Most of the exercises are taken from J. Matoušek's "Lectures on Discrete Geometry," Springer 2002.
Legend: $(*)$ highly recommended; $(-)$ easy

Exercise 1 $(-)$ This exercise is provided for those who are not familiar with metric spaces and normed spaces. Given a normed space $(X, \|\cdot\|)$, we define $\mu\colon X \times X \to \mathbb{R}_+$ as $\mu(x,y) = \|x-y\|$. Show that (X,μ) is a metric space.

Exercise 2 $(-)$ Show that every finite metric space is a shortest-path metric on a graph.

Exercise 3 Prove that every n-point metric space can be embedded into ℓ_∞^{n-1} with distortion 1. Note: In the lecture we have shown that every n-point metric space can be embedded into ℓ_∞^n with distortion 1. The goal of this exercise is to improve the dimension by one.

Exercise 4 $(*-)$ Complete the proof of $O(\log n)$-embeddability of any n-point metric space (X,μ) into ℓ_2 by showing that the embedding $f\colon X \to \ell_2^{2^n}$ constructed in the lecture satisfies $\|f(x) - f(y)\|_2 \le \mu(x,y)$ for every $x, y \in X$.

Exercise 5 $(-)$ Let Q_d be a d-dimensional Hamming cube. Prove that the number of vertices of Q_d is 2^d and the number of edges of Q_d is $d2^{d-1}$.

Exercise 6 $(-)$ For any four points $x_1, x_2, x_3, x_4 \in \mathbb{R}^d$ it holds that $\|x_1-x_3\|_2^2 + \|x_2-x_4\|_2^2 \le \|x_1-x_2\|_2^2 + \|x_2-x_3\|_2^2 + \|x_3-x_4\|_2^2 + \|x_4-x_1\|_2^2$. Prove it.

Exercise 7 $(*)$ Prove that a d-dimensional Hamming cube (with unit edge-weight) can be embedded into ℓ_2 with distortion $\sqrt{d}$. Hint: You may want to be natural, I believe.

Exercise 8 The *diameter* of a graph $G = (V,E)$ is the maximum length of a shortest path between two vertices, and is denoted by $\mathrm{diam}(G)$. Prove that every graph G (with unit edge-weight) can be embedded into ℓ_2 with distortion $\mathrm{diam}(G)$. Hint: You may like the triviality, I believe.

Exercise 9 $(*)$ The *Laplacian matrix* L_G of a graph G is an $n \times n$ matrix (where $n = |V(G)|$), with both rows and columns indexed by the vertices of G, defined as

$$(L_G)_{uv} = \begin{cases} \deg(u) & \text{if } u = v, \\ -1 & \text{if } u \neq v, \text{ and } \{u,v\} \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Remember that $\deg(u)$ represents the degree of u, namely the number of edges incident to u. Notice that L_G is symmetric and positive semidefinite. Namely, L_G has n non-negative real eigenvalues $0 \le \mu_1 \le \mu_2 \le \cdots \le \mu_n$. The goal of this exercise is to show that if G is r-regular (the degree of each vertex is exactly r) and $\mu_2 \ge \beta$ for some constants r and β, then every D-embedding of the shortest-path metric on G (with unit edge-weight) into ℓ_2 must satisfy $D \ge c \log n$ for some constant $c > 0$ which only depends on r and β, not on n. The result is due to Linial, London, and Rabinovich (1995).

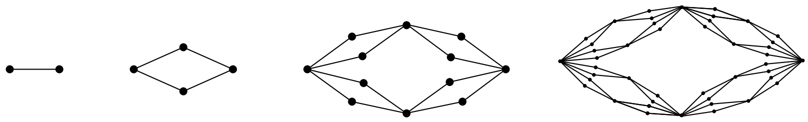
1. Show that $\mu_1 = 0$ and $\mu_2 = \min\{x^\top L_G x \colon x \in \mathbb{R}^n, \|x\|_2 = 1, \sum_{v \in V} x_v = 0\}$. Note that the coordinates of $x \in \mathbb{R}^n$ are indexed by the vertices of G. This requires some linear algebra, of course. For the latter, you may need to use a fact that a symmetric real matrix has real eigenvalues only and the corresponding eigenvectors form an orthogonal basis.

2. Let $f\colon V \to \ell_2^k$ be a D-embedding. As in the lecture, let μ be the shortest-path metric on G with unit edge-weight, and ν the metric on V defined as $\nu(x,y) = \|f(x) - f(y)\|_2$. Further, let $E = E(G)$ and $F = \binom{V(G)}{2}$. Prove that $\mathrm{ave}_2(\mu, E) = 1$ and $\mathrm{ave}_2(\mu, F) = \Omega(\log n)$. Here, the constant hidden in the order notation may depend on r (and β). This implies that $R_{E,F}(\mu) = \Omega(\log n)$.

3. In the same set-up as above, prove that $R_{E,F}(\nu) = O(1)$ (again the constant hidden in the order notation may depend on r and β). You may first observe that it suffices to show that $\sum_{\{u,v\}\in F} \nu(u,v)^2 = O(n \sum_{\{u,v\}\in E} \nu(u,v)^2)$. Then, use the first part of this exercise at some point.

Summing up, we obtain $D \geq R_{E,F}(\mu)/R_{E,F}(\nu) = \Omega(\log n)$. Note: A series of r-regular graphs with a lower-bounded second eigenvalue exist, and they are called constant-degree expanders.

Exercise 10 $(*)$ Let $G_0, G_1, \ldots$ be the graphs below.



G_0 G_1 G_2 G_3

In general, G_{i+1} is constructed from G_i by replacing each edge by a square with two new vertices. Prove that any embedding of the shortest-path metric on G_k (with unit edge-weight) into ℓ_2 has distortion at least $\sqrt{k+1}$. Hint: As a proof from the lecture find suitable E and F, and use the short diagonals lemma. Note: This shows that the distortion of an embedding of a planar-graph metric into ℓ_2 can be $\Omega(\sqrt{\log n})$. The result is due to Newman and Rabinovich (2003).

Exercise 11 $(*)$ Prove that every n-point metric space can be embedded into ℓ_p with distortion $O(\log n)$ where $p \geq 1$ is arbitrary. Hint: Modify the embedding used for ℓ_2 in the lecture. You may use the following *Hölder's inequality*: for every p, q with $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $1/p + 1/q = 1$ and for all $x, y \in \mathbb{R}^d$ it holds that $\|x\|_p \|y\|_q \geq \sum_{i=1}^d |x_i y_i|$.

Exercise 12 $(*)$ Prove that every tree metric can be isometrically embedded into ℓ_1.

Exercise 13 Give an example of a finite metric space that is not a planar-graph metric. Warning: A shortest-path metric on a non-planar graph can be a planar-graph metric. You have to *prove* your metric space is not a planar-graph metric.

Exercise 14 Let $G = (V, E)$ be a complete binary tree of height h. The goal of this exercise is to give an $O(\sqrt{\log h})$-embedding of G (with unit edge-weight) into ℓ_2, which is due to Bourgain (1986). Note that $h = O(\log n)$. We construct an embedding $f\colon V \to \ell_2^{n-1}$ as follows. First, we consider the coordinates of ℓ_2^{n-1} are indexed by the vertices of G except for the root. For each non-root vertex $u \in V$ we define

$$f(v)_u = \begin{cases} \sqrt{\text{depth}(v) - \text{depth}(u) + 1} & \text{if } u \text{ is an ancestor of } v, \\ 0 & \text{otherwise.} \end{cases}$$

Note that depth(v) is the distance from the root to v. Prove that the distortion of f is $O(\sqrt{\log h})$.

Exercise 15 The *diameter* of a finite set X in a normed space $(\mathbb{R}^d, \|\cdot\|)$ is defined as $\max\{\|x-y\| \colon x, y \in X\}$. Show that the diameter of an n-point set in ℓ_∞^d can be computed in $O(dn)$ time. Remark: This shows that the space ℓ_∞^d is "nice" in terms of the diameter computation.