Making, avoiding and probabilistic intuition in positional games

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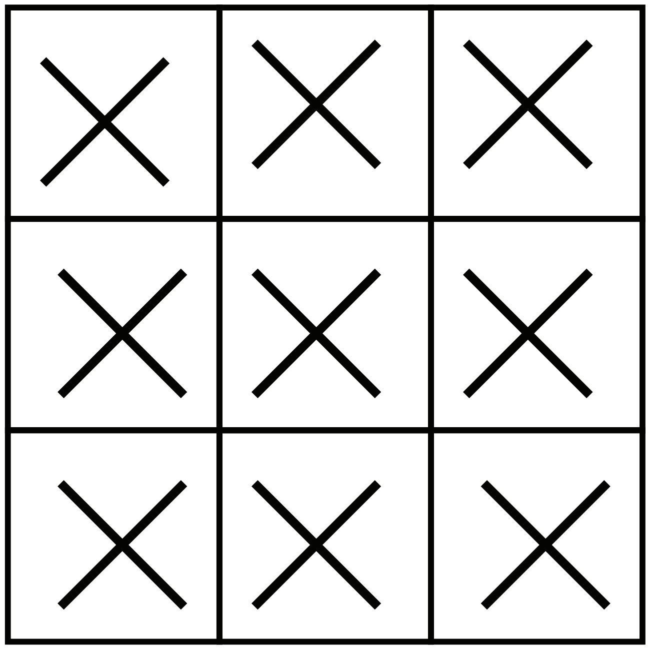
Positional Games

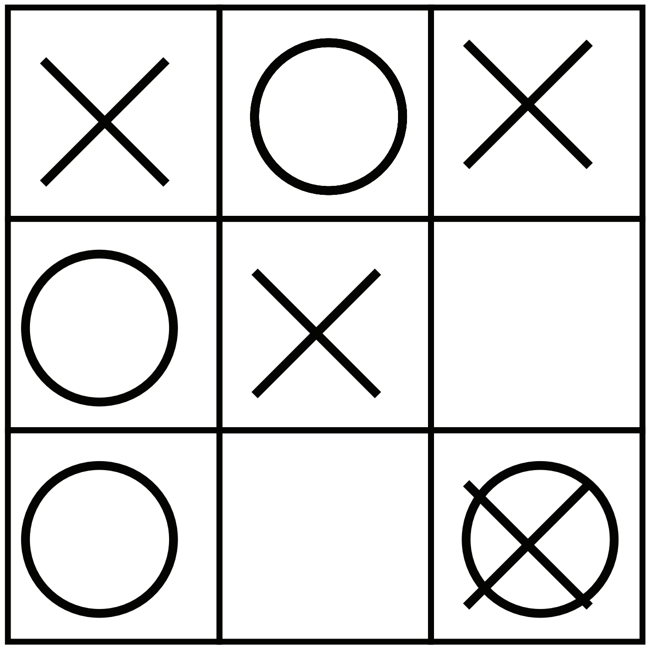
- Set X, the board
- Family $\mathcal{F} \subseteq 2^X$, the winning sets
- Player I and II alternately claim one unclaimed element of the board
- Who is the **WINNER**?
- **STRONG** GAME:

  Whoever occupies a winning set **first**
- EXAMPLES: **Tic-tac-toe**

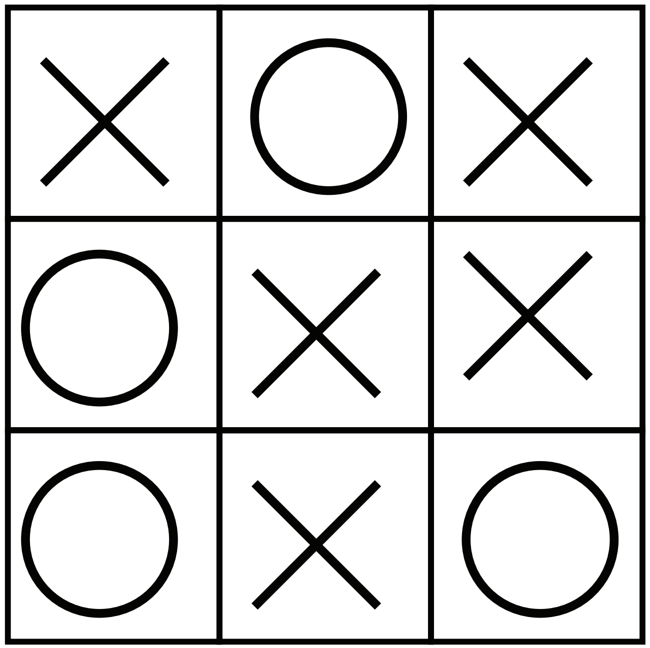
Winning sets in Tic-tac-toe



Winning Tic-tac-toe



Drawing in Tic-tac-toe



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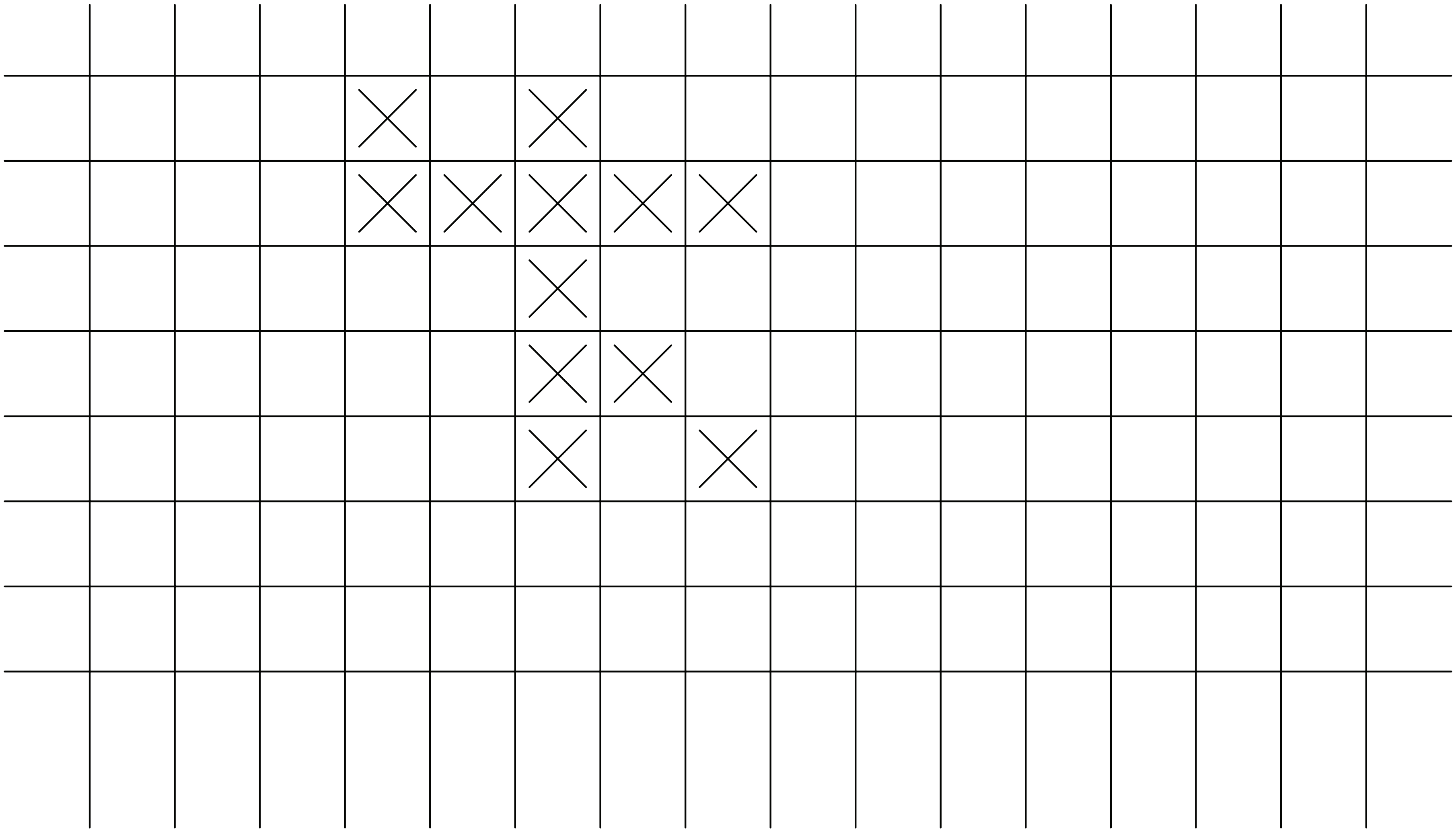
- EXAMPLES: Tic-tac-toe

  5-in-a-row

Winning sets in 5-in-a-row



Weak game

- In a strong game both players have to
  occupy and
  prevent the other from occupying

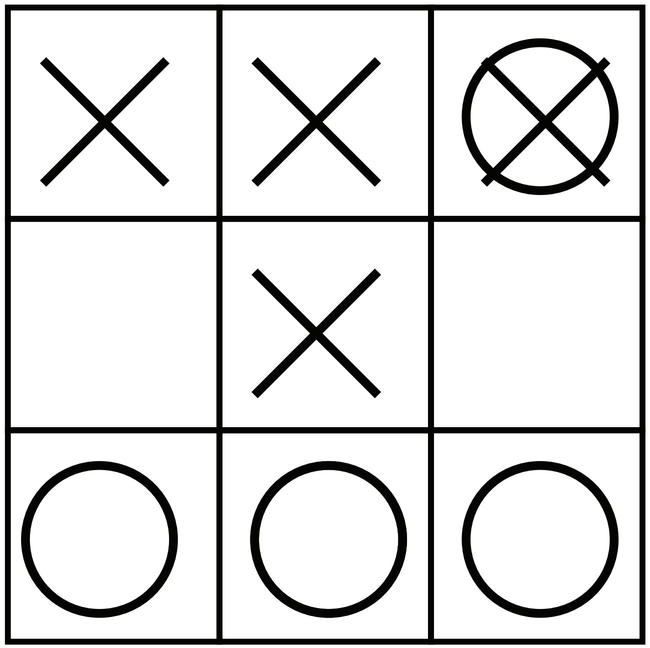
In a weak game these jobs are separated

- **WEAK** GAME:

  Player I (Maker) wins if he fully occupies a winning set, otherwise Player II (Breaker)

- EXAMPLE: Hex

“Weak” Tic-tac-toe



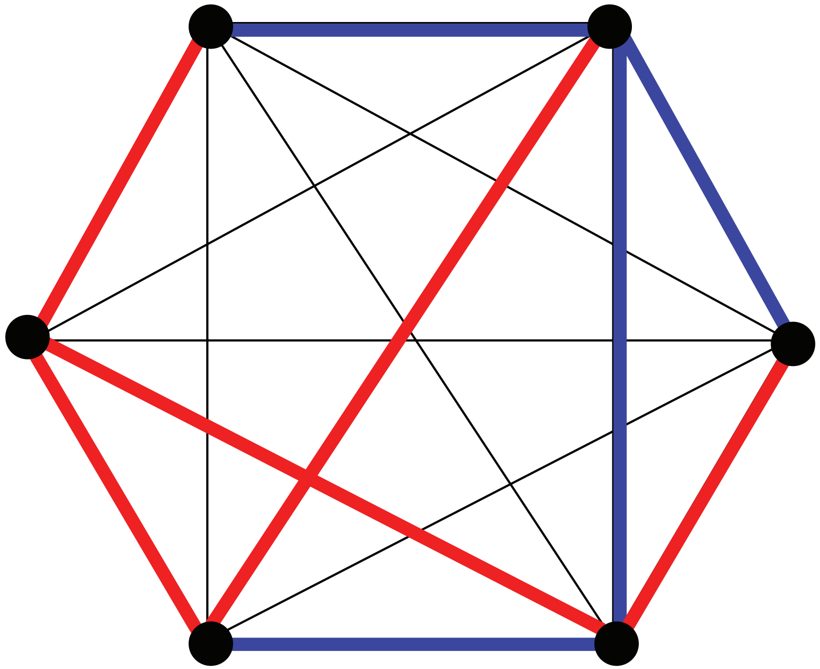
"Reasons" for study

- Beautiful theory
  - Recent book of Beck: "Tic-Tac-Toe Theory"
- Positional games motivated the technique of derandomization in the theory of algorithms
  - First application of the method of conditional expectations (Erdős-Selfridge)
  - First algorithmization of the Lovász Local Lemma (Beck)

Graph Games

- Board $E(K_n)$
- **Examples:**
  - Connectivity game $\mathcal{T}_n$

Playing “Connectivity”



Maker WON!!!

Maker's strategy $\mathcal{T}_n$

- Build a tree by joining one isolated vertex in each round to his component
- Why can Maker always do this?
- The game will end after $n-1$ moves
- If Maker's current component has k vertices, there are $k(n-k) \leq n-1$ potential edges to choose from. There is a free one.

  (Breaker selected at most $n-2$ edges)

Graph Games

- Board $E(K_n)$
- **Examples:**
  - Connectivity game $\mathcal{T}_n$
  - Perfect matching game $\mathcal{M}_n$
  - Hamiltonicity game $\mathcal{H}_n$

Graph Games

- **Connectivity**

  Maker wins (easily) for every *n*

- **Hamiltonicity**

  Maker wins for large *n* (Chvatal-Erdős)

Biased games

- Chvatal-Erdős: Give Breaker a break (a bias)

  (m:b) game: Maker takes m edges at once

  Breaker takes b edges at once

- $b_{\mathcal{F}}$ is the threshold bias of game $\mathcal{F}$ if
  - Maker wins the (1:b) game for every $b \leq b_{\mathcal{F}}$
  - Breaker wins the (1:b) game for every $b > b_{\mathcal{F}}$
- $b_{\mathcal{F}}$ **exists**

Biased graph games

- **Connectivity**

  Chvatal-Erdős

  Maker wins $\left(1:\left(\frac{1}{4}-\varepsilon\right)\frac{n}{\log n}\right)$

  Breaker wins $\left(1:(1+\varepsilon)\frac{n}{\log n}\right)$

  Beck

  Maker wins $\left(1:(\log 2-\varepsilon)\frac{n}{\log n}\right)$

- **Hamiltonicity**

  Bollobás-Papaioannou

  Maker wins $\left(1:\frac{c\log n}{\log\log n}\right)$

  Beck: Maker wins

  $$\left(1:\left(\frac{\log 2}{27}-\varepsilon\right)\frac{n}{\log n}\right)$$

Improvement

Theorem. (2006+) Maker can build a Hamilton cycle playing against a bias of

$$\left(\log 2-\varepsilon\right)\frac{n}{\log n}$$

Big open question: What is the asymptotic threshold bias for minimum degree 1 (isolation of a vertex)?

How to build a connected graph against a large bias?

- *Proof* (Beck)

- Ideas:
  - A criterion
  - Solving a kind of “dual” problem

Criteria

- Erdős-Selfridge: Breaker has a winning strategy in the (1:1)-game on $\mathcal{F}$ provided

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2}$$

- Beck: Breaker has a winning strategy in the (*p*:*q*) game on $\mathcal{F}$ provided

$$\sum_{A \in \mathcal{F}} (1+q)^{-|A|/p} < \frac{1}{q+1}$$

Proof of Erdős-Selfridge

- *Existence* of winning final position is easily proved by probabilistic argument
- But how to achieve it against a skilled adversary?
- Appropriate definition of the “danger” of a situation for Breaker.
  Then try to minimize the danger.
  - Assume first that Breaker starts the game.

Danger function

- $B_i=\{x_1,\ldots,x_i\}$, $M_i=\{y_1,\ldots,y_i\}$

$$f_A(i)=\begin{cases} 0 & \text{if } B_i \cap A \neq \{\} \\ 2^{-|A\setminus M_i|} & \text{if } B_i \cap A = \{\} \end{cases}$$

- Breaker wins iff every $A \in \mathcal{F}$ has danger < 1
- Cumulative danger of the position

$$F(i)=\sum_{A\in\mathcal{F}} f_A(i)$$

- At the beginning $F(0)<1$
- Let’s keep it that way!
- How?
- GREEDILY!
- Breaker’s strategy: Select $b_{i+1} \in X \setminus M_i \setminus B_i$ which **decreases the cumulative danger the most**!

- That is: $\sum_{\substack{x \in A \\ A \cap B_i = \{\}}} 2^{-|A \setminus M_i|}$ is maximized for $x = b_{i+1}$
- Hence

$$F(i+1) = F(i) - \sum_{\substack{b_{i+1} \in A \\ A \cap B_i = \{\}}} 2^{-|A \setminus M_i|} + \sum_{\substack{m_{i+1} \in A \\ A \cap B_i = \{\}}} 2^{-|A \setminus M_i|} - \sum_{\substack{b_{i+1}, m_{i+1} \in A \\ A \cap B_i = \{\}}} 2^{-|A \setminus M_i|}$$

$$\leq F(i) \leq \cdots\cdots\cdots \leq F(0) < 1$$

Criteria

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$$\sum_{A \in \mathcal{F}} (1+q)^{-|A|/p} < \frac{1}{q+1}$$

How to make a connected graph?

- **Make** a spanning tree!

or rather

- **Put an edge into every cut!!!**
- Play **Breaker** on the following family:

$$C_n = \{\{xy : x \in S, y \in V \setminus S\} : S \subseteq V\}$$

Using Beck's criterion

- For $b = (\log 2 - \varepsilon) n / \log n$

$$\sum_{A \in C_n} (1+1)^{-|A|/b} = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{k} 2^{-k(n-k)/b} = o(1)$$

Evaluating $\sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{k} 2^{-k(n-k)/b}$ for $b = (\log 2 - \varepsilon) n / \log n$

- For $k \le \sqrt{n}$

$$n^k 2^{-kn(1-o(1))/b} \le \exp\{k(\log n - (1+\varepsilon - o(1))\log n)\} < \frac{1}{3^k}$$

- For $k > \sqrt{n}$

$$\left(\frac{en}{k}\right)^k 2^{-kn/2b} \le \exp\left\{k \log\left(e\sqrt{n}\right) - \frac{k}{2}\log n(1+\varepsilon)\right\} < \frac{1}{3^k}$$

Theorem (Beck) Maker can build a spanning tree while playing against a bias of

$$\left(\log 2-\varepsilon\right)\frac{n}{\log n}$$

Improvement:

Theorem (2006+) Maker can build a Hamilton cycle while playing against a bias of

$$\left(\log 2-\varepsilon\right)\frac{n}{\log n}$$

Main tools

- Beck’s criterion for Breaker’s win
- New pseudorandom criterion for Hamiltonicity --- involving only two “positive” conditions
- A thinning trick

Hamiltonicity criterion

Let $d = \frac{\log\log\log n}{\log\log\log\log n}$

P1 For every $S \subseteq V$, if $|S| \le \frac{n \log\log n \log d}{d \log n \log\log\log n}$

Then $|N(S)| \ge d|S|$

P2 There is an edge in G between any two disjoint subsets $A, B \subseteq V$ if

$$|A|,|B| \ge \frac{n \log\log n \log d}{4130 \log n \log\log\log n}$$

Then G is hamiltonian.

When dumb players are playing…

- Maker/Breaker are random edge generators
- What is the largest bias against which DumbMaker still beats DumbBreaker (almost always)?
- At the end DumbMaker's graph is a random graph *G(n,M)* with $M = \binom{n}{2} \Big/ (b+1)$ edges

- What is the smallest *M=M(n)* such that almost all graphs with *M* edges and *n* vertices are connected / having a perfect matching / Hamiltonian?

The model G(n,M)

- *G(n,M)* is the probability space of graphs where each graph with *n* vertices and *M* edges occurs with the same probability.
- Introduced by Erdős and Rényi in 1960
- Studied extensively ever since

Thresholds in random graphs

Theorem (Bollobás, Thomason)

Let *P* be a monotone graph property. Then there exists a threshold function $M_P(n)$ such that

Pr($G(n,M'(n))$ has property *P*) → 0

for every $M' \ll M$ and

Pr($G(n,M''(n))$ has property *P*) → 1

for every $M'' \gg M$

Theorem (Erdős-Rényi) M_p = *nlog n* for connectivity

Theorem (Pósa) M_p = *nlogn* for hamiltonicity

Clever game vs. Dumb game

- For $\mathcal{F} = \mathcal{T}_n, \mathcal{M}_n$, and $\mathcal{H}_n$ the largest bias of Clever Breaker against which CleverMaker succeeds is approximately *equal to* the largest bias of DumbBreaker against which DumbMaker succeeds a.a.

$$b_{\mathcal{F}} \approx n^2/2M_{\mathcal{F}} = \Theta\left(\frac{n}{\log n}\right)$$

- **QUESTION**: Is this a coincidence?
- **ANSWER**: We don't know (yet)

How far does the random graph intuition go?

- **QUESTION:** Is the random graph intution tight up to constant factor?

Sharp threshold

Theorem (Erdős, Rényi) For every $\varepsilon > 0$

Pr($G(n,(1/2-\varepsilon)n\log n)$ is connected) → 0

Pr($G(n,(1/2+\varepsilon)n\log n)$ is connected) → 1

Theorem (Komlós-Szemerédi) For every $\varepsilon > 0$

Pr($G(n,(1/2-\varepsilon)n\log n)$ is hamiltonian) → 0

Pr($G(n,(1/2+\varepsilon)n\log n)$ is hamiltonian) → 1

How far does the random graph intuition go?

- **QUESTION:** Is the random graph intution tight up to constant factor?
- Is $b_{\mathcal{T}}$ or $b_{\mathcal{H}}$ equal to $(1+o(1))\frac{n}{\log n}$?
- The general answer to the above question is negative

Further threshold biases

- **Theorem** (2006+)

$$b_{\mathcal{NP}} = n/2 + o(n)$$

($\mathcal{NP}$ denotes the family of non-planar graphs)

$$b_{\mathcal{M}^k} = n/2 + o(n)$$

($\mathcal{M}^k$ is the family of graphs containing a K_k-minor)

$$b_{\mathcal{NC}_k} = \Theta(n)$$

($\mathcal{NC}_k$ denotes the family of non-k-colorable graphs)

“Sharp” random graph intuition fails

For $\mathcal{NP}$ and $\mathcal{M}^k$ the “clever-bias” is ≈$n/2$, while the “dumb-bias” is ≈n

Planar graphs

- **Definition**. A graph *G* is **planar** if there is an embedding of *G* in the plane such that no two edges cross.
- **Euler's formula.** Let *G* be a planar graph with a plane embedding. Then

#vertices + #faces = #edges +2

- **Corollary.** Let G be a planar graph with girth k. Then $$e(G) \leq \frac{k}{k-2}(n-2)$$

How to build a nonplanar graph?

- **Trivial:**

  If $b < n/6$, then any strategy will do

- Maker has at least $3n$ edges at the end, so he won

How to build a nonplanar graph

Let $b = \left(\frac{1}{2} - \varepsilon\right) n$

\# of edges of Maker: $(1+\alpha_n(\varepsilon))n = \dfrac{\binom{n}{2}}{\left(\frac{1}{2} - \varepsilon\right) n + 1}$

Let $k = k(\varepsilon)$ be the smallest s.t. $\left(1 + \frac{\alpha}{2}\right) > \frac{k}{k-2}$

- **GOAL** of Maker:
- Avoid creating cycles of length < *k*, in the first $(1+\alpha/2)n$ moves

- If Maker succeeds in doing this, he won:

He has a graph of girth at least *k* with

at least $\left(1+\frac{\alpha}{2}\right)n > \frac{k}{k-2}n$ edges

- **GOAL** of Maker:
- Avoid creating cycles of length < k, in the first $(1+\alpha/2)n$ moves

- **Strategy:** Claim edge (u,v) such that
  - (u,v) does not close a cycle of length $< k$
  - Degrees of u and v are $< n^{1/(k+1)}$

- Works, since in the time-interval of our interest there exist $\Omega(n^2)$ unclaimed edges

How to prevent our opponent from building a non-planar graph?

- Proof: Let $b=n/2-1$. How can Breaker win?
- He will **force** Maker to build a spanning tree! (which is planar)

  - (Assume that n is even. Note that Maker has exactly $n-1$ edges at the end)

Enforcing a spanning tree

- More generally: assume G consists of $b+1$ pairwise edge-disjoint spanning trees.
- Then Breaker can make sure that Maker has a spanning tree at the end of the game.

  - Note K_n can be partitioned into $n/2$ spanning trees.

Breaker's strategy

- Maintain spanning trees $T_1, T_2, \ldots, T_{f+1}$ such that
- Maker's graph = $\cap E(T_i)$
- Breaker's graph = $E(K_n) \setminus \cup E(T_i)$
- Unclaimed edges = $\cup E(T_i) \setminus \cap E(T_i)$

- What did just happen here?

- Breaker “enforced” that his opponent did something.

Avoider/Enforcer games

- Avoider wins the game if he does NOT occupy any of the "winning sets" (which thus could be called "losing sets"), otherwise Enforcer wins

- $f_{\mathcal{F}}$ is the threshold bias of the Avoider/Enforcer game $\mathcal{F}$ if
  - Enforcer wins the (*1:f*) game for every $f \leq f_{\mathcal{F}}$
  - Avoider wins the (*1:f*) game for every $f > f_{\mathcal{F}}$

Occurences

- The goal of Maker is to build a graph from a monotone decreasing family.
  - planarity game
- Building a pseudorandom graph (useful for various Maker/Breaker games)
  - Making lots of edge-disjoint Hamilton cycles

First surprise

- Random graph intuition fails

BADLY!

Theorem.(2006+) Avoider loses the (*1:b*) game on $\mathcal{T}_n$ as soon as *b* is such that he has at least *n-1* edges at the end. I.e.

$$f_{\mathcal{T}_n} = \begin{cases} \lfloor n/2 \rfloor & n \text{ is odd} \\ \lfloor n/2 \rfloor - 1 & n \text{ is even} \end{cases}$$

Second surprise

- We do not even know whether a threshold bias exists at all!
- Sometimes it doesn’t!

What do we know?

- The lower threshold bias $f_{\mathcal{F}}^{-}$ is the largest integer such that Enforcer wins the *(1:f)* AE-game $\mathcal{F}$ for every *f* $\leq f_{\mathcal{F}}^{-}$
- The upper threshold bias $f_{\mathcal{F}}^{+}$ is the largest integer such that Avoider wins the *(1:f)* AE-game $\mathcal{F}$ for every *f* $> f_{\mathcal{F}}^{+}$

Theorem (2006+) For all the discussed games,

$$b_{\mathcal{F}} \leq f_{\mathcal{F}}^{-}$$

Criterion

- **Theorem** Avoider has a winning strategy in the (*p*:*q*)-game on $\mathcal{F}$ provided

$$\sum_{A\in\mathcal{F}}\left(1+\frac{1}{p}\right)^{-|A|} < \left(1+\frac{1}{p}\right)^{-p}$$

- **Remark** Formula does not depend on *q*
- **Open Problem** Obtain a useful criterion for *q>1*.

What we don't know

- **Burning open problems.**

  Upper bounds on $f_{\mathcal{F}}^{+}$

  Prove $f_{\mathcal{F}}^{+} = \Theta\left(f_{\mathcal{F}}^{-}\right)$ for "nice" games

Thank you for your attention!