B(y,r_{j-1})\cap A_j\neq\emptyset] \\
&\quad\geq \frac{1}{q}\sum_{j=1}^{i}(r_j-r_{j-1})\cdot\frac{1}{4}\cdot\frac{1}{4}
\end{aligned}
$$

Summing up...

$$\sum_{S \subseteq X} p_S |\mu(x,S) - \mu(y,S)|$$

$$\geq \frac{1}{q} \sum_{j=1}^{i} (r_j - r_{j-1}) \Pr[B^\circ(x, r_j) \cap A_j = \emptyset] \cdot \Pr[B(y, r_{j-1}) \cap A_j \neq \emptyset]$$

$$\geq \frac{1}{q} \sum_{j=1}^{i} (r_j - r_{j-1}) \cdot \frac{1}{4} \cdot \frac{1}{4}$$

$$= \frac{1}{16q} \sum_{j=1}^{i} (r_j - r_{j-1})$$

$$
\begin{aligned}
&\sum_{S \subseteq X} p_S |\mu(x, S) - \mu(y, S)| \\
\geq\ & \frac{1}{q} \sum_{j=1}^{i} (r_j - r_{j-1}) \Pr[B^\circ(x, r_j) \cap A_j = \emptyset] \cdot \Pr[B(y, r_{j-1}) \cap A_j \neq \emptyset] \\
\geq\ & \frac{1}{q} \sum_{j=1}^{i} (r_j - r_{j-1}) \cdot \frac{1}{4} \cdot \frac{1}{4} \\
=\ & \frac{1}{16q} \sum_{j=1}^{i} (r_j - r_{j-1}) \\
=\ & \frac{1}{16q} (r_i - r_0) \quad \text{(telescopic sum)}
\end{aligned}
$$

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&\sum_{S\subseteq X} p_S|\mu(x,S)-\mu(y,S)| \\
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\geq\ & \frac{1}{16q}\left(\frac{1}{2}\mu(x,y)-0\right)
\end{aligned}$$

Summing up...

$$
\begin{aligned}
&\sum_{S \subseteq X} p_S |\mu(x,S) - \mu(y,S)| \\
&\geq \frac{1}{q} \sum_{j=1}^{i} (r_j - r_{j-1}) \Pr[B^\circ(x, r_j) \cap A_j = \emptyset] \cdot \Pr[B(y, r_{j-1}) \cap A_j \neq \emptyset] \\
&\geq \frac{1}{q} \sum_{j=1}^{i} (r_j - r_{j-1}) \cdot \frac{1}{4} \cdot \frac{1}{4} \\
&= \frac{1}{16q} \sum_{j=1}^{i} (r_j - r_{j-1}) \\
&= \frac{1}{16q} (r_i - r_0) \quad \text{(telescopic sum)} \\
&\geq \frac{1}{16q} \left(\frac{1}{2} \mu(x,y) - 0 \right) \\
&= \frac{1}{32q} \mu(x,y)
\end{aligned}
$$

Contents

Theorem (Linial, London, and Rabinovich '95)

An embedding of a shortest-path metric on an n-vertex constant-degree expander (with unit weight) into ℓ_2 needs $\Omega(\log n)$ distortion.

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Proof: exercise (with guides)

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An embedding of a shortest-path metric on an n-vertex constant-degree expander (with unit weight) into ℓ_2 needs $\Omega(\log n)$ distortion.

Proof: exercise (with guides)

Instead, we now prove the following

Theorem (Enflo '69)

An embedding of a shortest-path metric on an n-vertex Hamming cube (with unit weight) into ℓ_2 needs $\Omega(\sqrt{\log n})$ distortion.

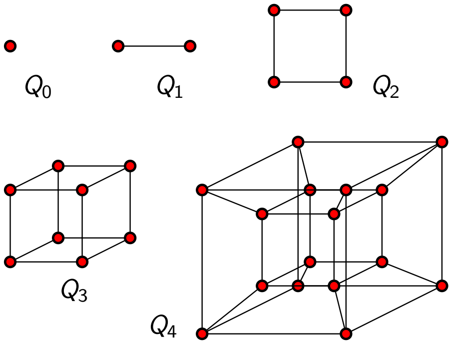
A proof (below) should be a hint to the exercise above.

Hamming cubes

d a positive integer

Definition

A *d-dimensional Hamming cube* Q_d is a graph defined as
Vertex set $V(Q_d) = \{0,1\}^d$
Edge set $E(Q_d) = \{\{u,v\} \colon u \;\&\; v$ differ at exactly one coord.$\}$



See Q_d itself as a shortest-path metric space on Q_d w/ unit weight

Theorem

$d \geq 2$ a natural number

$f\colon V(Q_d) \to \ell_2$ a D-embedding $\Rightarrow D \geq \sqrt{d}$

See Q_d itself as a shortest-path metric space on Q_d w/ unit weight

Theorem

$d \geq 2$ a natural number
$f\colon V(Q_d) \to \ell_2$ a D-embedding $\Rightarrow D \geq \sqrt{d}$

Note: $n = |V(Q_d)| = 2^d$, $\therefore \sqrt{d} = \sqrt{\log_2 n}$

Theorem (in words)

There exists an n-point metric space that cannot be embedded into ℓ_2 with distortion better than $\sqrt{\log_2 n}$

Set-up

- (X, μ), (X, ν) two metric spaces
- $E, F \subseteq \binom{X}{2}$ non-empty sets of 2-element subsets of X

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- $\mathrm{ave}_2(\mu, E) = \sqrt{\frac{1}{|E|} \sum_{\{x,y\} \in E} \mu(x,y)^2}$ (root mean square)
- $R_{E,F}(\mu) = \frac{\mathrm{ave}_2(\mu, F)}{\mathrm{ave}_2(\mu, E)}$

Proof: Observation

For a D-embedding $f\colon X \to \ell_2^k$, let $\nu(x, y) = \|f(x) - f(y)\|_2$

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- $\therefore\ \text{ave}_2(\mu, F) \leq \dfrac{1}{r}\text{ave}_2(\nu, F)$

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To prove the theorem it is enough to show the following

Claim

$$\frac{R_{E,F}(\mu)}{R_{E,F}(\nu)} \geq \sqrt{d} \text{ for some } E, F \subseteq \binom{V(Q_d)}{2}$$

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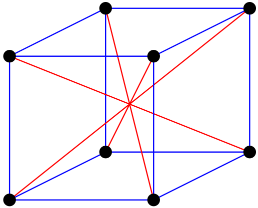

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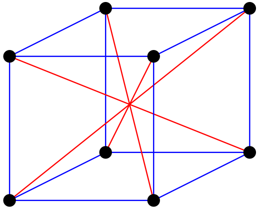

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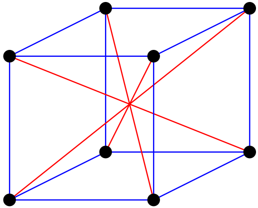
- Show $R_{E,F}(\mu) = d$



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$\dfrac{R_{E,F}(\mu)}{R_{E,F}(\nu)} \geq \sqrt{d}$ for some $E, F \subseteq \binom{V(Q_d)}{2}$

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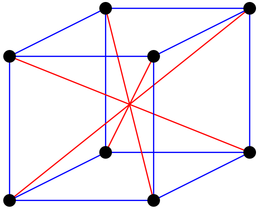
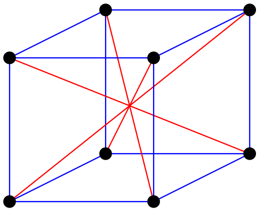
- Show $R_{E,F}(\mu) = d$
- Show $R_{E,F}(\nu) \leq \sqrt{d}$



Reminder

$E = E(Q_d)$ and $F = \{\{v, \overline{v}\} \colon v \in V(Q_d)\}$

- $\mu(x, y) = 1$ for every $\{x, y\} \in E$



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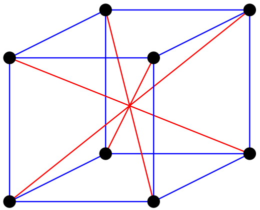

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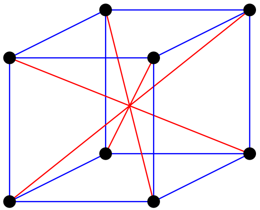

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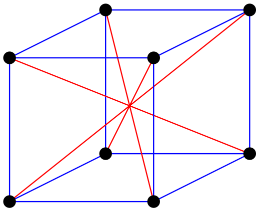

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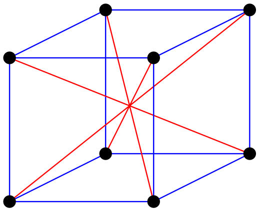
- $\therefore \mathrm{ave}_2(\mu, E) = 1$
- $\mu(x, y) = d$ for every $\{x, y\} \in F$
- $\therefore \mathrm{ave}_2(\mu, F) = d$
- $\therefore R_{E,F}(\mu) = \dfrac{\mathrm{ave}_2(\mu, F)}{\mathrm{ave}_2(\mu, E)} = d$



$$R_{E,F}(\nu) = \frac{\mathrm{ave}_2(\nu, F)}{\mathrm{ave}_2(\nu, E)}$$

For bounding $R_{E,F}(\nu)$

$$
\begin{aligned}
R_{E,F}(\nu) &= \frac{\mathrm{ave}_2(\nu, F)}{\mathrm{ave}_2(\nu, E)} \\
&= \frac{\sqrt{\frac{1}{|F|}\sum_{\{u,v\}\in F} \nu(u,v)^2}}{\sqrt{\frac{1}{|E|}\sum_{\{u,v\}\in E} \nu(u,v)^2}}
\end{aligned}
$$

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&= \sqrt{\frac{\frac{1}{2^{d-1}}\sum_{\{u,v\}\in F} \nu(u,v)^2}{\frac{1}{d2^{d-1}}\sum_{\{u,v\}\in E} \nu(u,v)^2}}
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\end{aligned}
$$

$\therefore$ Enough to show $\sum_{\{u,v\}\in F} \nu(u,v)^2 \leq \sum_{\{u,v\}\in E} \nu(u,v)^2$.

Notation

$\nu^2(E) = \sum_{\{u,v\}\in E} \nu(u,v)^2$, $\nu^2(F) = \sum_{\{u,v\}\in F} \nu(u,v)^2$

Subclaim

$\nu^2(F) \leq \nu^2(E)$

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Subclaim

$\nu^2(F) \leq \nu^2(E)$

Proof outline:

- Induction on d
- When $d = 2$, a direct calculation
- When $d > 2$, make use of a product structure of cubes

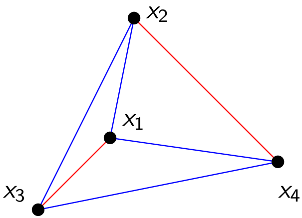
To prove the case $d = 2$ we use the following

Lemma (short diagonals lemma)

For any four points x_1, x_2, x_3, x_4 in a Euclidean space

$$\|x_1 - x_3\|_2^2 + \|x_2 - x_4\|_2^2 \le \|x_1 - x_2\|_2^2 + \|x_2 - x_3\|_2^2 + \|x_3 - x_4\|_2^2 + \|x_4 - x_1\|_2^2$$

Proof: Exercise



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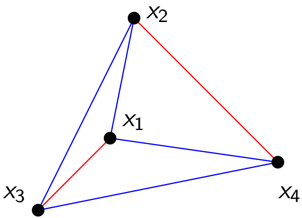
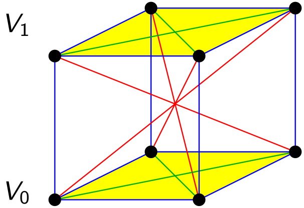


Proof of Subclaim when $d = 2$

- Set $x_i = f(v_i)$ and use the lemma above
- LHS $= \nu^2(F)$ and RHS $= \nu^2(E)$

- Partition the vertex set $V = \{0,1\}^d$ into



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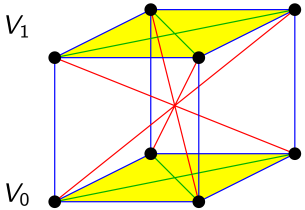

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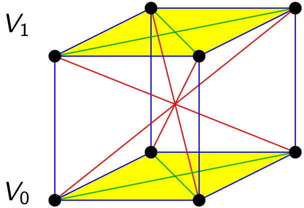

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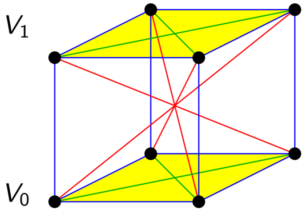
- V_i induces a $(d-1)$-dimensional Hamming cube in Q_d



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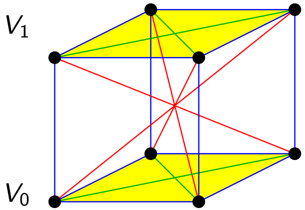
- E_i = the edge set of the Hamming cube induced by V_i
  F_i = the set of pairs w/ dist. $d-1$ in the induced cube



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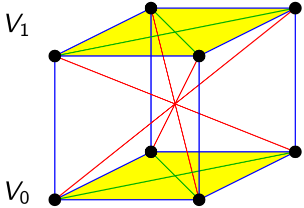

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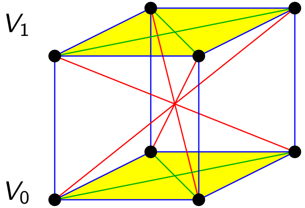
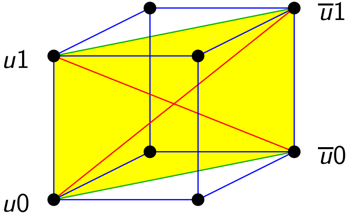
- $\nu^2(F_i) \le \nu^2(E_i)$ by induction hypothesis
- $E_{01} = E \setminus (E_0 \cup E_1)$



- Each pair in F has a form $\{(u,0),(\overline{u},1)\}$



- Each pair in F has a form $\{(u, 0), (\overline{u}, 1)\}$
- By the short diagonals lemma
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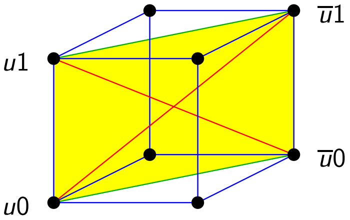
  $\nu(u0, \overline{u}0)^2 + \nu(\overline{u}0, \overline{u}1)^2 + \nu(\overline{u}1, u1)^2 + \nu(u1, u0)^2$



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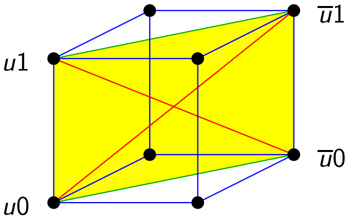
  $\nu^2(F) \leq \nu^2(E_{01}) + \nu^2(F_0) + \nu^2(F_1)$



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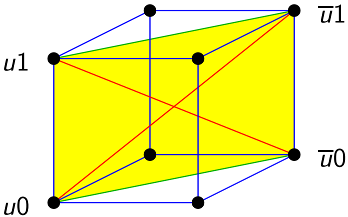
- By adding up the inequalities over all u
  $\nu^2(F) \leq \nu^2(E_{01}) + \nu^2(F_0) + \nu^2(F_1)$
- $\therefore \nu^2(F) \leq \nu^2(E_{01}) + \nu^2(E_0) + \nu^2(E_1) = \nu^2(E)$



Contents

Summary

Every n-point metric space can be embedded

- into ℓ_∞ isometrically,
- into ℓ_2 with distortion $O(\log n)$ and this is tight.

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Every n-point metric space can be embedded

- into ℓ_∞ isometrically,
- into ℓ_2 with distortion $O(\log n)$ and this is tight.

We may wonder about

- ℓ_1 and ℓ_p?
- restricted classes of metric spaces?

Theorem (Bourgain '85)

For any fixed $p \in [1, \infty)$, every n-point metric space can be embedded into ℓ_p with distortion $O(\log n)$

What about ℓ_p?

Theorem (Bourgain '85)

For any fixed $p \in [1, \infty)$, every n-point metric space can be embedded into ℓ_p with distortion $O(\log n)$

Proof idea:

- Modify the proof for ℓ_2 (exercise)

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Proof idea:

- Modify the proof for ℓ_2 (exercise)
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Theorem (a consequence of Dvoretzky's thm)

Every n-point set in ℓ_2 can be isometrically embedded into ℓ_p for any $p \in [1, \infty)$

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Tightness?

Theorem (Matoušek '97)

The distortion of $\Omega(\log n)$ is needed (again by expanders)

$\mathcal{G}$ a class of (connected) graphs

Def.: $\mathcal{G}$-metric

A finite metric space (X, μ) is a $\mathcal{G}$-metric
if $\exists$ a graph $G \in \mathcal{G}$ and an edge-weight function w s.t. X can be isometrically embedded into the shortest-path metric on G with w

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- Tree metric: $\mathcal{G}$ the class of trees
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- Planar-graph metric: $\mathcal{G}$ the class of planar graphs
- ...

Tree: a connected graph with no cycle

Theorem

Every n-point tree metric can be

- isometrically embedded into ℓ_1 (exercise)

Tree: a connected graph with no cycle

Theorem

Every n-point tree metric can be

- isometrically embedded into ℓ_1 (exercise)
- embedded into ℓ_p with distortion $O((\log \log n)^{\min\{1/2, 1/p\}})$ (Matoušek '99)

Tree: a connected graph with no cycle

Theorem

Every n-point tree metric can be

- isometrically embedded into ℓ_1 (exercise)
- embedded into ℓ_p with distortion $O((\log \log n)^{\min\{1/2,1/p\}})$ (Matoušek '99)
  - this is tight (Bourgain '86)

A bad example is a complete binary tree with unit weight.

Outerplanar graph: a graph that can be drawn on the plane with no edge crossings and all edges incident to the outer face

Theorem

Every n-point outerplanar-graph metric can be

- isometrically embedded into ℓ_1 (Okamura and Seymour '81)

Outerplanar graph: a graph that can be drawn on the plane with no edge crossings and all edges incident to the outer face

Theorem

Every n-point outerplanar-graph metric can be

- isometrically embedded into ℓ_1 (Okamura and Seymour '81)
- embedded into ℓ_2 with distortion $O(\sqrt{\log n})$ (Rao '99)

As far as I surveyed, tightness for ℓ_2 doesn't seem to be known

Planar graph: a graph that can be drawn on the plane with no edge crossings

Theorem

Every n-point planar-graph metric can be embedded into ℓ_2 with distortion $O(\sqrt{\log n})$ (Rao '99)

Planar graph: a graph that can be drawn on the plane with no edge crossings

Theorem

Every n-point planar-graph metric can be embedded into ℓ_2 with distortion $O(\sqrt{\log n})$ (Rao '99)

- this is tight (Newman and Rabinovich '03)

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Theorem

Every n-point planar-graph metric can be embedded into ℓ_2 with distortion $O(\sqrt{\log n})$ (Rao '99)

- this is tight (Newman and Rabinovich '03)

Conjecture (Gupta, Newman, Rabinovich, and Sinclair '04)

A planar-graph metric can be embedded into ℓ_1 with constant distortion

- They proved it is the case for series-parallel graphs
- They further conjecture that the shortest-path metric on an H-minor-free graph (for H fixed) can be embedded into ℓ_1 with constant distortion

The girth of a graph: the length of a shortest cycle

Theorem (Linial, Magen, and Naor '02)

An embedding of the shortest-path metric on an r-regular graph ($r \geq 3$) of girth g with unit weight into ℓ_2 requires the distortion $\Omega(\sqrt{g})$

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Conjecture (Linial, London, and Rabinovich '95)

The tight bound is $\Omega(g)$

Note: Upper bound $O(g)$ is easy when r is constant

(a consequence of one exercise)

No time to mention ...

- Dimension into ℓ_2, ℓ_∞, ℓ_p and trade-offs
- Embeddings into probabilistic tree metrics
- Efficiency of the construction
- Other important special metrics (e.g., edit distance, Hausdorff distance)
- Dimension reduction (e.g., Johnson–Lindenstrauss Lemma)
- Algorithmic applications
- ...

One of the most developing subjects in discrete and computational geometry

Contents

Contents

[End of Lecture]

- Introduction (10 min)
- Embedding into ℓ_∞ (10 min)
- Embedding into ℓ_2 (30 min)
- Lower bound for ℓ_2 (30 min)
- Remarks (10 min)
- Exercises (?? min)

[End of Lecture]