Approximation algorithms for geometric intersection graphs

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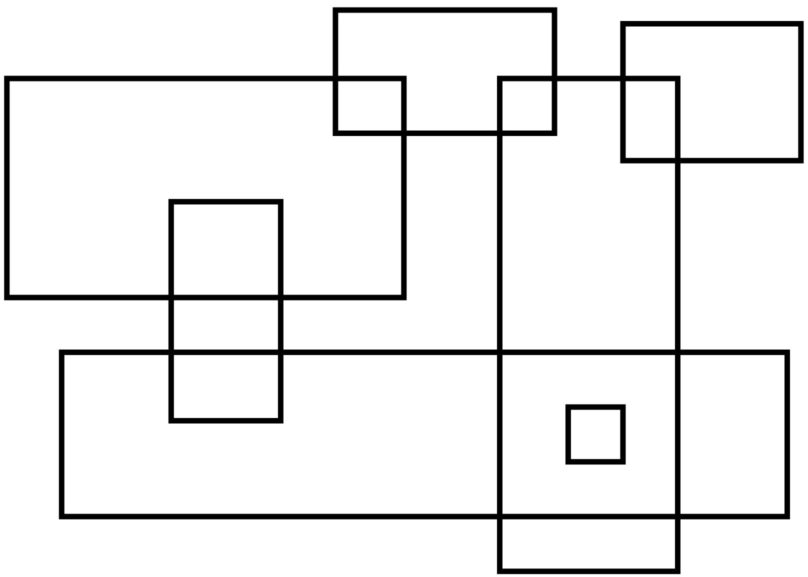
Outline

- Introduction
- Independent sets in disk graphs
- Vertex covers in disk graphs
- Vertex coloring disk graphs
- Rectangle intersection graphs
- Dominating sets in unit disk graphs
- Some open problems

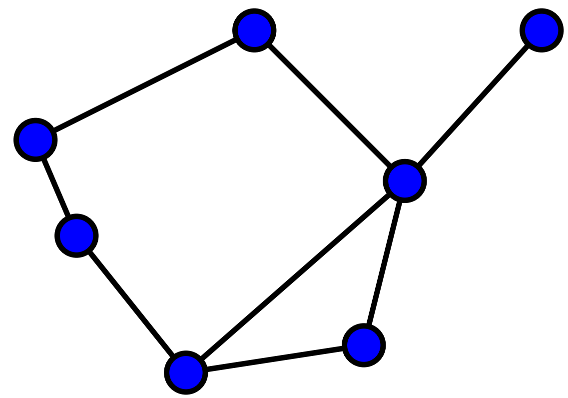
What are geometric intersection graphs?

- **vertices** = geometric objects
- **edges** = non-empty intersection between objects

Example: a rectangle intersection graph



geometric representation



intersection graph

Popular geometric intersection graphs

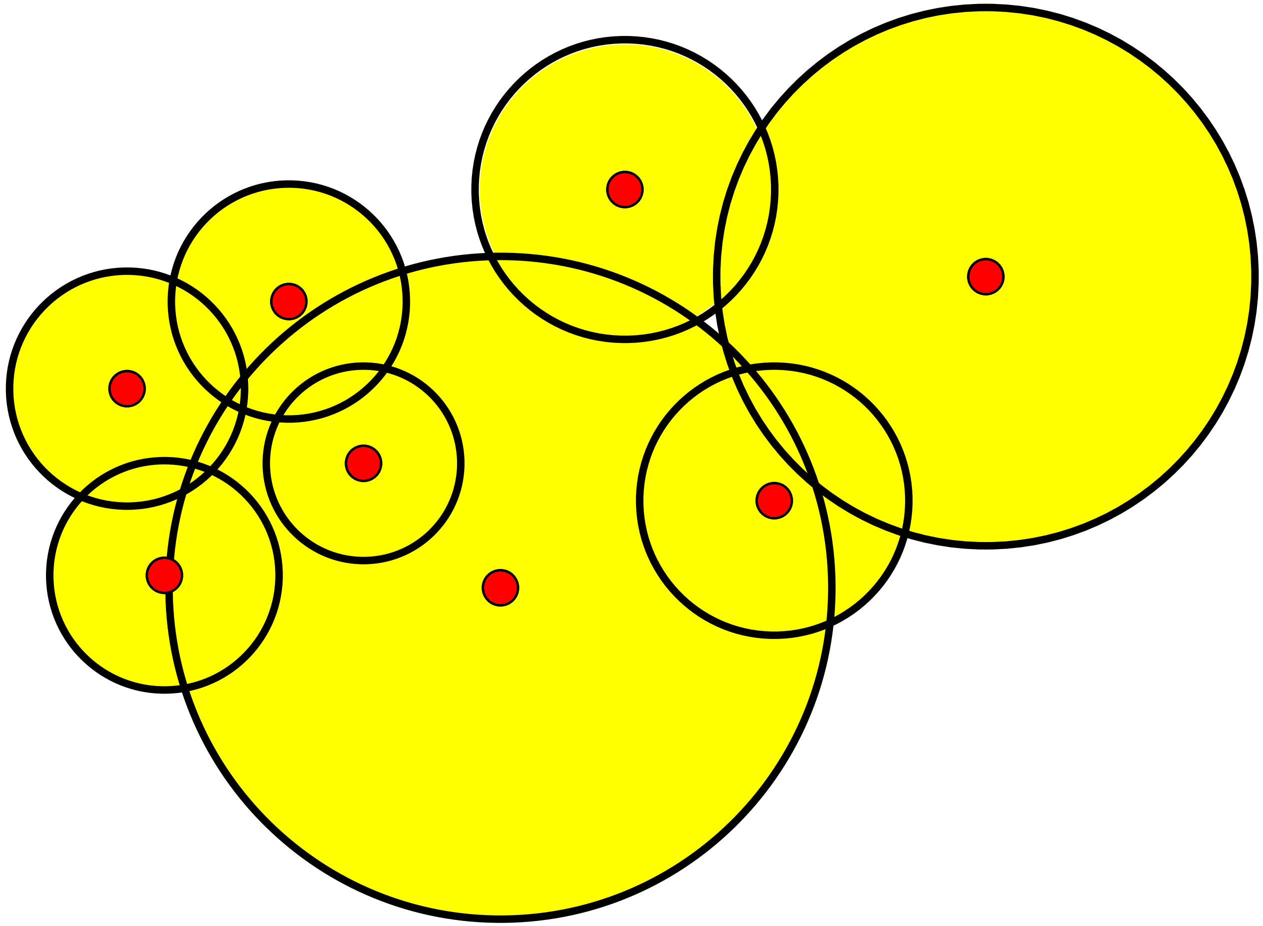
- disks (➜ **disk graphs**), squares
- "fat" objects
- ellipses, rectangles (axis-aligned), arbitrary convex objects
- line segments, curves, higher-dimensional objects

The **recognition problem is typically NP-hard**!!

Some Applications:

- ⇨ Wireless networks (frequency assignment problems)
- ⇨ Map labeling
- ⇨ Resource allocation (e.g. admission control in line networks)

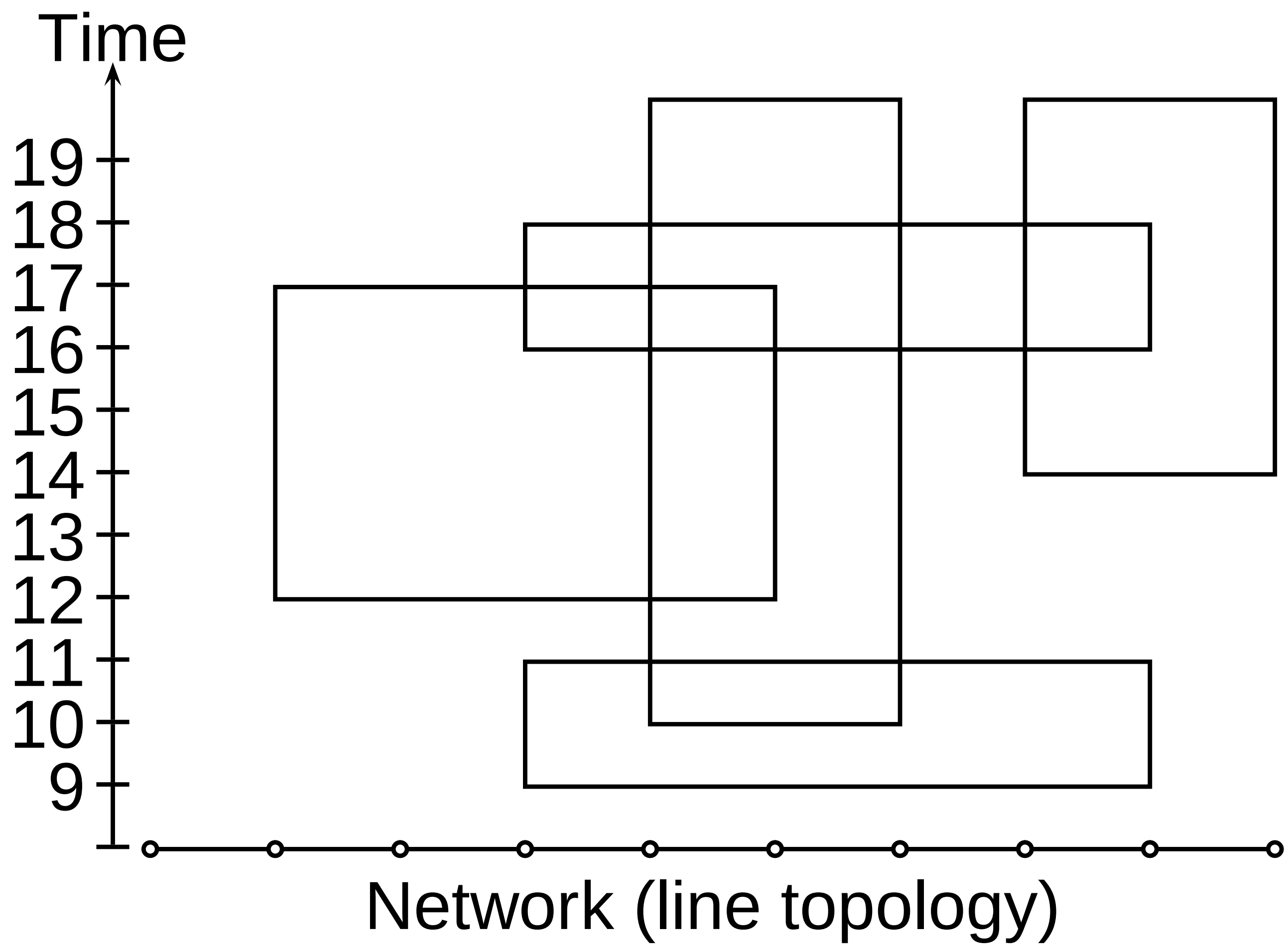
Application: Wireless networks



Application: Map labeling

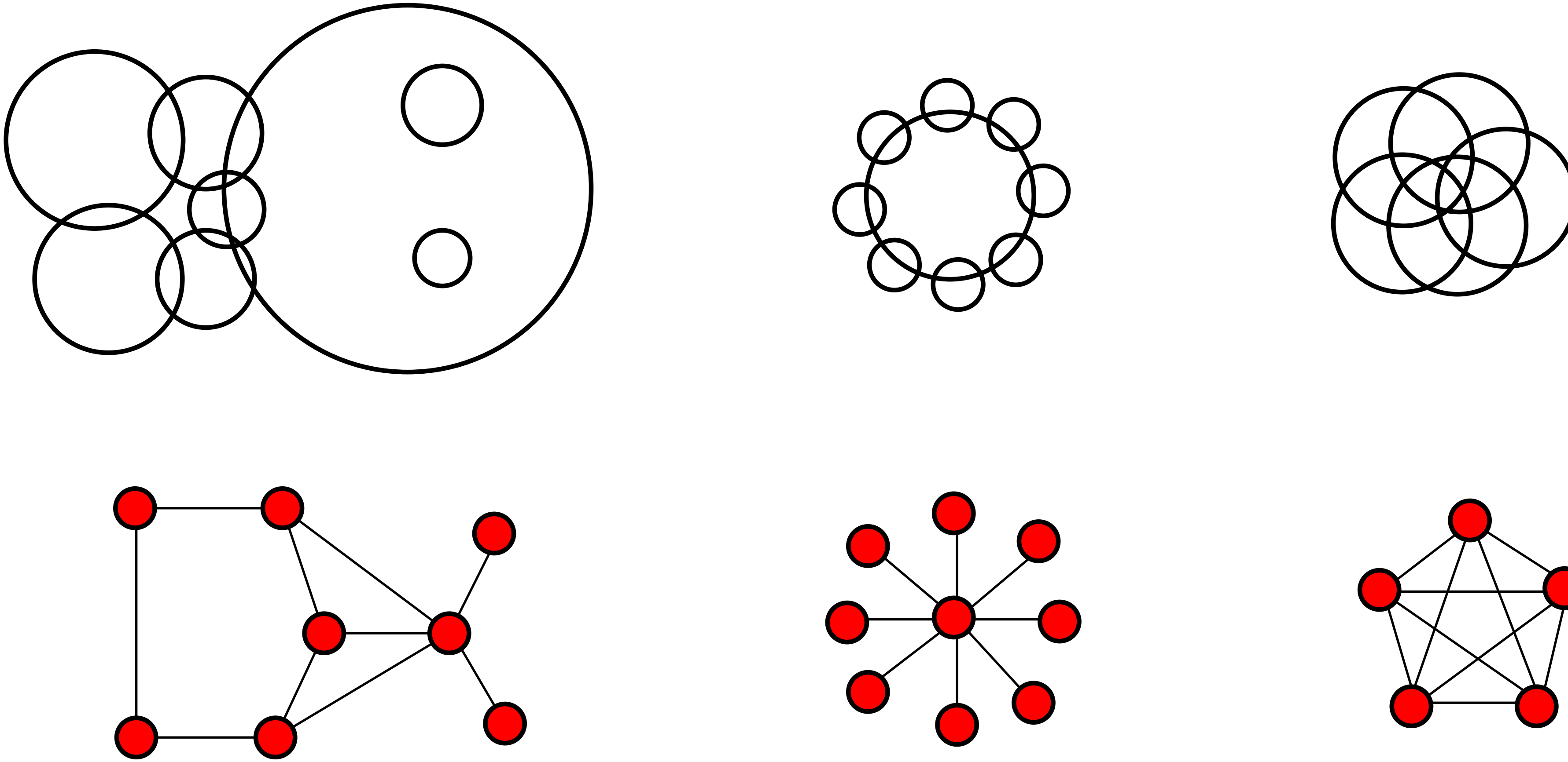
(illustration taken from a paper by van Kreveld, Strijk, Wolff)

Application: Call admission control



Disk graphs

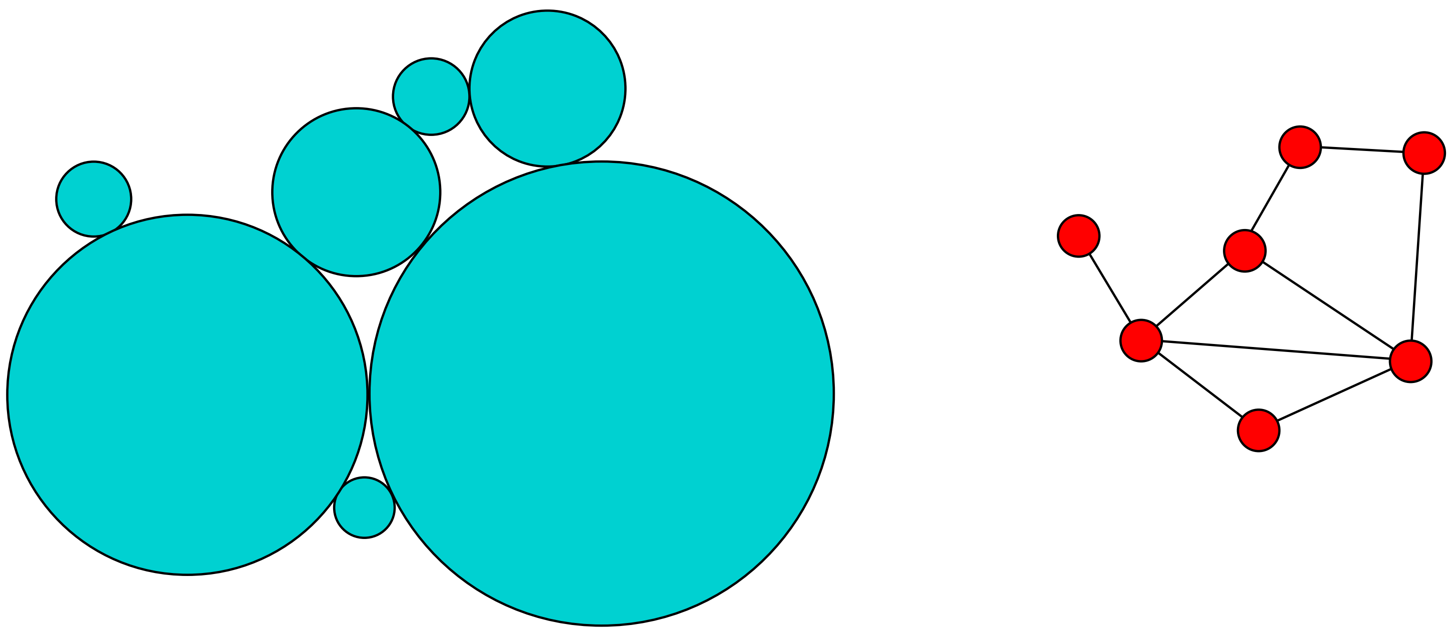
. . . are the intersection graphs of disks in the plane:



Subclasses of disk graphs

- **Unit disk graphs**: all disks have diameter 1

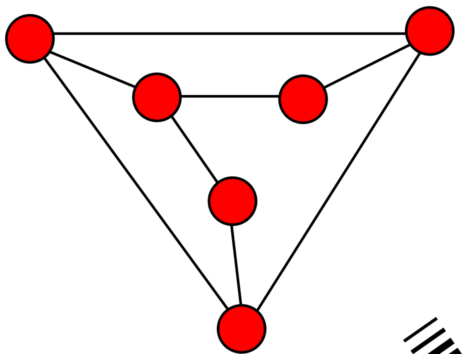
- **Coin graphs**: touching graphs of disks whose interiors are disjoint



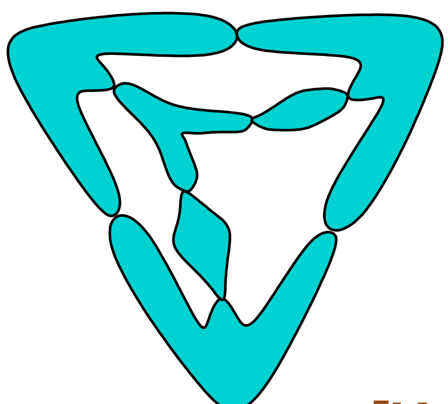
Coin graphs are planar, but surprisingly . . .

…every planar graph is a coin graph

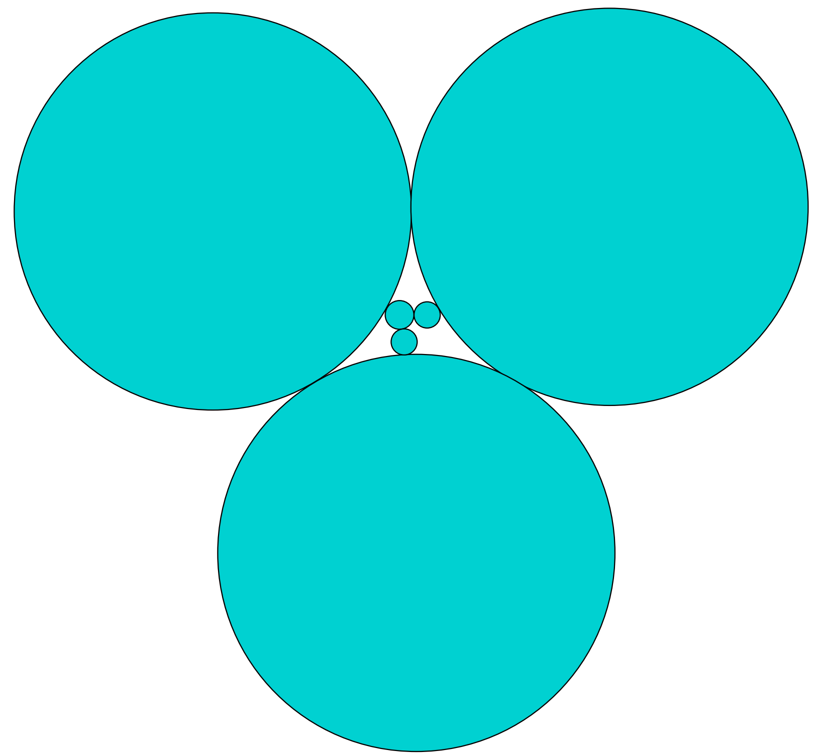
planar graph:



touching graph of “blobs”:



touching graph of disks:

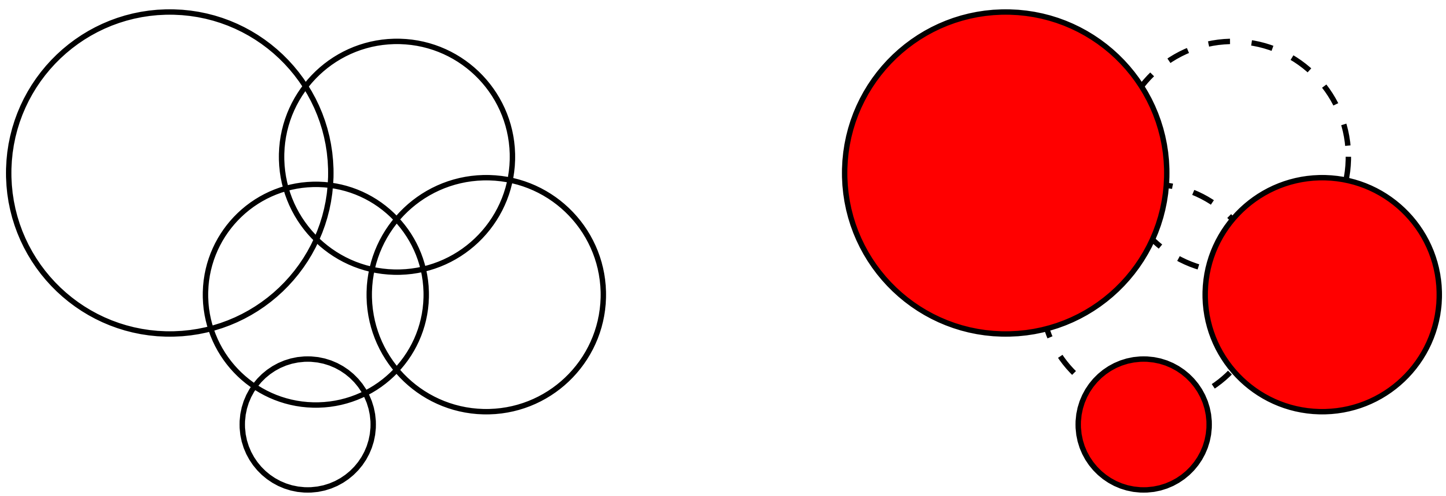


[Koebe, 1936]

Maximum Independent Set

Maximum Independent Set (MIS)

Input: a set $\mathcal{D}$ of disks in the plane
Feasible solution: subset $A \subseteq \mathcal{D}$ of disjoint disks
Goal: maximize $|A|$



In the weighted case (MWIS), each disk is associated with a positive weight.

Approximation algorithms for MIS

An algorithm for MIS is a **ρ-approximation algorithm** if it

- runs in **polynomial time** and
- always outputs an independent set of **size at least** OPT/ρ, where OPT is the size of the optimal independent set.

A **polynomial-time approximation scheme (PTAS)** is a family of $(1+\varepsilon)$-approximation algorithms for every constant $\varepsilon > 0$.

For MWIS, the definitions are analogous.

MIS in unit disk graphs

The problem is $\mathcal{NP}$-hard [Clark, Colbourn, Johnson'90].
Let's try the **greedy algorithm**:

Algorithm GREEDY
$I = \emptyset$;
for all given disks D **do**
    if D is disjoint from the disks in I **then**
        $I = I \cup \{D\}$;
return I;

Analysis of the greedy algorithm

① Compare the greedy solution I with the optimal solution I^*.

② "Charge" every disk in I^* to a disk in I.

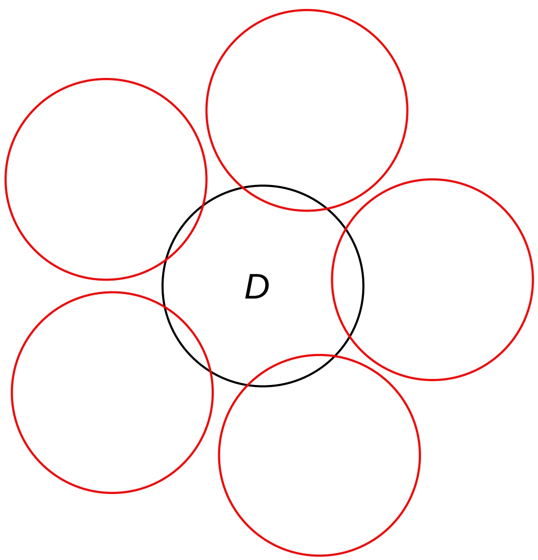
③ Bound the number of disks charged to the same disk in I.

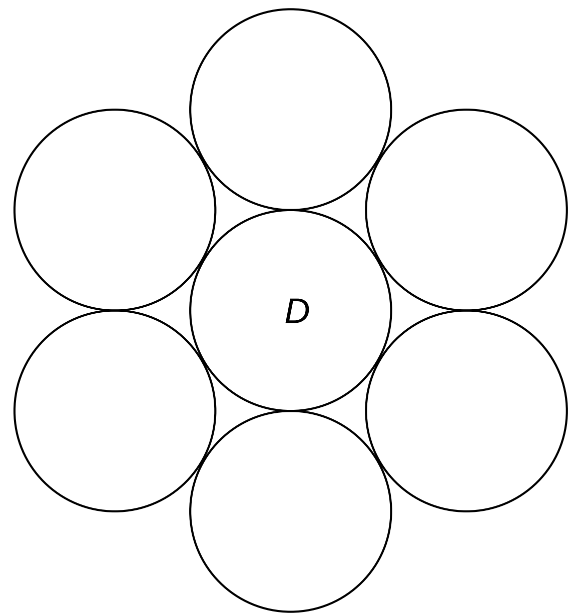
Charging rules for a disk $D \in I^*$:

- If D is in I, charge D to itself.
- If D is not in I, then charge it to any disk that intersects D and was accepted by GREEDY before it processed D.

How often can a disk D in I be charged?

If D is also in I^*, D is charged only once.
If D is not in I^*, it is charged by disks in I^* that intersect D.
These disks are disjoint, so there can be at most 5 such disks:





➥ $|I^*| \leq 5|I|$ and **GREEDY is a 5-approximation algorithm**.

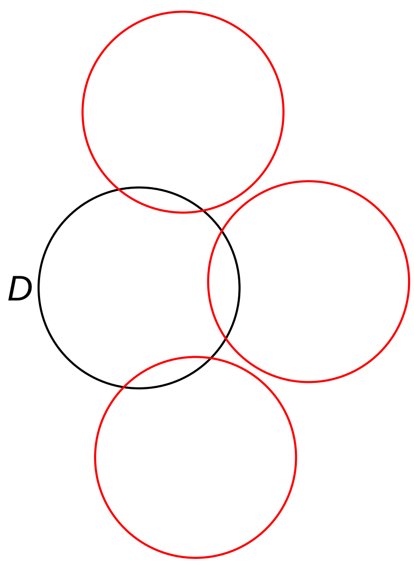
An improved greedy algorithm

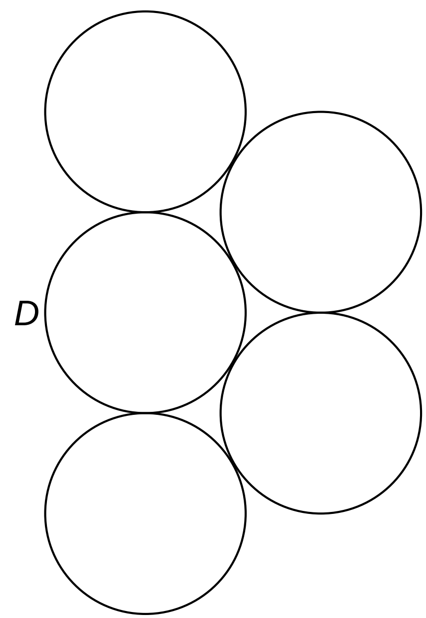
Algorithm LEFTMOST-GREEDY
$I = \emptyset$;
for all given disks D in order of increasing x-value **do**
  if D is disjoint from the disks in I **then**
    $I = I \cup \{D\}$;
return I;

Claim. LEFTMOST-GREEDY is a 3**-approximation algorithm** for MIS in unit disk graphs.

Analysis of LEFTMOST-GREEDY

Use the same charging argument.
Note: A disk D in I receives charge from disks in I^* that are processed **after** D by LEFTMOST-GREEDY.
Therefore, each disk is charged at most three times:





Do we need the representation?

GREEDY did not need to know the representation, but what about LEFTMOST-GREEDY?

For getting ratio 3 we needed only the following:

> When a disk D is selected, the disks intersecting D that are processed later contain at most three disjoint disks.

- ➥ We can still get ratio 3 if we can identify a disk whose neighborhood does not contain four disjoint disks!

LEFTMOST-GREEDY w/o representation

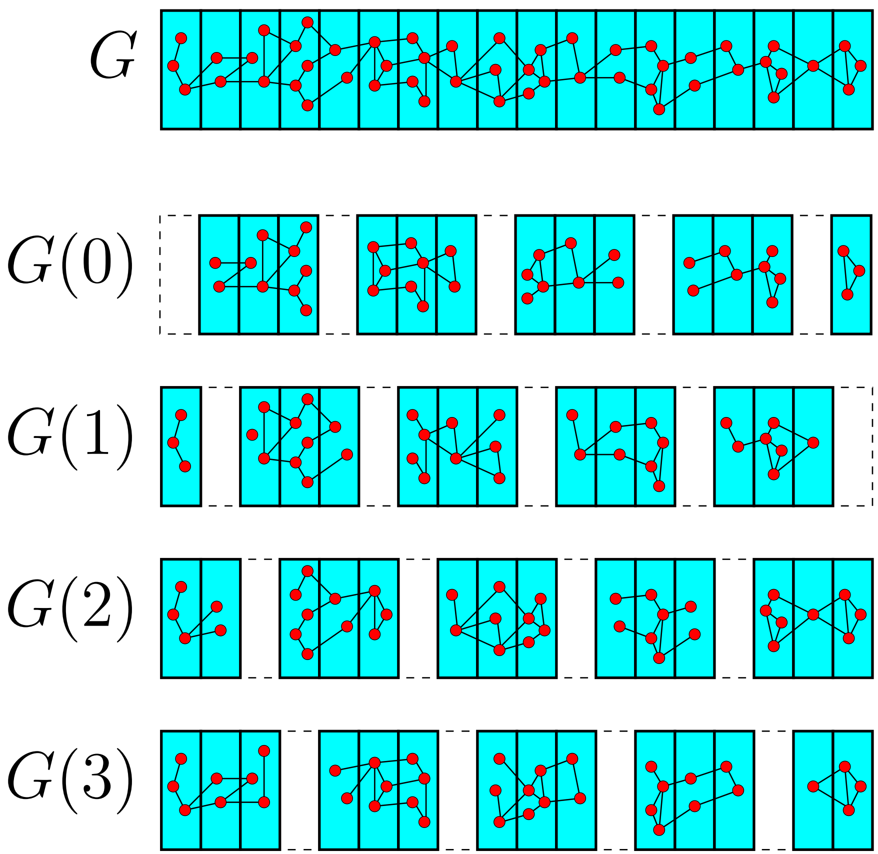
Given a graph $G = (V, E)$ that is the intersection graph of unit disks, the following is a 3**-approximation algorithm for MIS:**

$I = \emptyset$;
repeat
    $v =$ a vertex whose neighborhood does not have 4 independent vertices;
    $I = I \cup \{v\}$;
    delete v and its neighbors from the graph;
until the graph is empty;
return I;

The vertex v can be found in $O(|V|^5)$ time.

The shifting strategy

[Baker, 1984; Hochbaum and Maass, 1985]



❶ Partition graph into **slices**.

❷ Let $k > 0$ be a fixed integer.

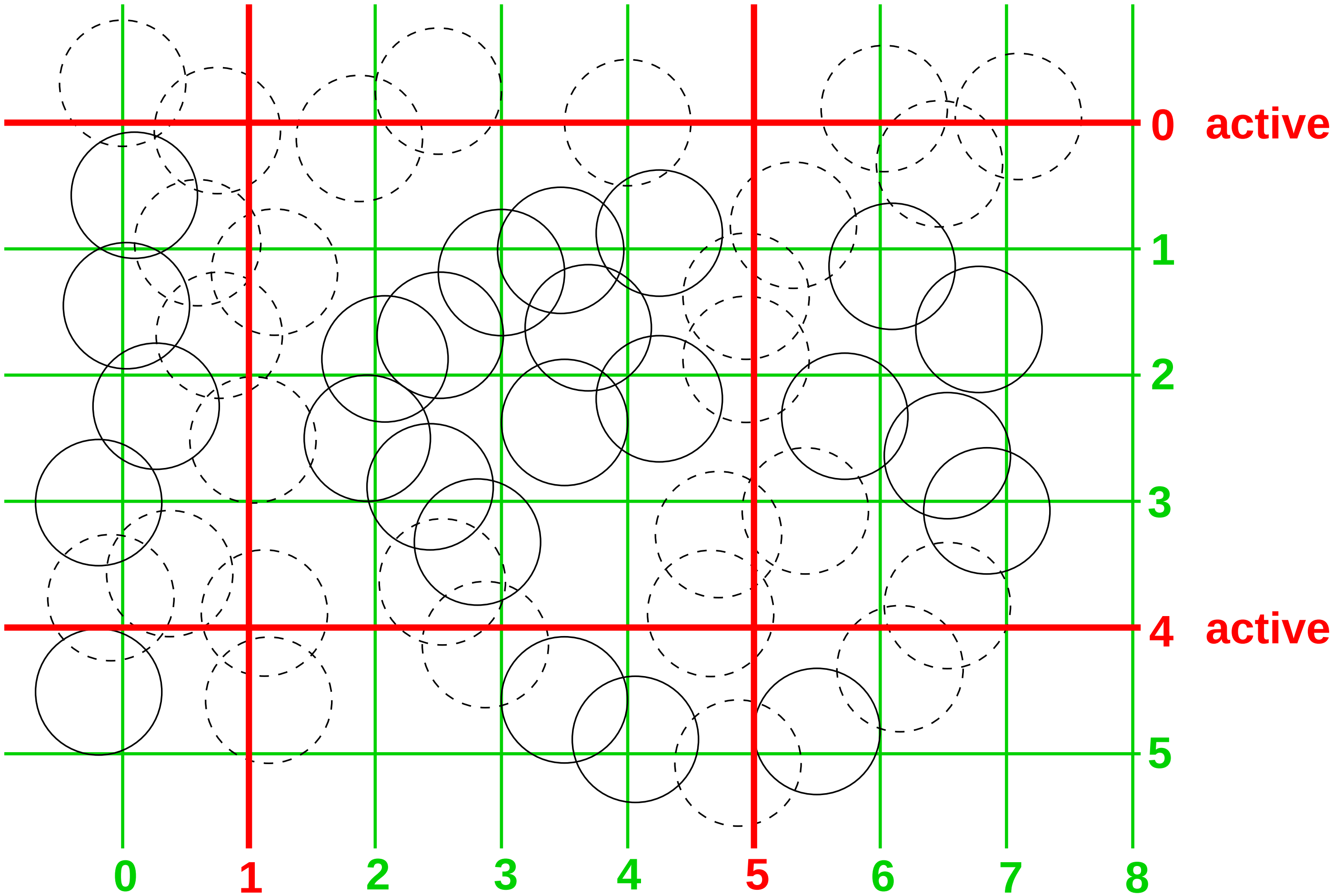
❸ Remove slices equal to ℓ modulo k and compute a maximum independent set in the graph $G(\ell)$, $0 \leq \ell < k$.

❹ Output the largest set found in this way.

The largest of these sets contains at least $(1 - \frac{1}{k})\mathrm{OPT}$ vertices.

Shifting for unit disk graphs

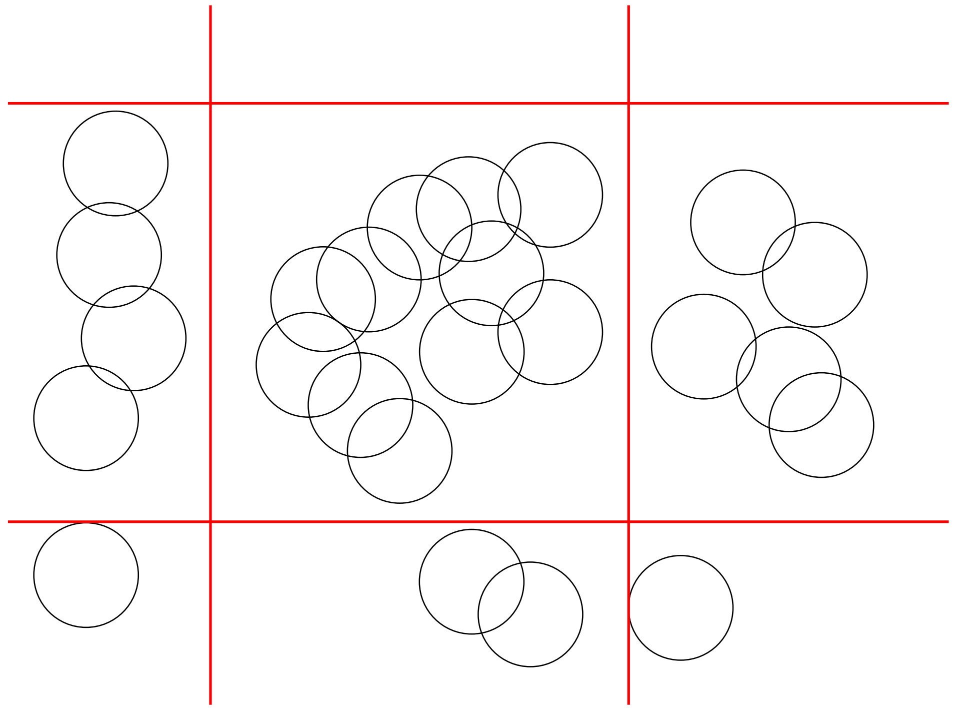
[Hochbaum and Maass, 1985]



Remove disks hitting active lines (and shift active lines).

Solving the Subproblems

Active lines partition the plane into squares that can be considered independently:



➥ Compute maximum independent set I in each square by brute-force enumeration. Since $|I| = O(k^2)$, time $n^{O(k^2)}$ suffices.

PTAS for MIS in unit disk graphs

❶ For $0 \le r, s < k$, get $\mathcal{D}(r,s)$ from $\mathcal{D}$ by deleting disks that

- ➜ hit a horizontal line equal to r modulo k or
- ➜ hit a vertical line equal to s modulo k.

❷ Compute the maximum independent set I_S in each $k \times k$ square S of $\mathcal{D}(r,s)$ by brute-force enumeration.

❸ The union of the sets I_S gives a maximum independent set in $\mathcal{D}(r,s)$.

❹ Output the largest independent set obtained in this way.

Running-time: $n^{O(k^2)}$ for n disks. (Can be improved to $n^{O(k)}$.)

Approximation: Computed solution has size at least $\left(1-\frac{2}{k}\right)\mathrm{OPT}$.

MIS in unit disk graphs: Summary

- $\mathcal{NP}$-hard [Clark, Colbourn, Johnson 1990].
- GREEDY gives a 5-approximation.
  [Marathe et al., 1995]
- LEFTMOST-GREEDY gives a 3-approximation. There is a variant that does not need the representation.
  [Marathe et al., 1995]
- The shifting strategy gives a PTAS. It needs the representation.
  [Hochbaum and Maass, 1985; Hunt III et al., 1998]

Recent related results

- [Nieberg, Hurink, Kern, 2004] PTAS for maximum weight independent set in unit disk graphs without given representation.
- [Marx, 2005] Maximum independent set in unit disk graphs is W[1]-hard. (➠ No FPT algorithm and no EPTAS unless FPT=W[1].)
- [van Leeuwen, 2005] Asymptotic FPTAS for maximum independent set (and various other problems) in unit disk graphs of bounded density.

MIS in general disk graphs

- The approximation ratio of GREEDY is only $|V| - 1$.
- But it helps to process the disks in the right order:

Algorithm SMALLEST-GREEDY
$I = \emptyset$;
for all given disks D in order of increasing diameter **do**
    if D is disjoint from the disks in I **then**
        $I = I \cup \{D\}$;
return I;

Analysis of SMALLEST-GREEDY

Again, charge disks in the optimal solution I^* to disks in the solution I computed by the algorithm.

- Every disk D in I receives charge only from disks in I^* that intersect D and were processed after D. There can be **at most five such disks**.

SMALLEST-GREEDY is a 5-approximation algorithm.

If the representation is not given: Find a vertex whose neighborhood does not contain an independent set of size 6, select it, and delete its neighbors.

Extending the shifting strategy

❶ Classify the disks into layers according to their sizes.

❷ Use the shifting strategy on all layers simultaneously.

❸ After removing all disks that hit active lines, use dynamic programming to compute a maximum independent set.

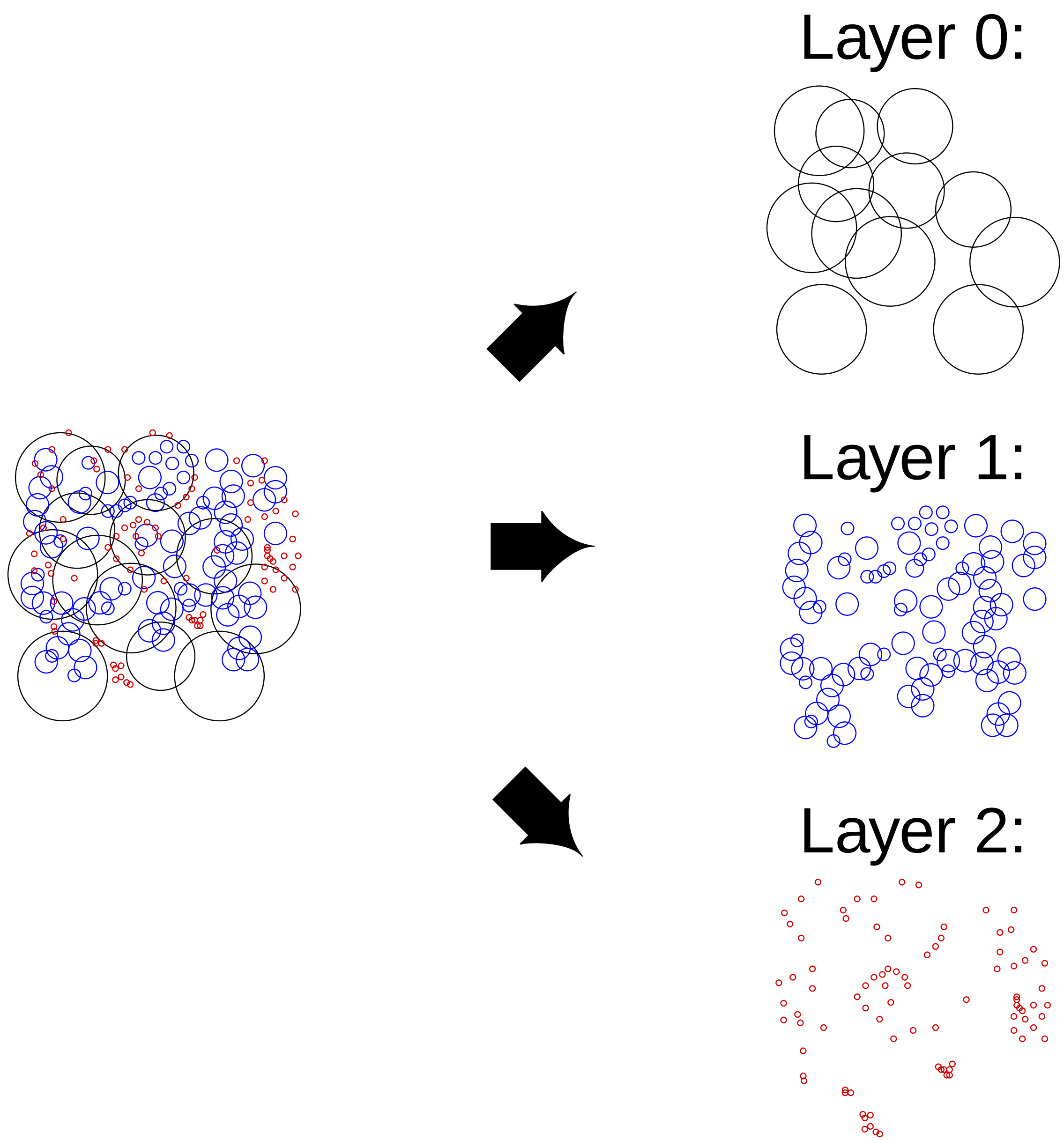
Classification into layers:

➢ Assume that the largest disk has diameter 1.

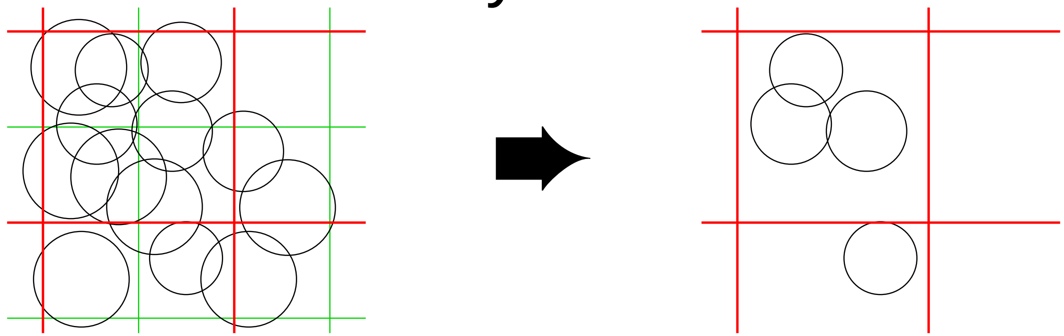
➢ Layer ℓ: disks with diameter d, $\frac{1}{(k+1)^\ell} \geq d > \frac{1}{(k+1)^{\ell+1}}$.

➢ Lines on layer ℓ are $\frac{1}{(k+1)^\ell}$ apart, every k-th line is active.

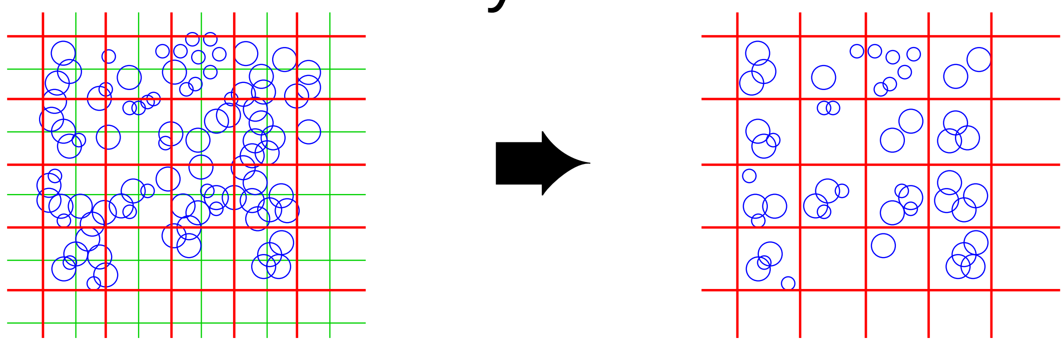
Partition into layers



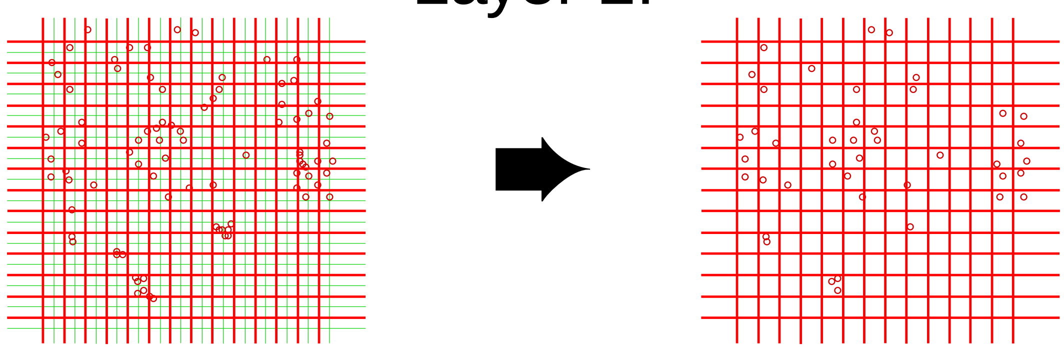
Layer 0:



Layer 1:



Layer 2:

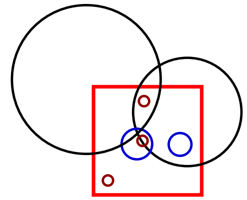


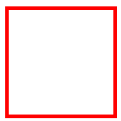
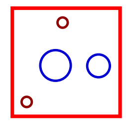
Dynamic programming table

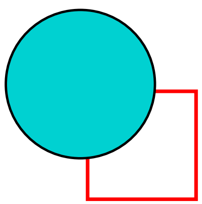
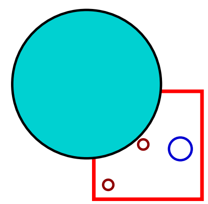
At square S on level ℓ, compute $\textsc{Table}_S$.
If I is an independent set of disks of level $< \ell$ intersecting S, then

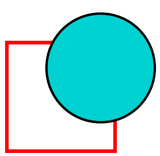
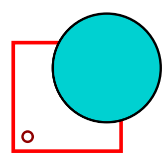
$$\textsc{Table}_S[I] = \begin{cases} \text{size of maximum independent set } I' \\ \text{of disks of level } \geq \ell \text{ in } S \text{ such that} \\ I \cup I' \text{ is an independent set.} \end{cases}$$

Example

S: 

TABLE$_S$ [] $= 4$ (note )

TABLE$_S$ [] $= 3$ (note )

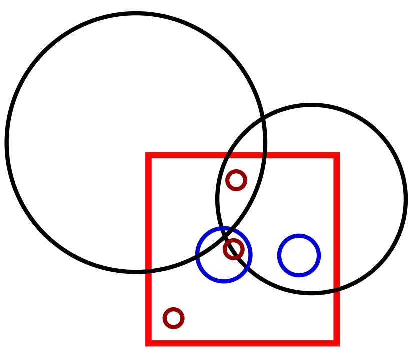
TABLE$_S$ [] $= 1$ (note )

Computing $\textsc{Table}_S$

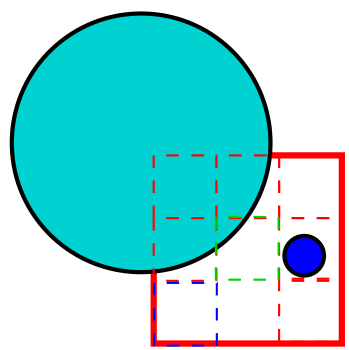
1. Enumerate all $n^{O(k^4)}$ independent sets J of disks of level $\leq \ell$ touching S.
2. Look up corresponding entries of $\textsc{Table}_{S'}$ for subsquares of S.
3. Update $\textsc{Table}_S[I]$ for $I = \{D \in J \mid D \text{ has level } < \ell\}$.

Example:

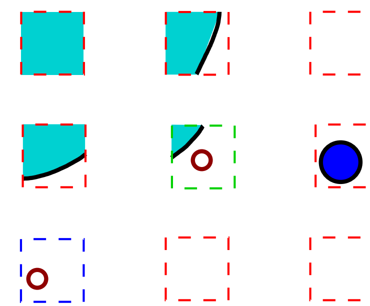
S:

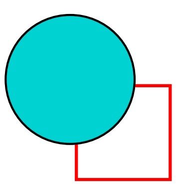
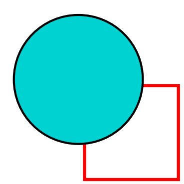
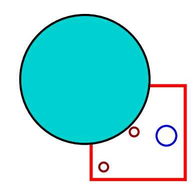


J:

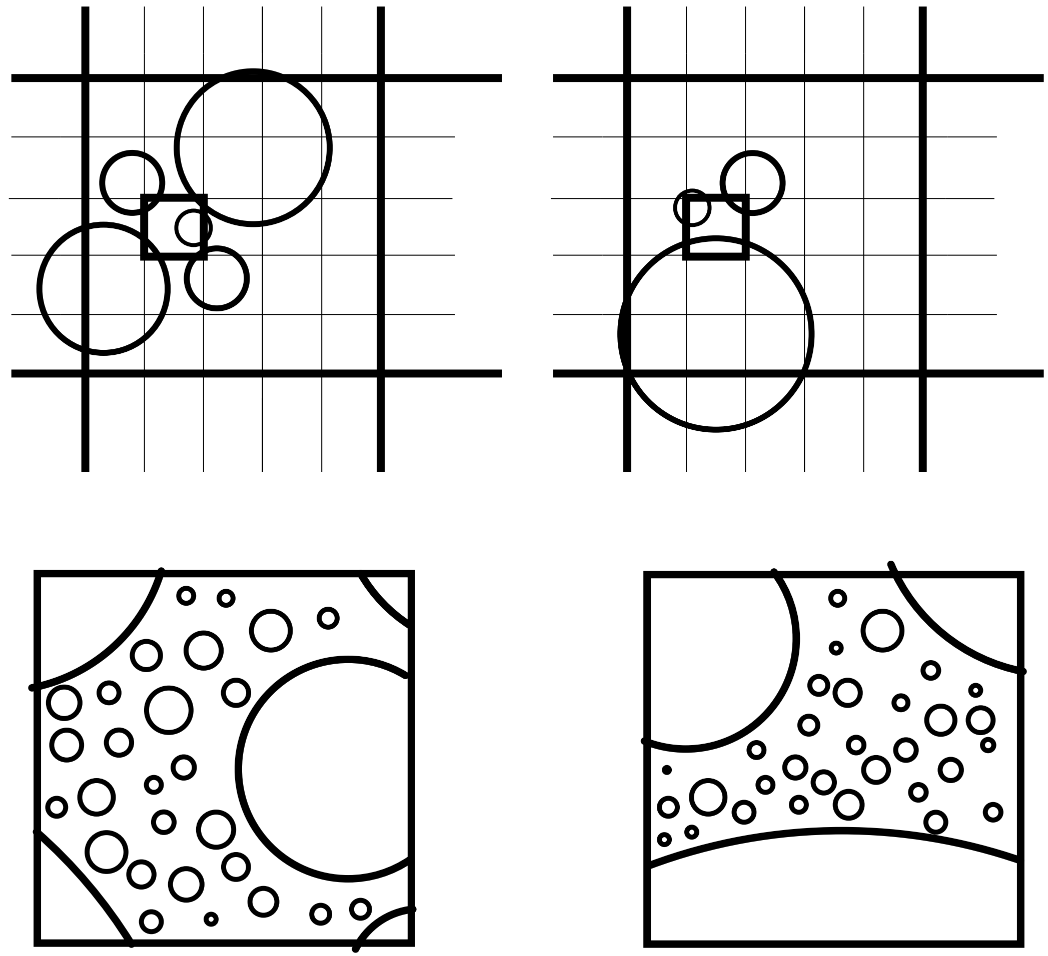


Lookups:



$\Rightarrow$ $\textsc{Table}_S$ [] $=$ max { $\textsc{Table}_S$ [] , 3 } (note )

Two more examples for lookups



The PTAS for MIS

❶ For $0 \le r, s < k$, get $\mathcal{D}(r,s)$ from $\mathcal{D}$ by deleting disks that

- ➜ hit a horizontal line equal to r modulo k on their level, or
- ➜ hit a vertical line equal to s modulo k on their level

❷ Compute dynamic programming tables for $\mathcal{D}(r,s)$ in all squares.

❸ The union of $\textsc{Table}_S[\emptyset]$ over all top-level squares gives a maximum independent set in $\mathcal{D}(r,s)$.

❹ Output the largest independent set obtained in this way.

Running-time: $n^{O(k^4)}$ for n disks. (Can be improved to $n^{O(k^2)}$.)

Approximation: Computed solution has size at least $\left(1-\frac{2}{k}\right)\mathrm{OPT}$.

MIS in disk graphs: Summary

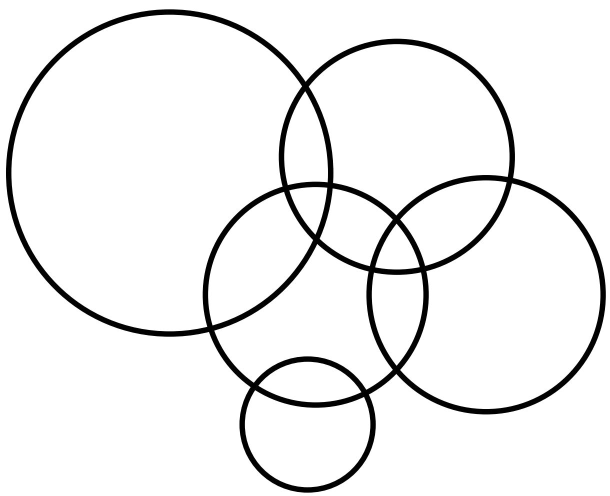
- SMALLEST-GREEDY is a 5-approximation algorithm. There is a variant that does not need the representation.
  [Marathe et al., 1995]
- The shifting strategy combined with dynamic programming gives a PTAS. It needs the representation.
  [E, Jansen, Seidel'01: $n^{O(k^2)}$; Chan'01: $n^{O(k)}$]

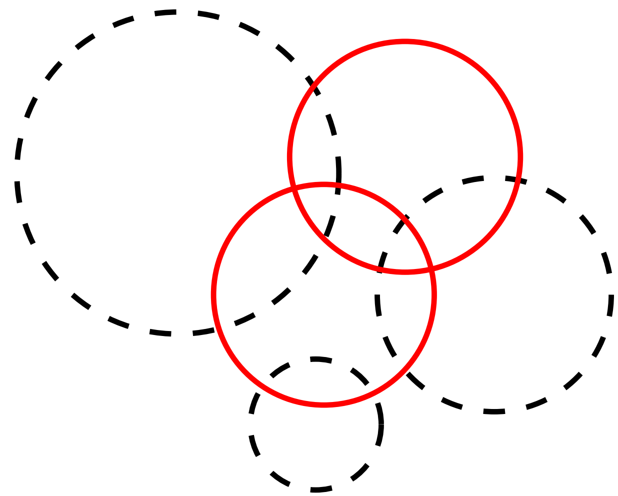
Note: These results can be adapted to **squares, regular polygons and other "disk-like" or fat objects**, also in **higher dimensions.** The PTAS works also for the **weighted version**.

Minimum Vertex Cover

The problem MINVERTEXCOVER

Input: a set $\mathcal{D}$ of disks in the plane
Feasible solution: subset $C \subseteq \mathcal{D}$ of disks such that, for any $D_1, D_2 \in \mathcal{D}$, $D_1 \cap D_2 \neq \emptyset \Rightarrow D_1 \in C$ or $D_2 \in C$.
Goal: minimize $|C|$





Approximating MinVertexCover

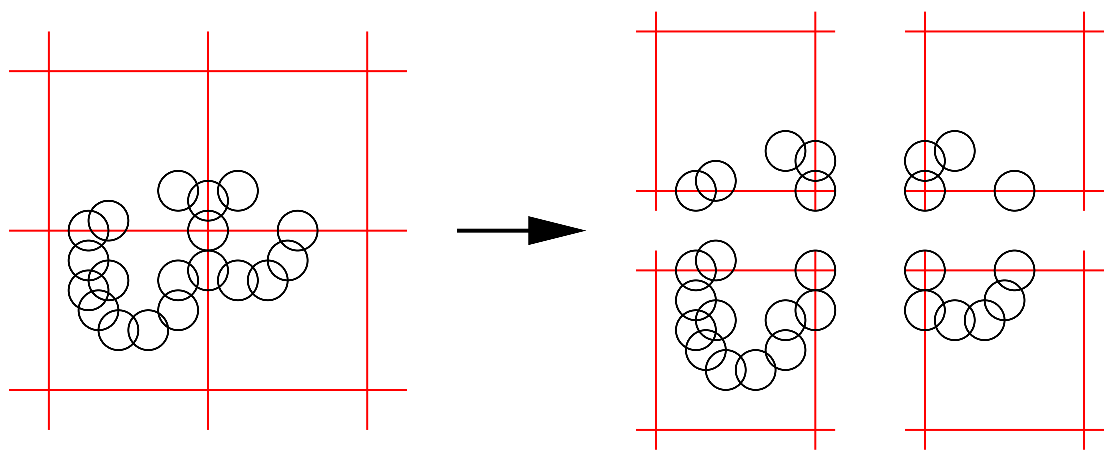
An algorithm for MinVertexCover is a **ρ-approximation algorithm** if it

- runs in **polynomial time** and
- always outputs a vertex cover of **size at most** $\rho \cdot \mathrm{OPT}$, where OPT is the size of the optimal vertex cover.

A **polynomial-time approximation scheme (PTAS)** is a family of $(1 + \varepsilon)$-approximation algorithms for every constant $\varepsilon > 0$.

PTAS idea for MinVertexCover

- **Fact:** I is an independent set $\Leftrightarrow \mathcal{D} \setminus I$ is a vertex cover
- To approximate MinVertexCover in unit disk graphs, we can again use the **shifting strategy**.
- Disks that hit an active line are considered in **all squares that they intersect** (at most 4 squares).



PTAS: MinVertexCover in unit disk graphs

❶ For $0 \le r, s < k$, partition the plane into squares via

- ➔ horizontal lines equal to r modulo k and
- ➔ vertical lines equal to s modulo k.

❷ Compute the minimum vertex cover C_S among the disks intersecting each $k \times k$ square S by computing a maximum independent set and taking the complement.

❸ The union of the sets C_S gives a candidate vertex cover (for each (r,s)).

❹ Output the smallest vertex cover obtained in this way.

Running-time: $n^{O(k^2)}$ for n disks. (Can be improved to $n^{O(k)}$.)

Analysis of PTAS for MINVERTEXCOVER

- Let C^* be an optimum vertex cover.
- For $0 \leq r, s < k$ let $C^*(r, s)$ be the disks intersecting active lines for (r, s) and let $\mathcal{S}(r, s)$ be the set of all $k \times k$ squares determined by these active lines.
- For a $k \times k$-square S, let C^*_S be the disks in C^* intersecting S and let $\mathrm{OPT}(S)$ be the optimum vertex cover of the disks intersecting S.

Candidate vertex cover computed by the algorithm for (r,s) has size

$$\begin{aligned}
\left|\bigcup_{S\in\mathcal{S}(r,s)} \mathrm{OPT}(S)\right| &\le \sum_{S\in\mathcal{S}(r,s)} |\mathrm{OPT}(S)| \\
&\le \sum_{S\in\mathcal{S}(r,s)} |C^*(S)| \\
&\le 3|C^*(r,s)| + |C^*|
\end{aligned}$$

For some choice of (r,s):

⇨ at most $\frac{1}{k}|C^*|$ disks of C^* intersect vertical active lines

⇨ at most $\frac{1}{k}|C^*|$ disks of C^* intersect horizontal active lines

For this choice, we have $|C^*(r,s)| \le \frac{2}{k}|C^*|$.

➥ Solution has size at most $\left(1+\frac{6}{k}\right)C^*$ for some choice of $(r,s$

MINVC in disk graphs: Summary

- PTAS for **unit disk graphs** using the shifting strategy (needs the representation). [Hunt III et al., 1994]
- $\frac{3}{2}$-approximation algorithm for **general disk graphs** (not needing the representation). [Malesińska, 1997]
- PTAS for **general disk graphs** using the shifting strategy and dynamic programming (needs the representation).
  [E, Jansen, Seidel'01]

Note: PTAS adapts to **squares, regular polygons etc.**, also in **higher dimensions.** Result holds for the **weighted version** as well.

Vertex Coloring

Coloring disk graphs

Goal: Assign a minimum number of colors to the disks such that intersecting disks get different colors!

> **Algorithm SMALLEST-DEGREE-LAST(graph G)**
> $v =$ a vertex with minimum degree in G;
> color $G \setminus \{v\}$ recursively;
> assign v the smallest available color;

Observation. Let D be the maximum degree of a vertex v at the time it was colored. Then the algorithm needs at most $D + 1$ colors.

Analysis for disk graphs

Let v be the vertex corresponding to the smallest disk.
Let $N(v)$ be the set of neighbors of v.
Note: At most 5 disks in $N(v)$ can get the same color.

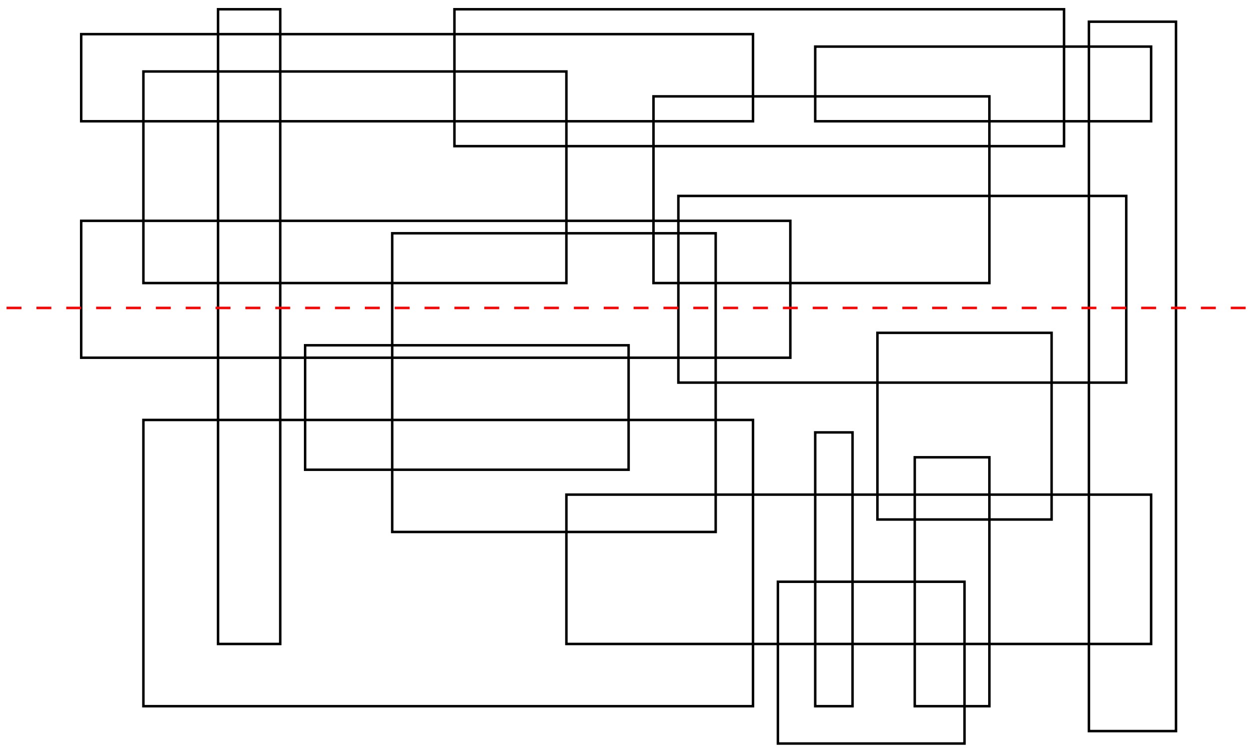
- ➥ Optimal number of colors OPT is at least $1 + \frac{|N(v)|}{5}$.
- ➥ $|N(v)| \leq 5 \cdot \mathrm{OPT} - 5$.
- ➥ So we must also have $D \leq 5\mathrm{OPT} - 5$.

The SMALLEST-DEGREE-LAST algorithm colors any disk graph with at most $5\mathrm{OPT} - 4$ colors. [Marathe et al. 1995; Gräf 1995]

Rectangle Intersection Graphs

MIS in Rectangle Graphs

⋆ **Idea:** find a “stabbing line” with at most half of the rectangles above and below.



Approximation algorithm for rectangles

Algorithm RECTANGLE-APPROX(set of rectangles R)
$\ell =$ stabbing line with at most $|R|/2$ rectangles above and below;
$R_{\mathrm{above}} =$ rectangles above stabbing line;
$R_{\mathrm{below}} =$ rectangles below stabbing line;
$R_{\mathrm{mid}} =$ rectangles intersecting stabbing line;
compute approximations I_1 and I_2 for R_{above} and R_{below} recursively;
compute optimal independent set I_0 for R_{mid};
return the larger of I_0 and $I_1 \cup I_2$;

Analysis of RECTANGLE-APPROX

Theorem The algorithm achieves approximation ratio $\log n$ for n rectangles.

Proof. by induction on the number of rectangles.
Let I^* be an optimal independent set.
Let I^*_0, I^*_1, I^*_2 be the rectangles in I^* that are on, above, below ℓ.
Case 1: $|I^*_0|$ is at least $|I^*|/\log n$.
Algorithm outputs a set of size at least

$$|I_0| \geq |I^*_0| \geq \frac{|I^*|}{\log n}.$$

Case 2: $|I_0^*|$ is smaller than $|I^*|/\log n$.
The algorithm outputs a set of size at least

$$
\begin{aligned}
|I_1 \cup I_2| &\geq \frac{\mathrm{OPT}(R_{\mathrm{above}})}{\log|R_{\mathrm{above}}|} + \frac{\mathrm{OPT}(R_{\mathrm{below}})}{\log|R_{\mathrm{below}}|} \\
&\geq \frac{\mathrm{OPT}(R_{\mathrm{above}})}{(\log n) - 1} + \frac{\mathrm{OPT}(R_{\mathrm{below}})}{(\log n) - 1} \\
&\geq \frac{|I_1^*| + |I_2^*|}{(\log n) - 1} = \frac{|I^*| - |I_0^*|}{(\log n) - 1} \\
&\geq \frac{|I^*| \cdot \left(1 - \frac{1}{\log n}\right)}{(\log n) - 1} = \frac{|I^*|}{\log n}
\end{aligned}
$$

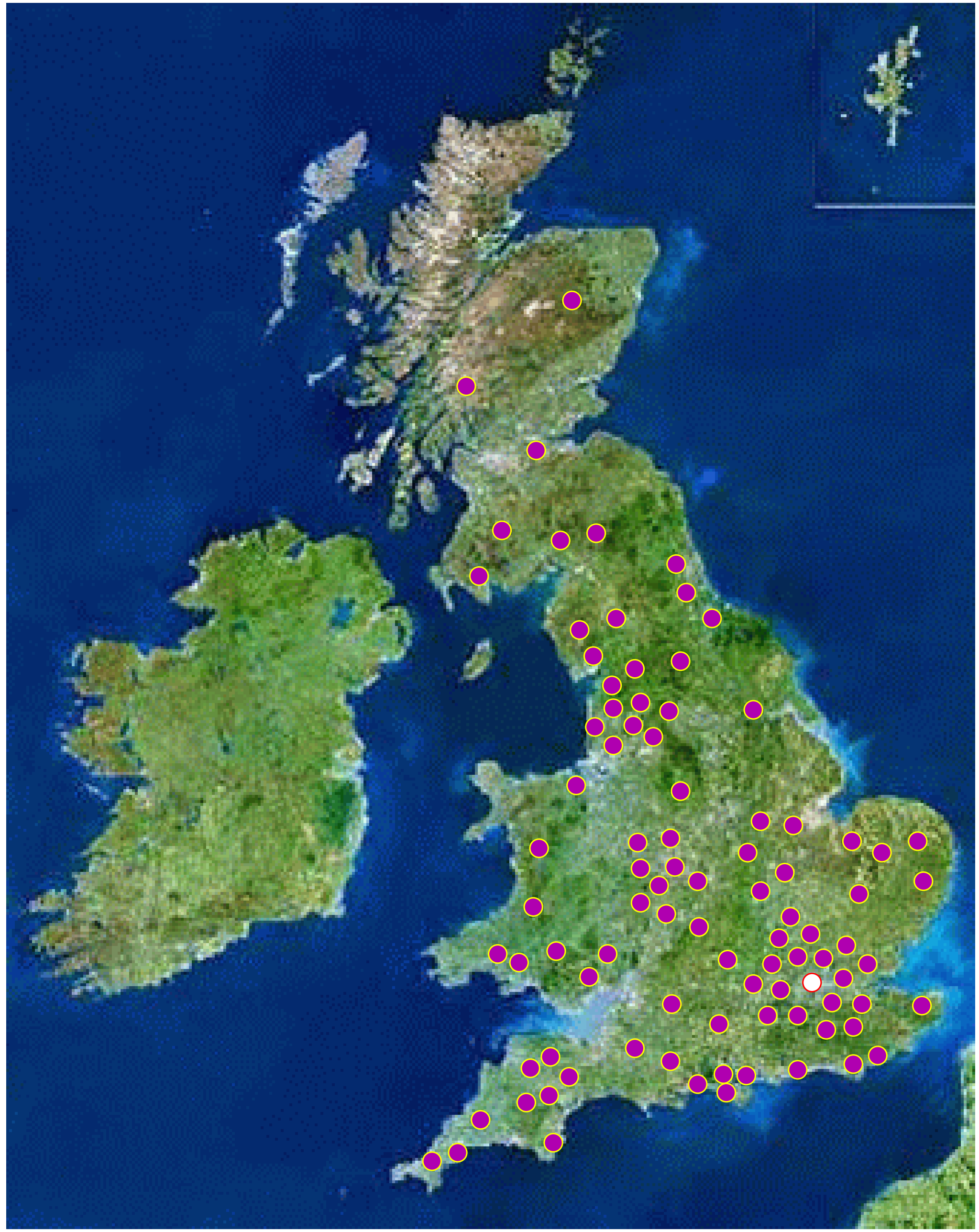
$\square$

MIS in rectangle graphs: Summary

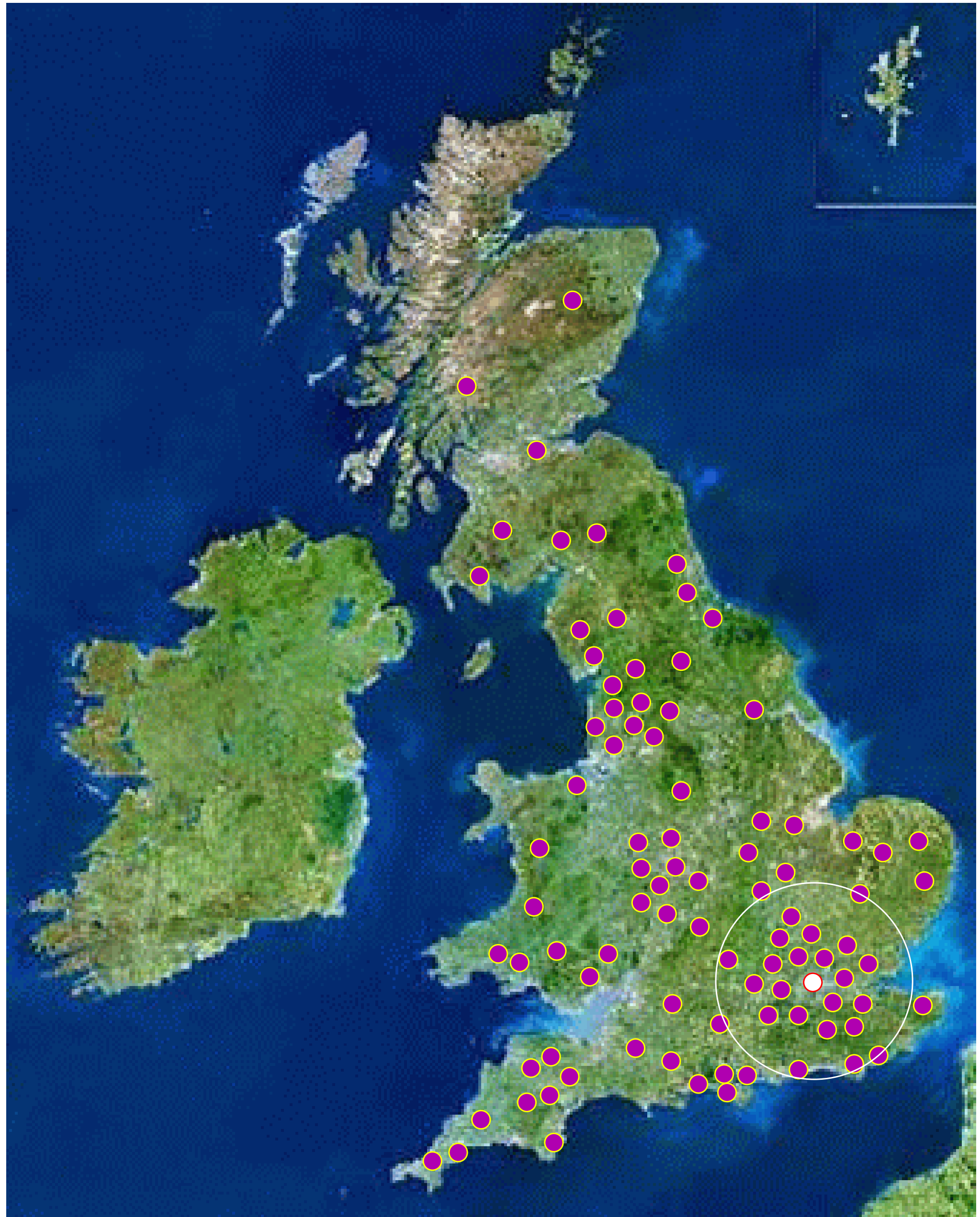
- There is an $O(\log n)$-approximation algorithm (with given representation).
[Agarwal et al., 1998; Khanna et al. 1998; Nielsen 2000]
- For every constant $c > 0$, there is an approximation algorithm with ratio $1 + \frac{1}{c}\log n$.
[Berman et al., 2001]
- If all rectangles have the same height, there is a PTAS.
[Agarwal et al., 1998]

Minimum Dominating Set

Flooding an Ad-Hoc Network



Flooding an Ad-Hoc Network



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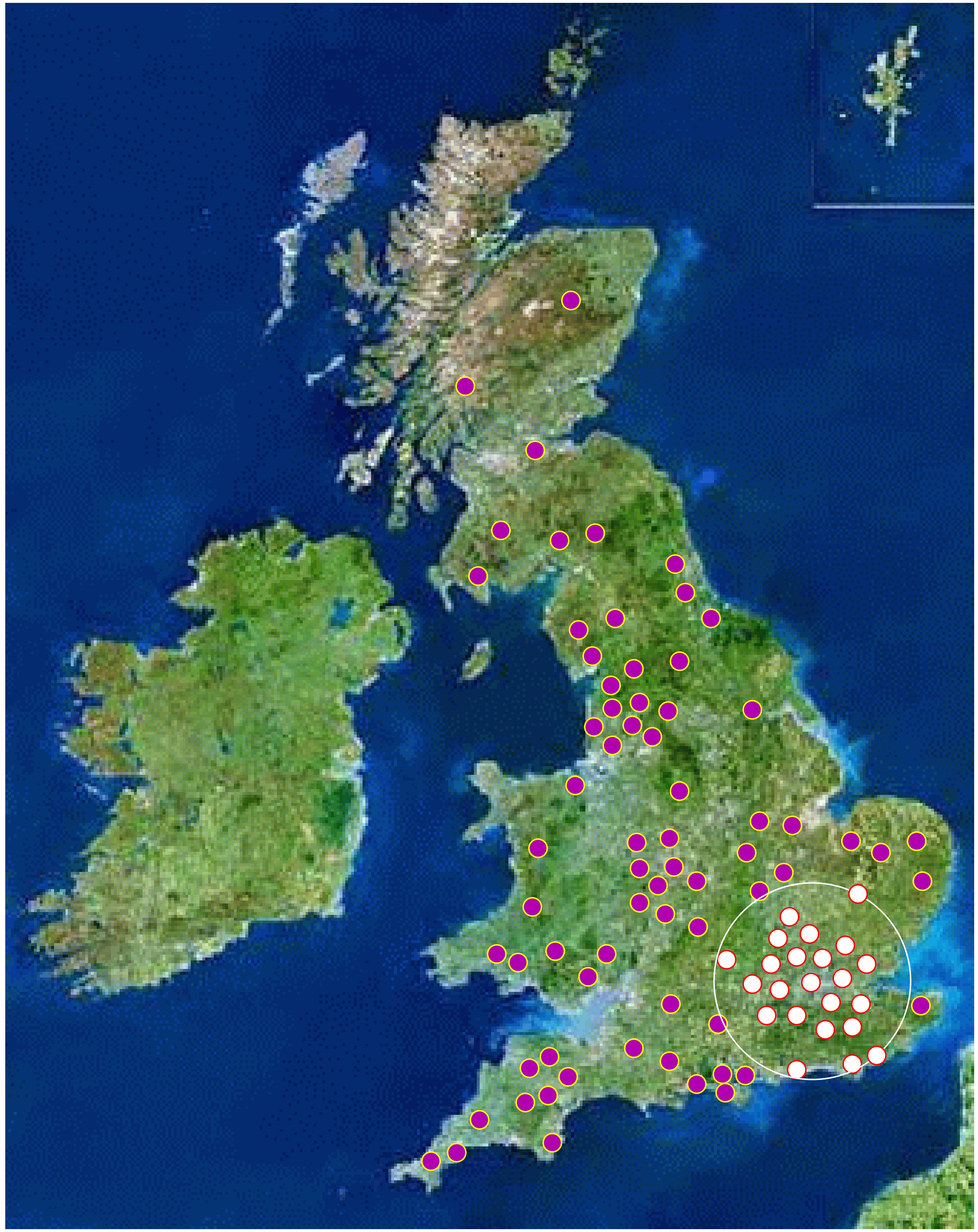
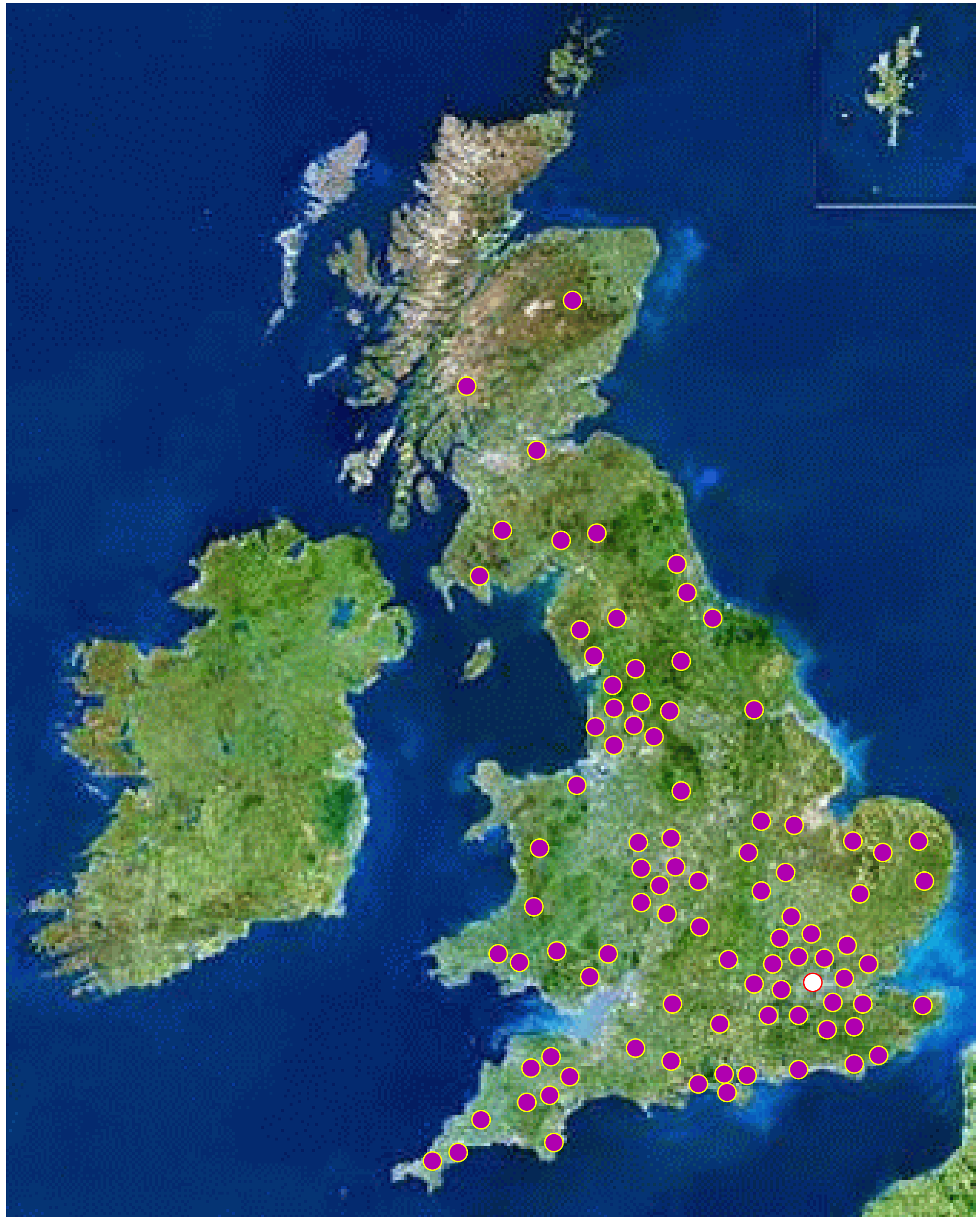


Efficient Flooding



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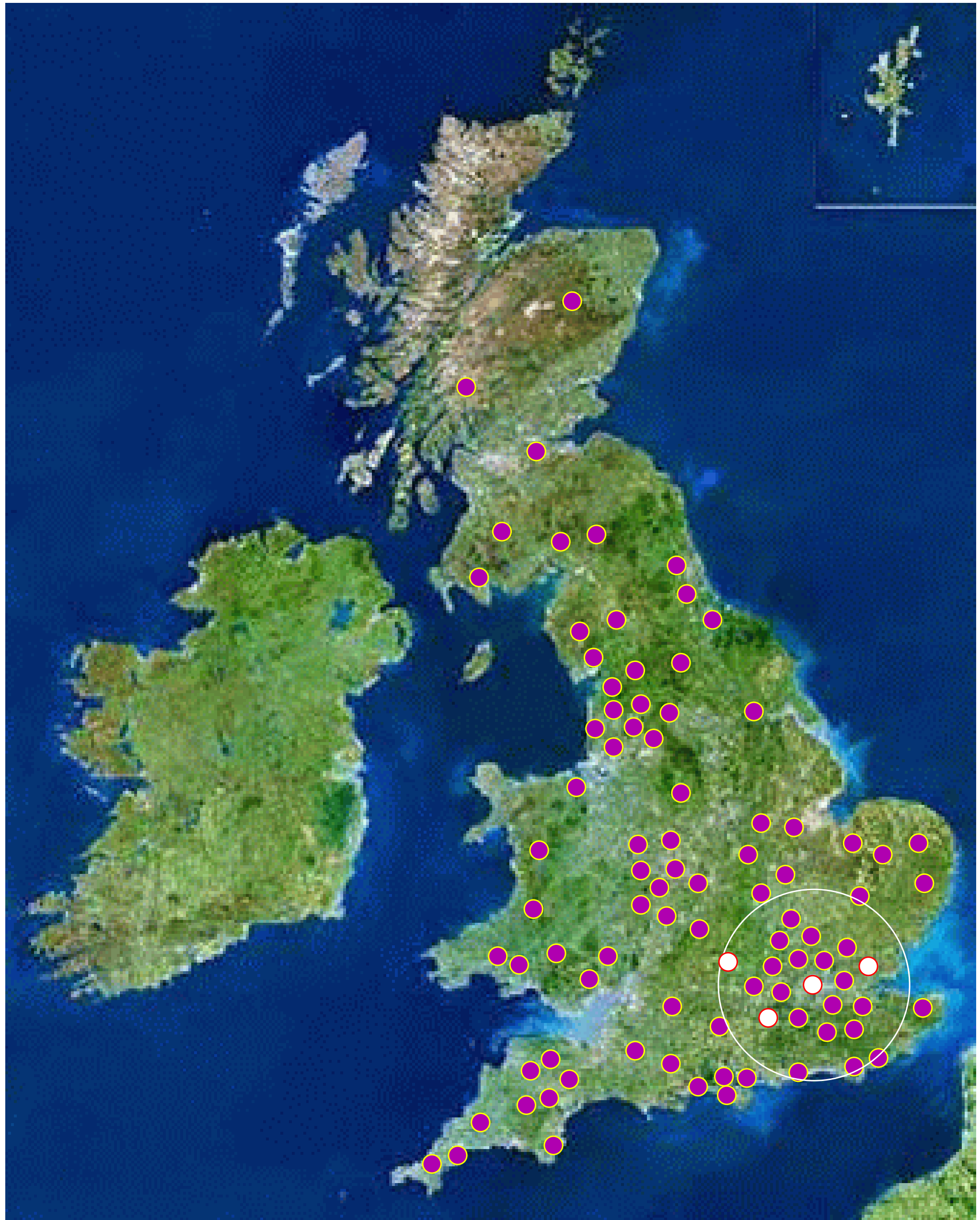

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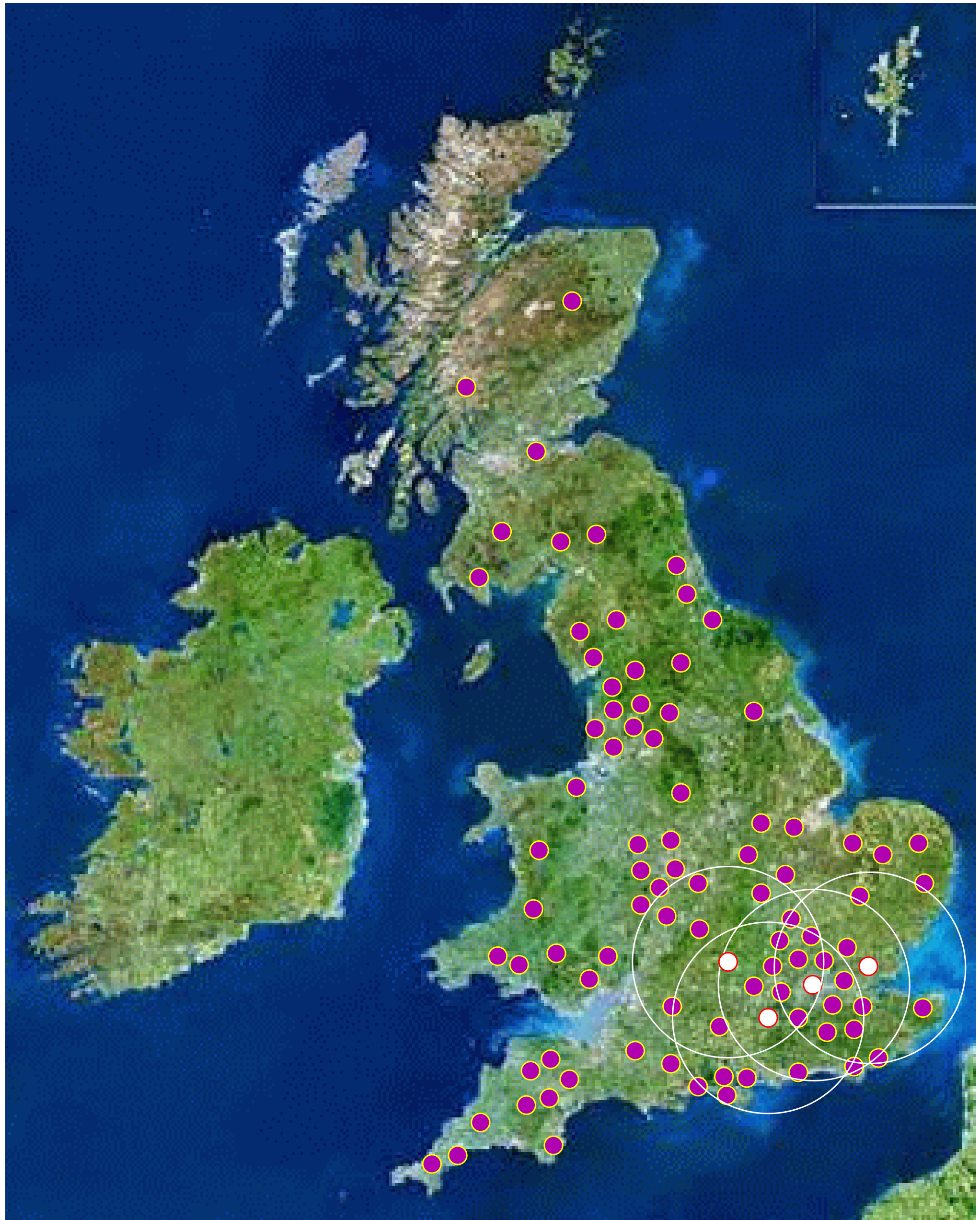

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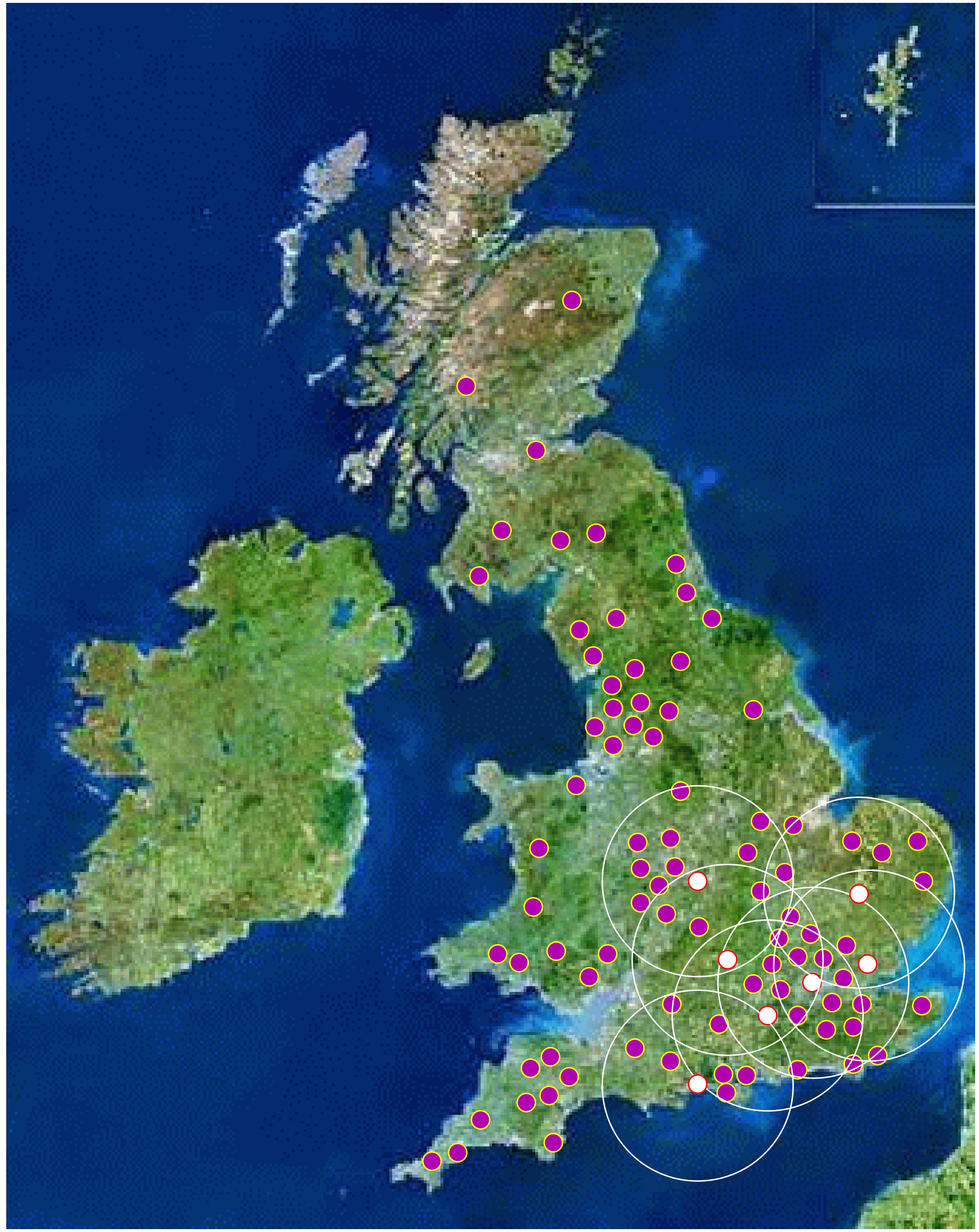

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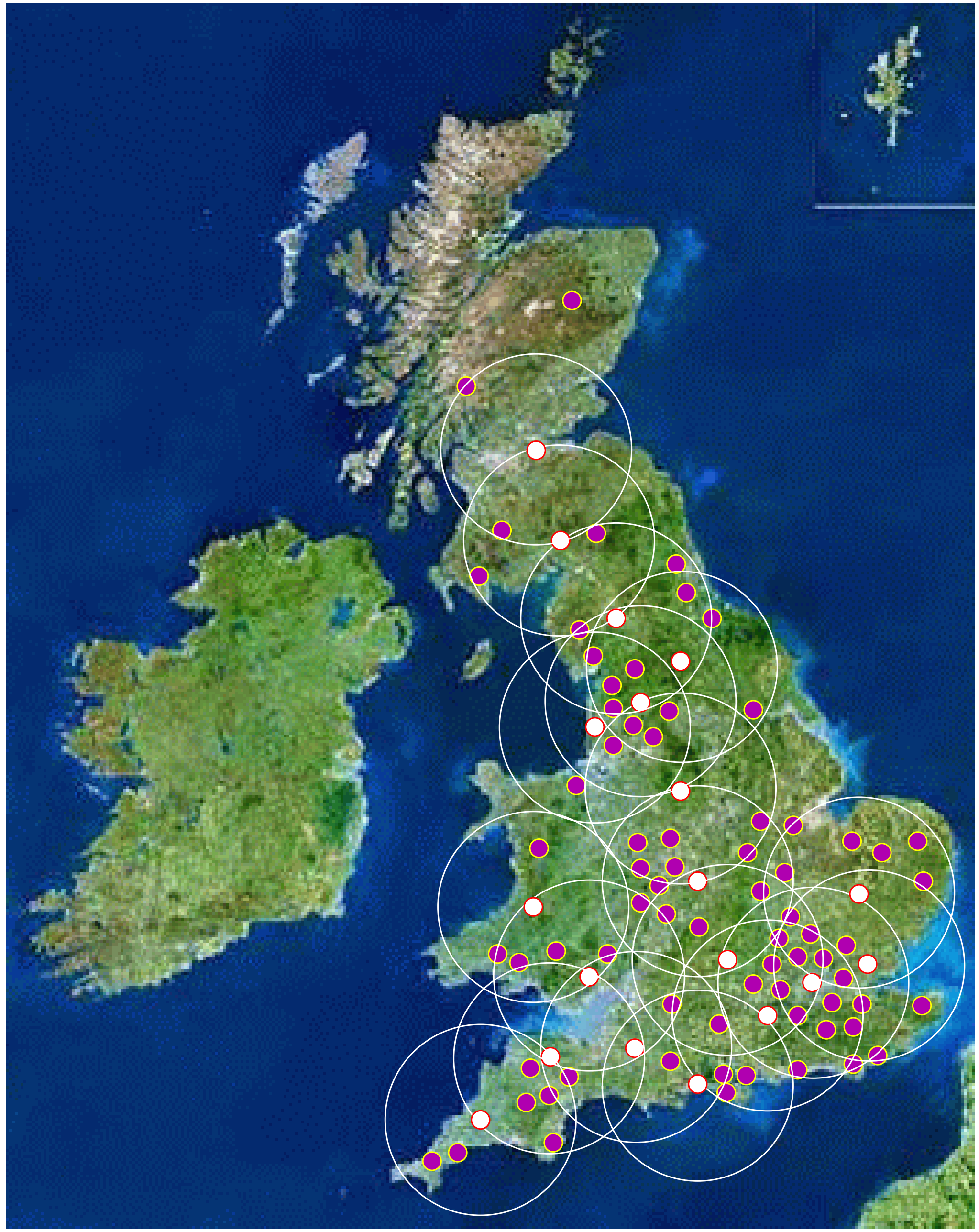
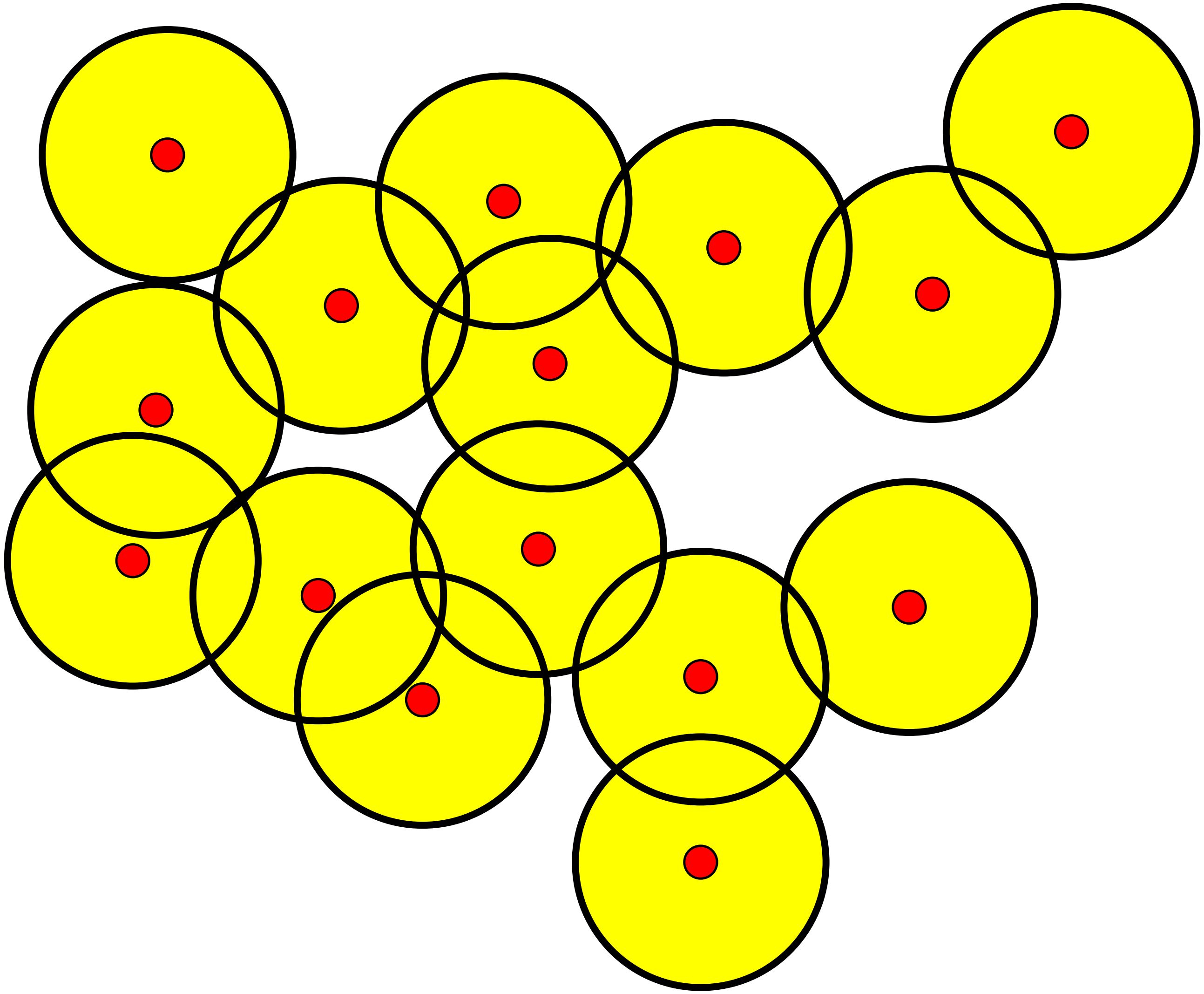


Routing Backbone

- For efficient flooding, we want to find a small subset of the nodes that can reach all other nodes. That subset is then the **routing backbone**. [Guha and Khuller, 1999]
- We can model the network as a graph.
  - Simple model: **Unit Disk Graph**
    Two nodes can reach each other if their distance is at most d, for some fixed value d.

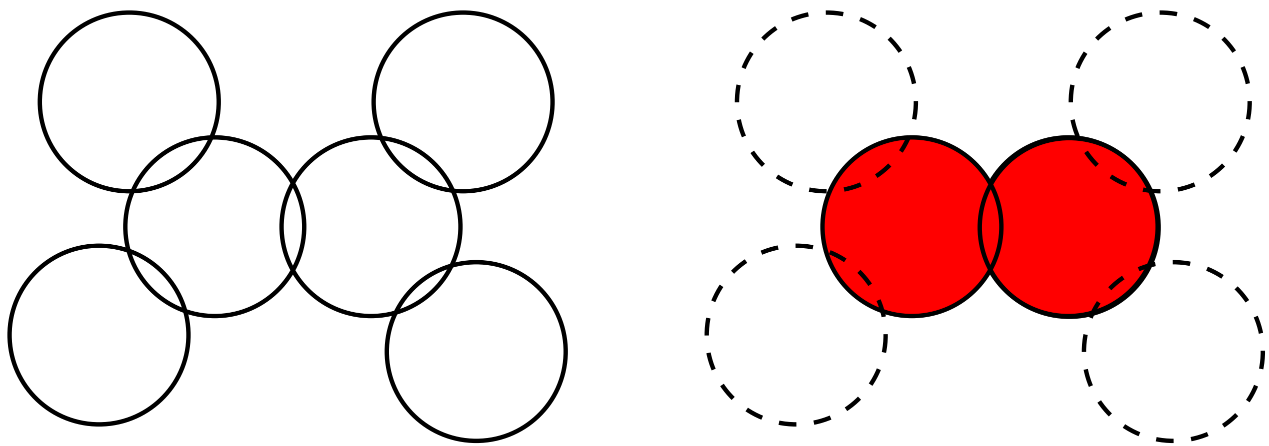
    Each node corresponds to a unit disk, and there is an edge between two nodes if the disks intersect.
- The problem of identifying a small routing backbone then becomes the minimum (connected) dominating set problem in unit disk graphs.

Unit Disk Graph



Minimum Dominating Set (MDS)

Input: a set $\mathcal{D}$ of unit disks in the plane
Feasible solution: subset $A \subseteq \mathcal{D}$ that dominates all disks
Goal: minimize $|A|$



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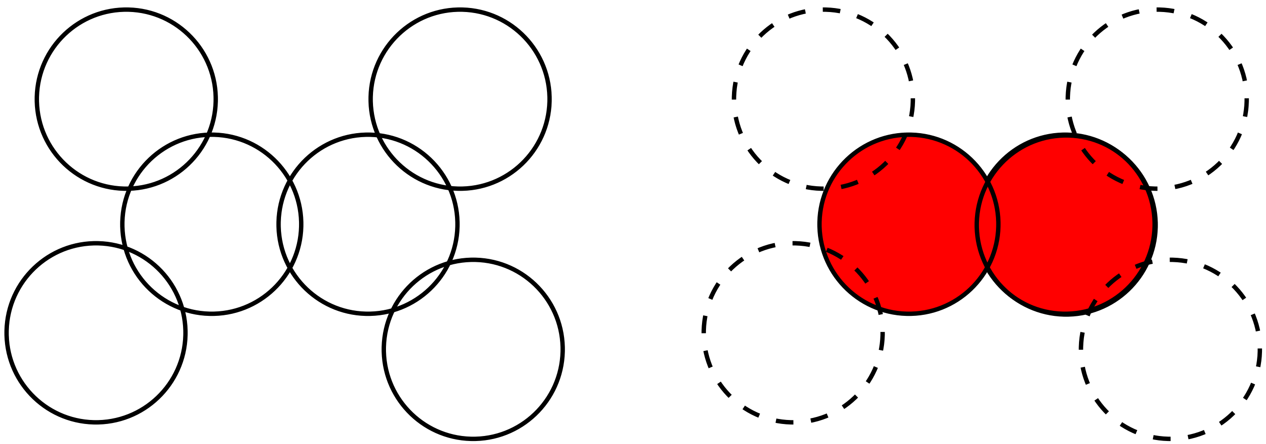

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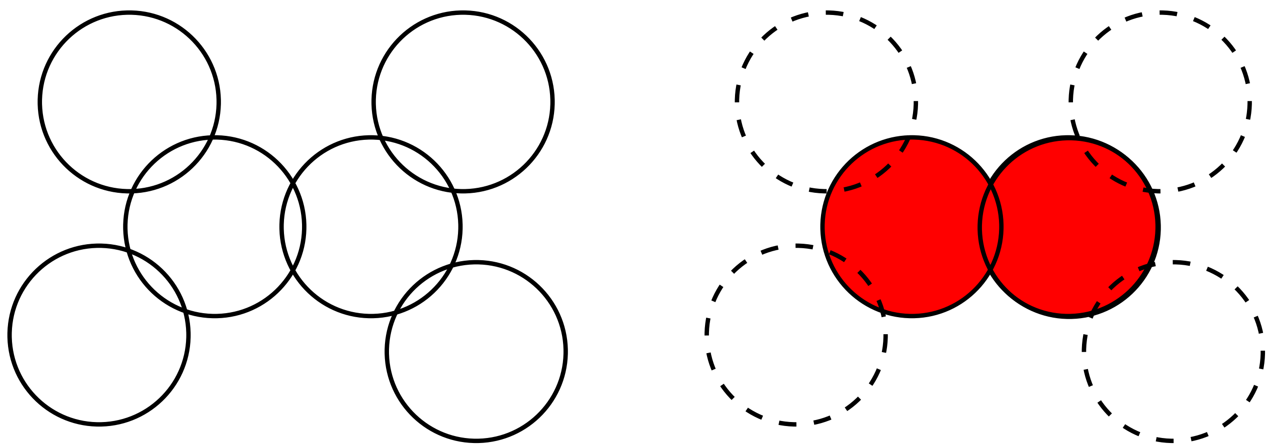


In the weighted case (MWDS), each disk is associated with a positive weight.

For Minimum (Weight) Connected Dominating Set (MCDS/MWCDS), the dominating set must induce a connected subgraph.

Approximation Algorithms

An algorithm for MWDS is a ρ-approximation algorithm if it runs in polynomial time and always outputs a solution of weight at most $\rho \cdot \mathrm{OPT}$, where OPT is the weight of an optimal solution.

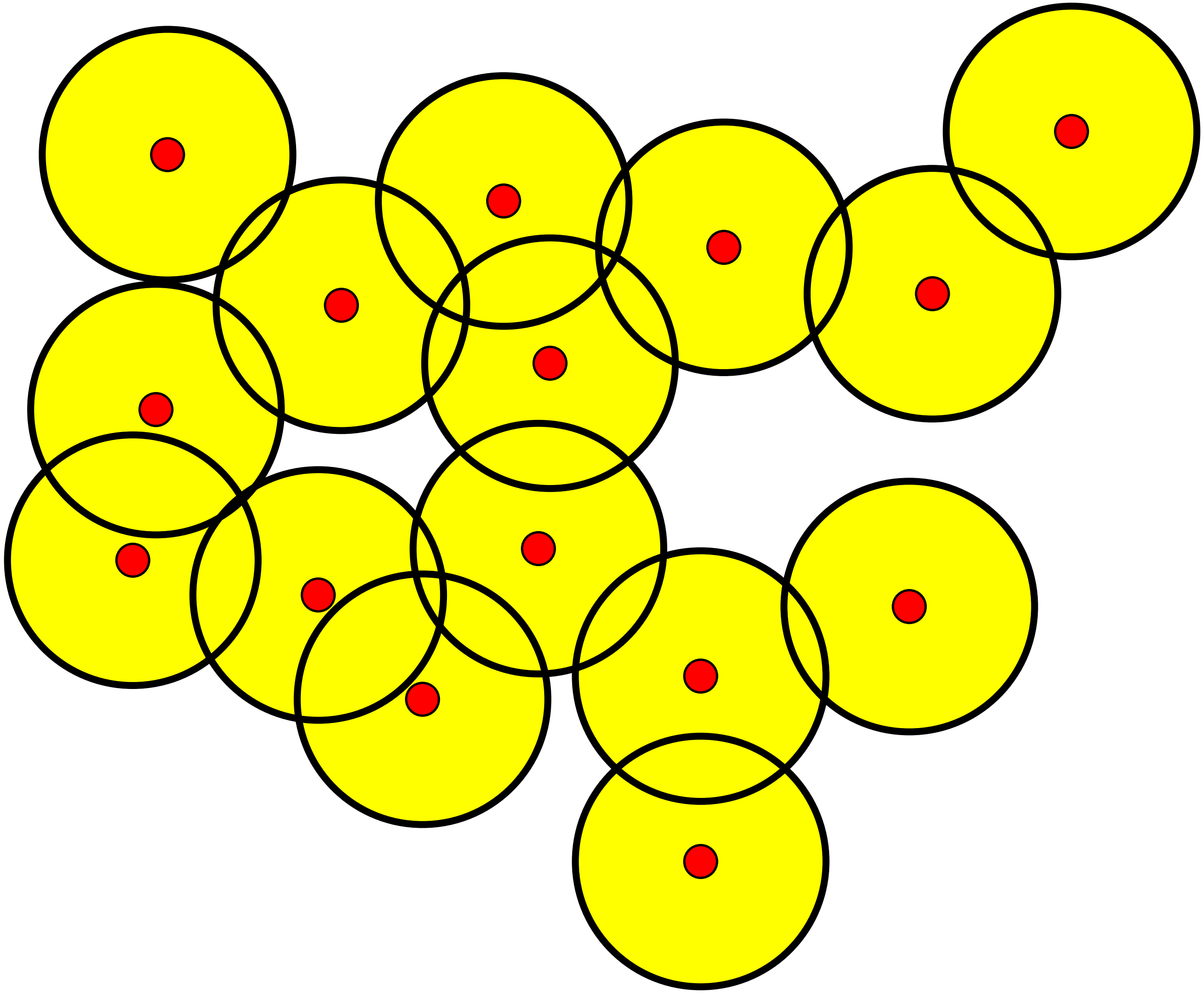
A polynomial-time approximation scheme (PTAS) is a family of algorithms containing a $(1+\varepsilon)$-approximation algorithm for every fixed $\varepsilon > 0$.

Remark: In practice, we are interested in distributed algorithms with fast running-time and good performance in realistic scenarios.

A simple algorithm for MDS

- Initialise $\mathcal{U}$ as the empty set.
- Repeat until no disk left:
  - pick an arbitrary disk D
  - insert D into the set $\mathcal{U}$
  - delete the disk D and all its neighbours from the instance
- Output the set $\mathcal{U}$ as dominating set

Example run



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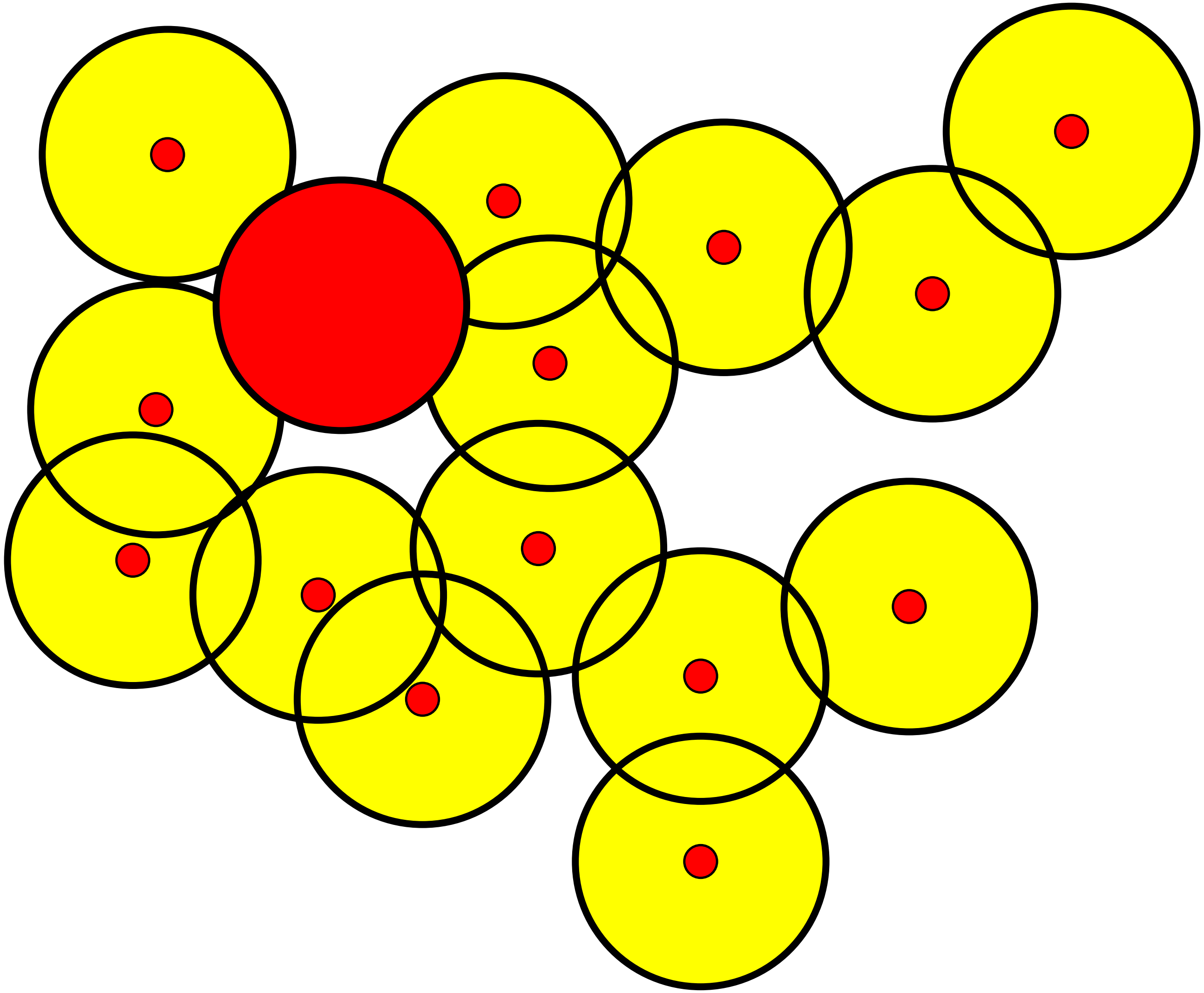

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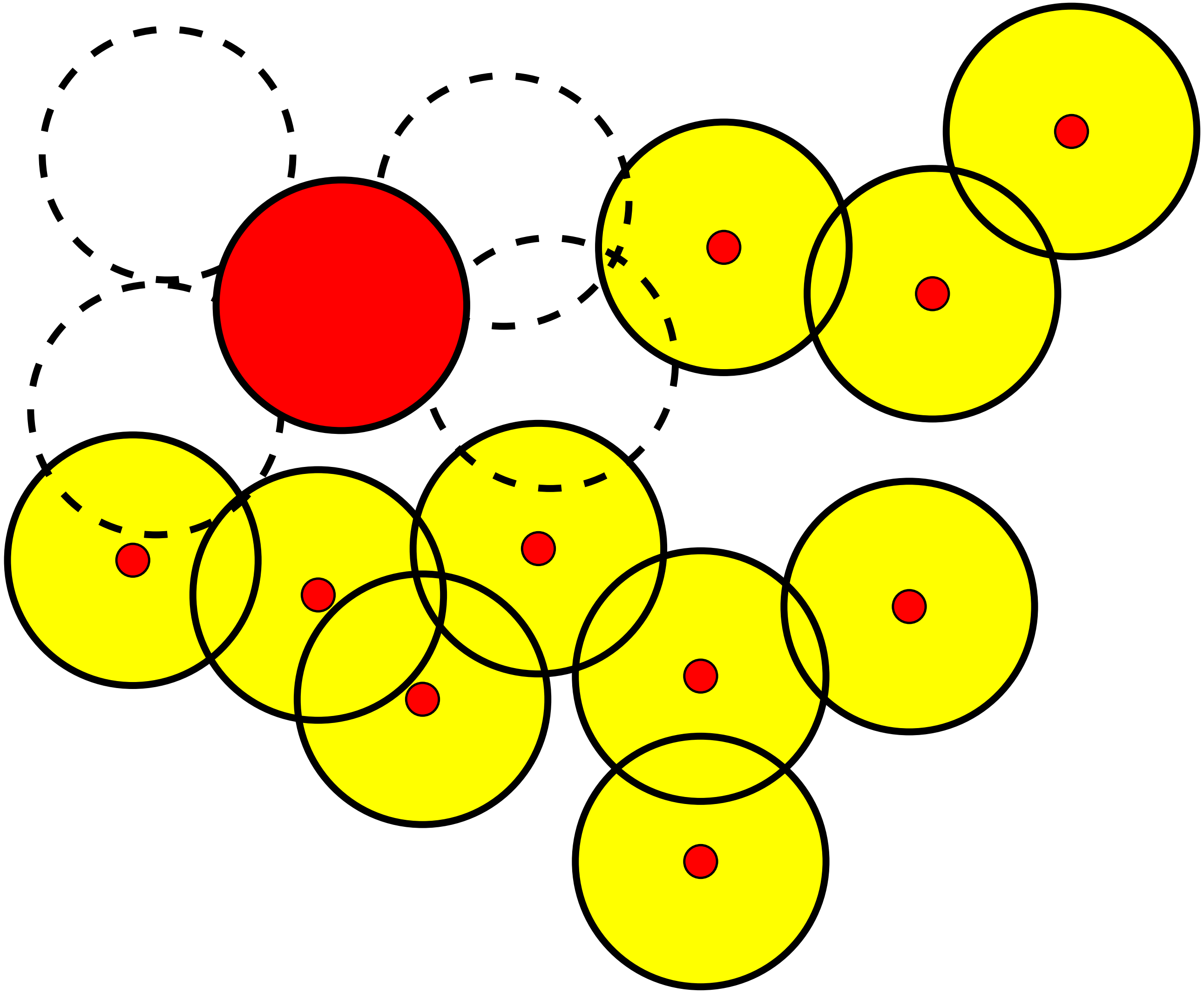

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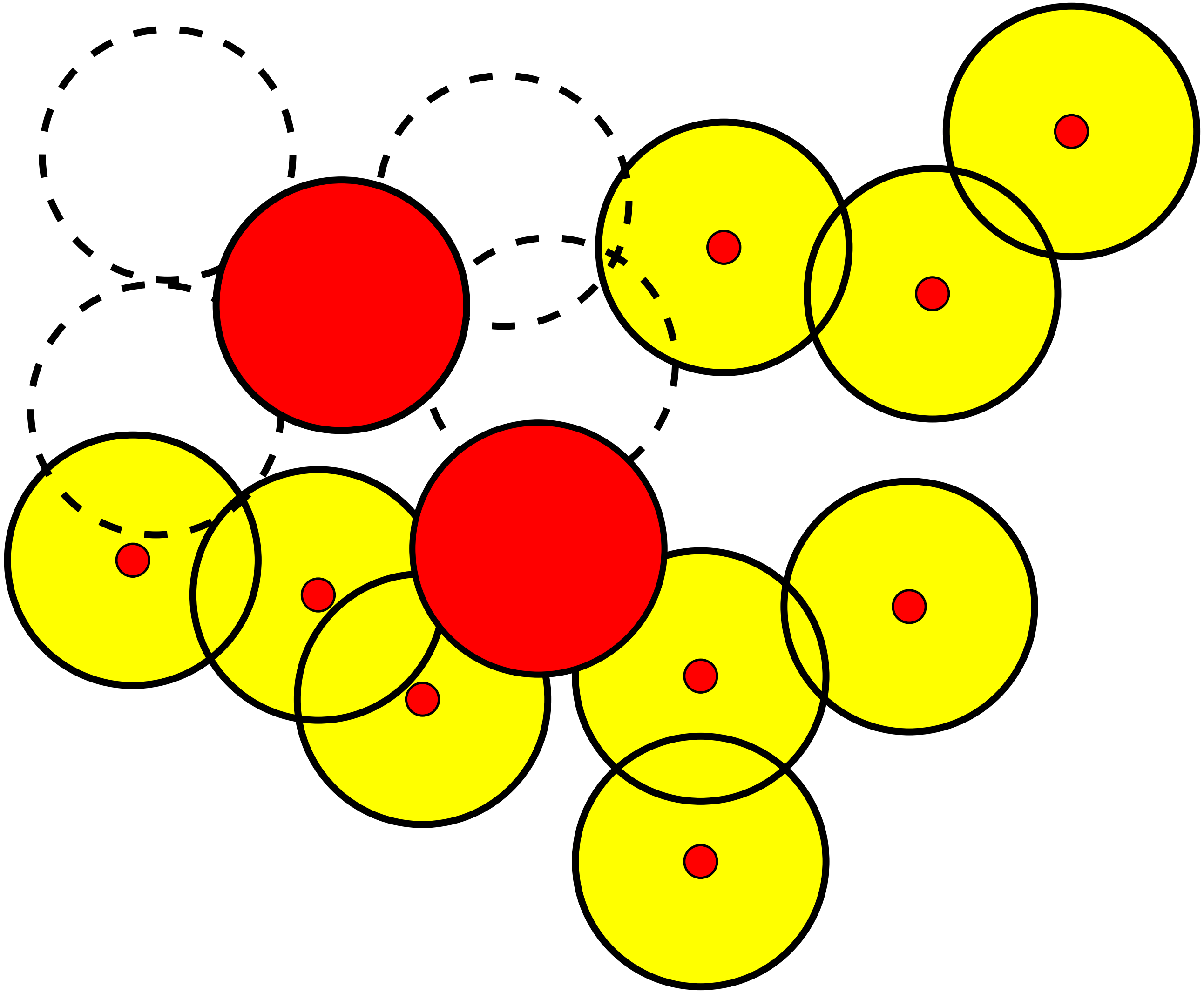

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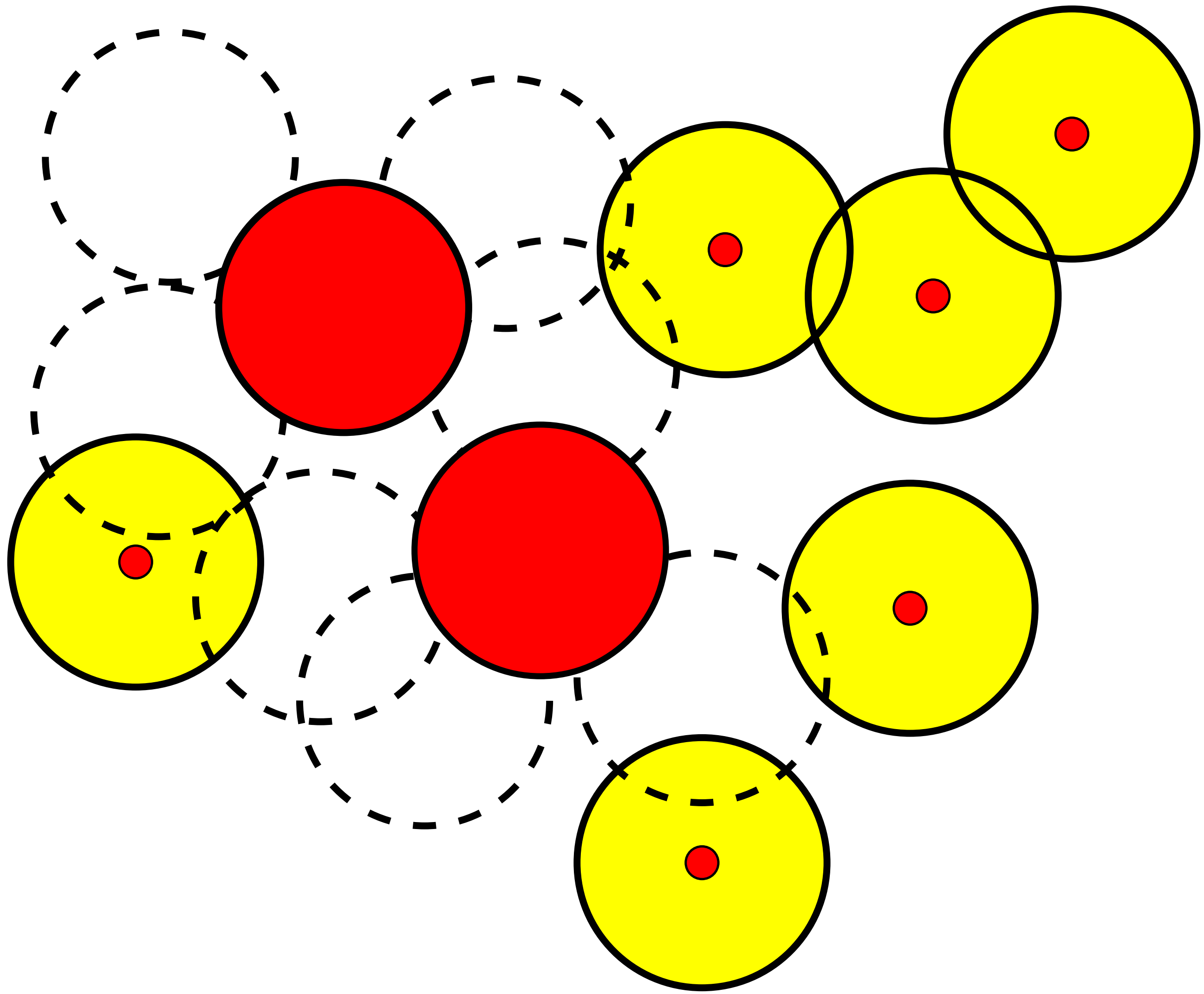

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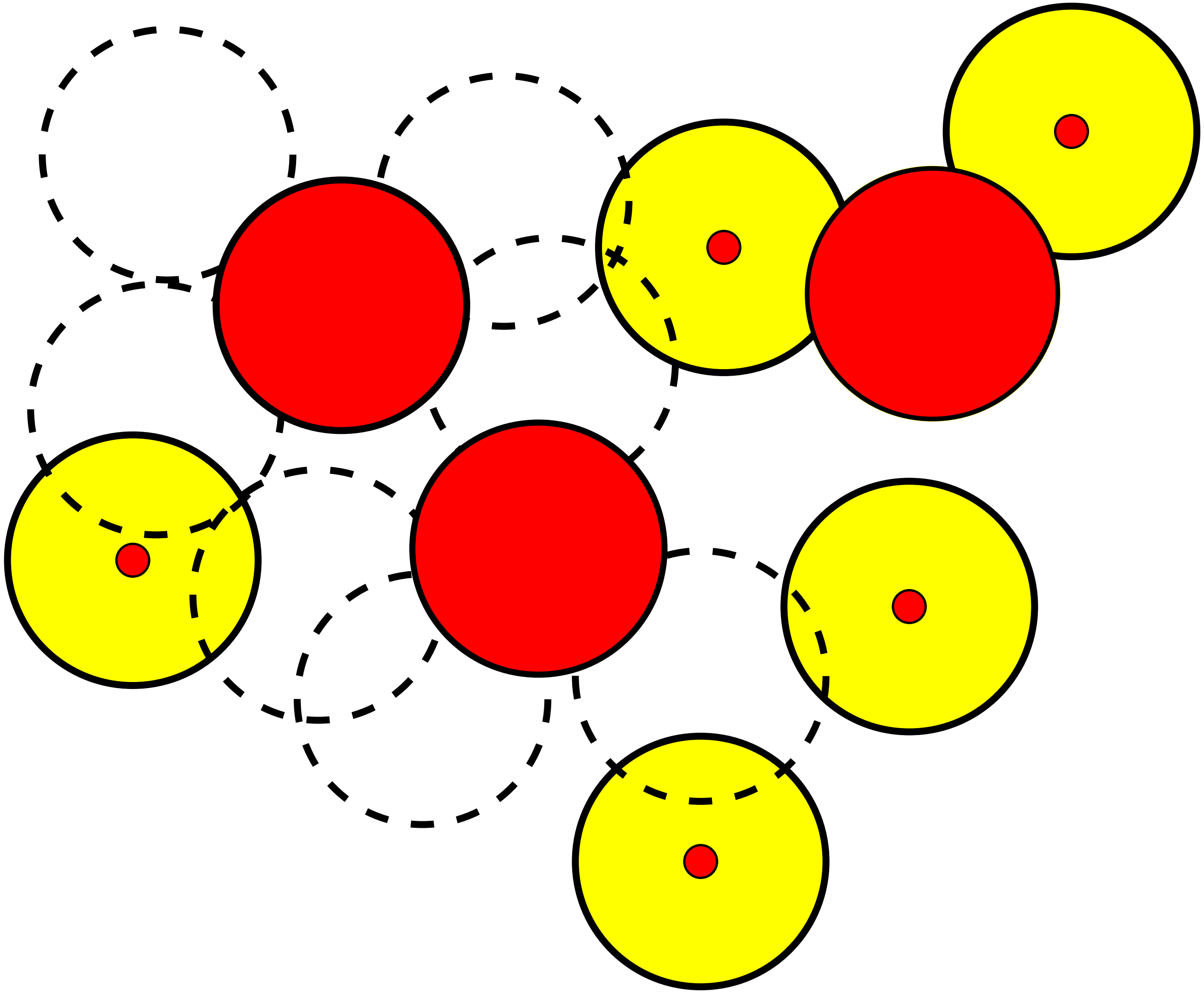

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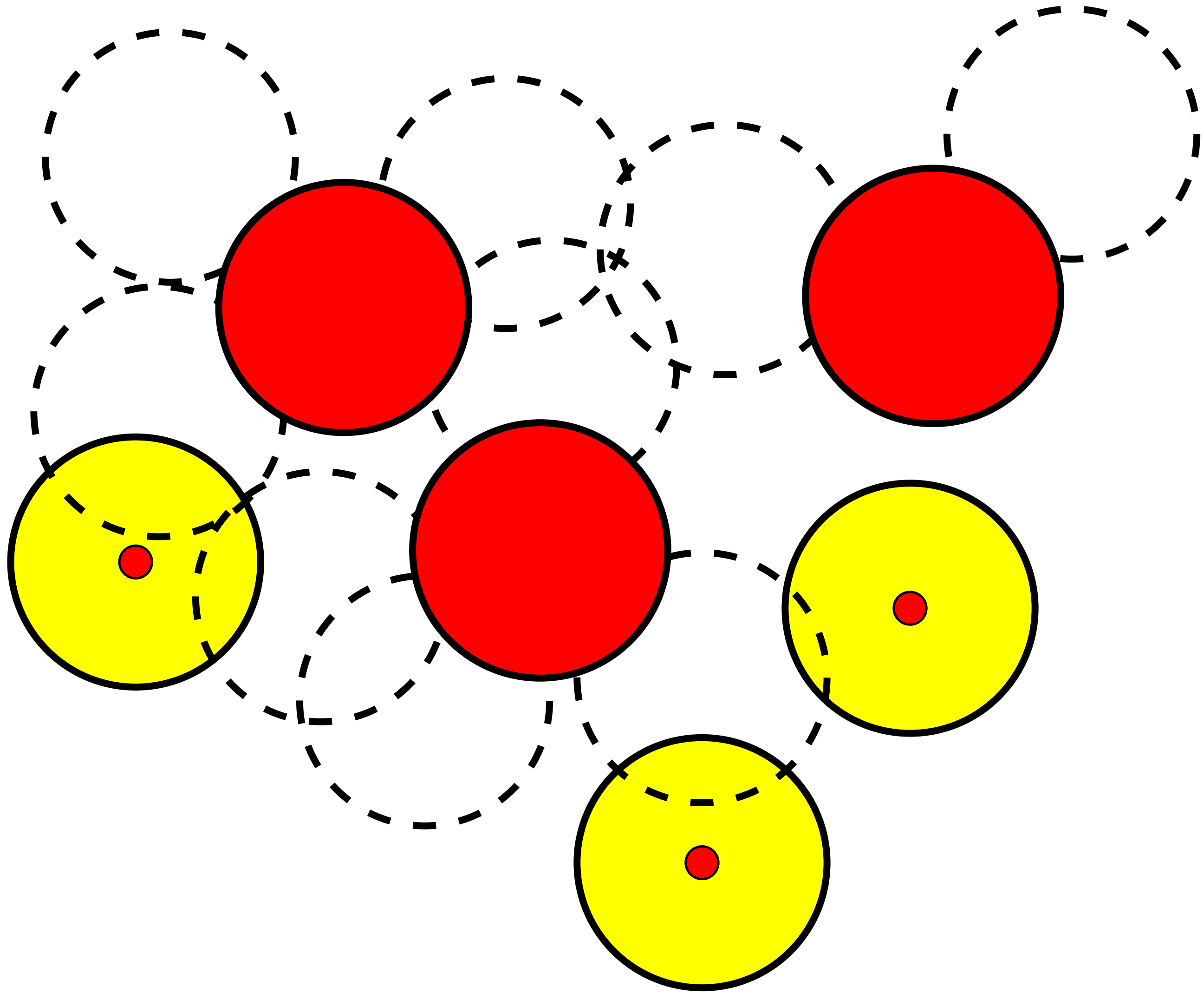

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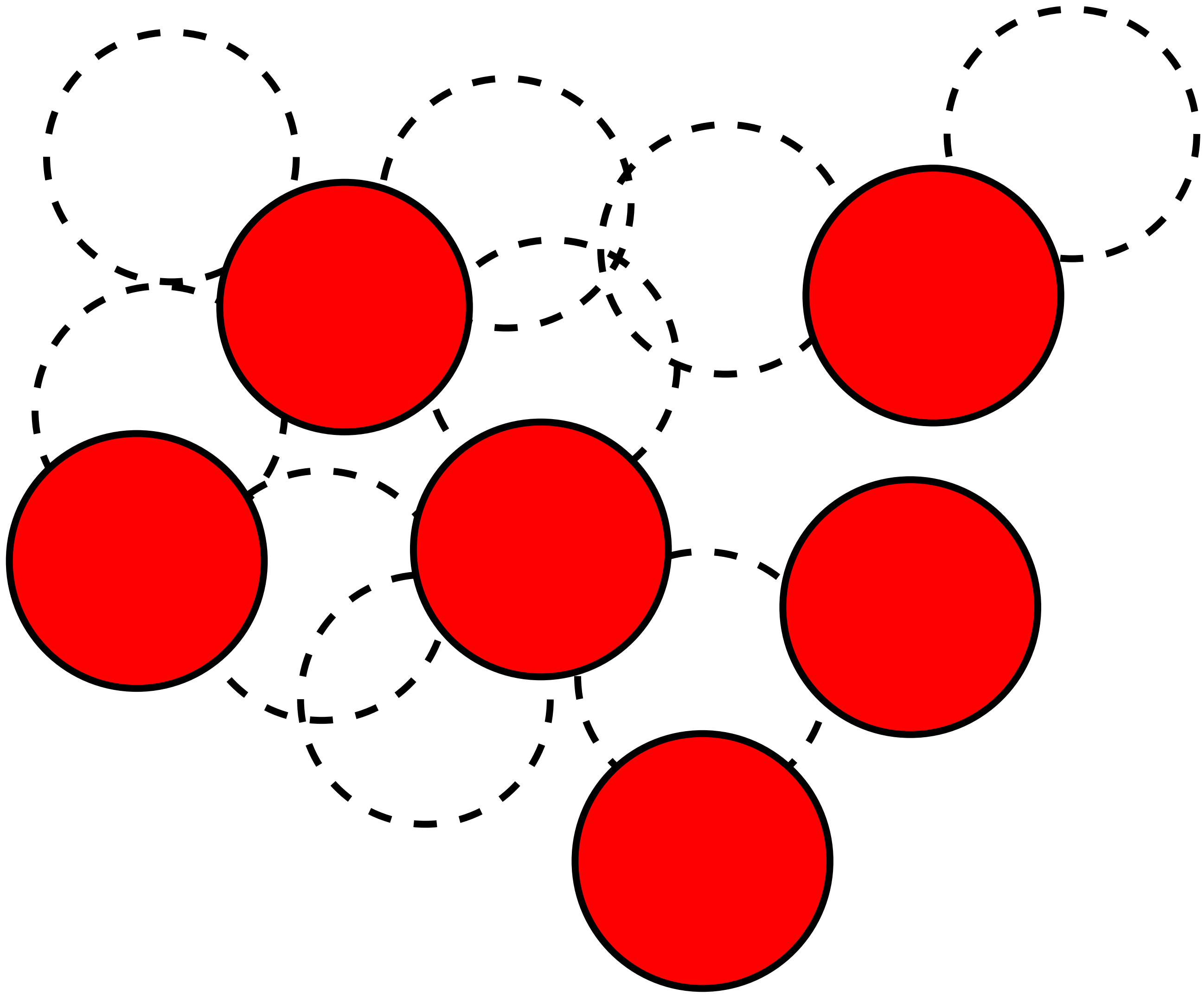

Analysis of the algorithm

- How much worse than the optimal dominating set can the solution produced by this algorithm be?

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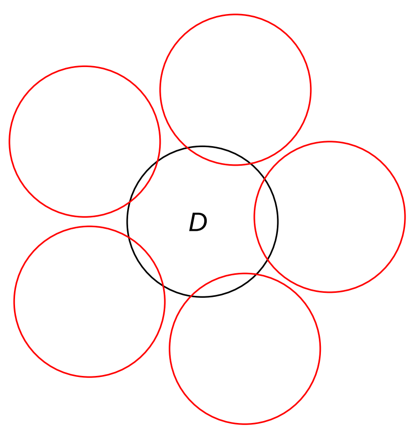
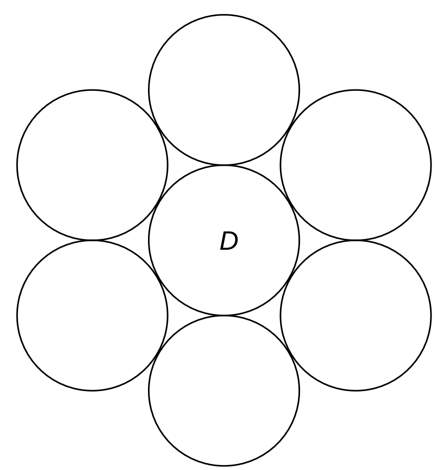
At most 5:





Simple approximation results

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There is a simple 10-approximation algorithm for MCDS in unit disk graphs.

Remark: There are also fast distributed approximation algorithms for dominating set problems in unit disk graphs or general graphs. (Gao et al., 2001, Kuhn & Wattenhofer, 2005)

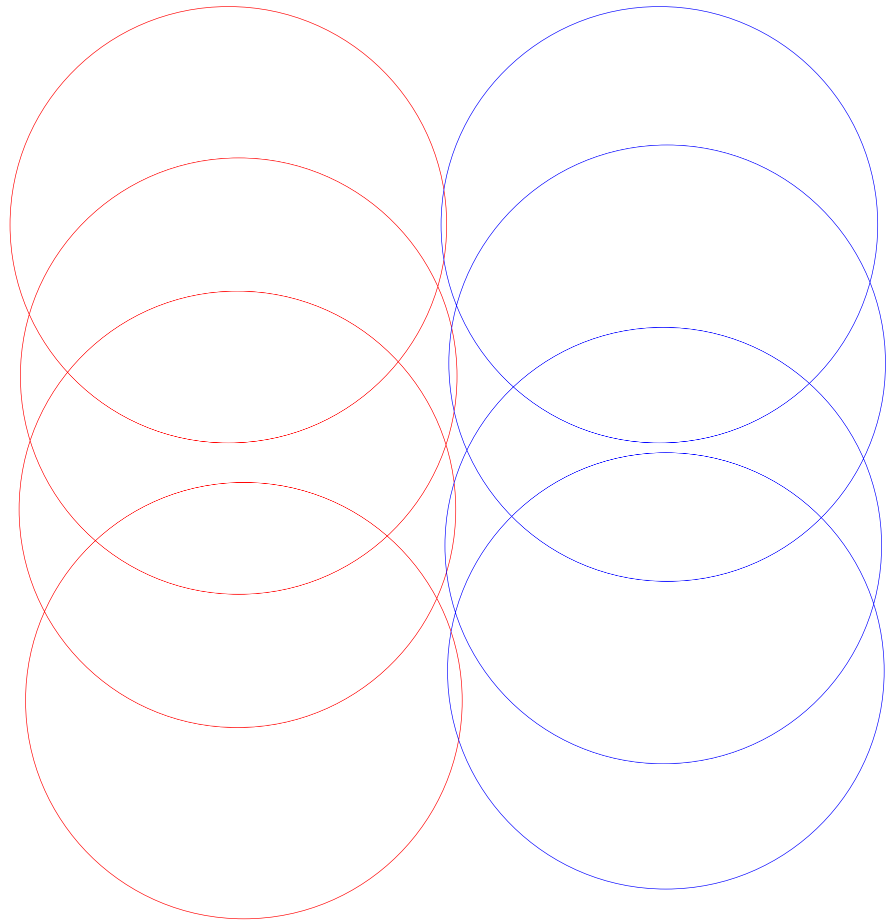
Known dom. set approximations

- In **arbitrary graphs**, ratio $\Theta(\log n)$ is best possible (unless $P = NP$) for MDS, MWDS, MCDS and MWCDS. [Feige '96; Arora and Sudan '97; Guha and Khuller '99]
- For **MDS in unit disk graphs**, a PTAS can be obtained using the shifting strategy [Hunt III et al., 1994]:
  - Any maximal independent set is a dominating set.
  - Therefore, the smallest dominating set in a constant-size square can be found in polynomial time by enumeration.
- PTAS for **MDS in unit disk graphs without representation** [Nieberg and Hurink, 2005]
- PTAS for **MCDS in unit disk graphs** [Cheng et al., 2003]
- **Question:** MWDS and MWCDS in unit disk graphs?

Shifting strategy doesn't seem to work

MWDS can be arbitrarily large for unit disks in an area of constant size:

small weight



large weight

➠ Brute-force enumeration does no longer work.

Constant-Factor Approximation

Theorem (Ambühl, E, Mihal'ák, Nunkesser, 2006) There is a constant-factor approximation algorithm for MWDS in unit disk graphs.

Ideas:

- Partition the plane into unit squares and solve the problem for each square separately.
- In each square, reduce the problem to the problem of covering points with weighted disks.
- Use enumeration techniques (guess properties of OPT) and dynamic programming to solve the latter problem.

The constant factor is 72.

The subproblem for each square

- Find a dominating set for the square:
  - Let $\mathcal{D}_S$ denote the set of disks with center in a 1×1 square S.
  - Let $N(\mathcal{D}_S)$ denote the disks in $\mathcal{D}_S$ and their neighbors.
  - **Task:** Find a minimum weight set of disks in $N(\mathcal{D}_S)$ that dominates all disks in $\mathcal{D}_S$.

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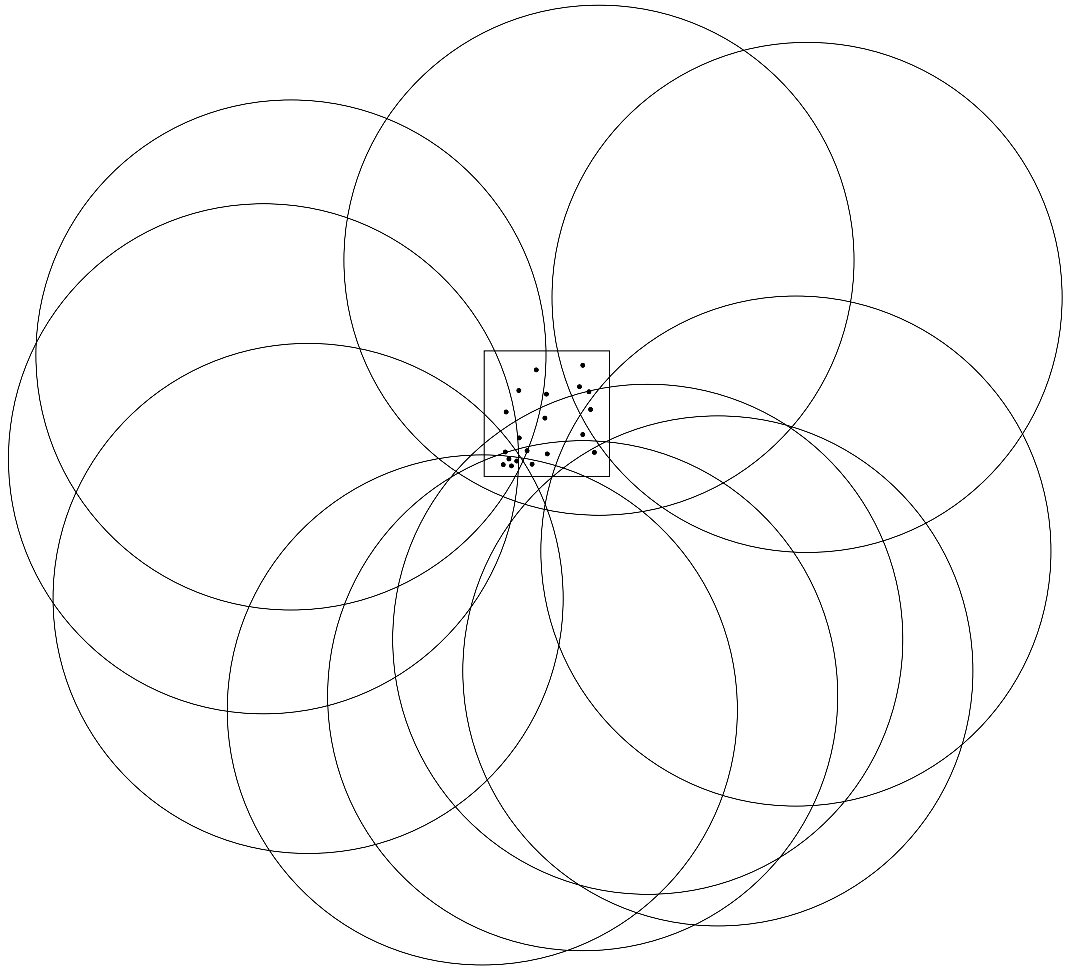
  - **Task:** Find a minimum weight set of disks in $N(\mathcal{D}_S)$ that dominates all disks in $\mathcal{D}_S$.
- Reduces (by guessing the max weight of a disk in OPT$_S$) to covering points in a square with weighted disks:
  - Let P be a set of points in a $\frac{1}{2} \times \frac{1}{2}$ square S.
  - Let $\mathcal{D}$ be a set of weighted unit disks covering P.
  - **Task:** Find a minimum weight set of disks in $\mathcal{D}$ that covers all points in P.

Covering points by weighted disks



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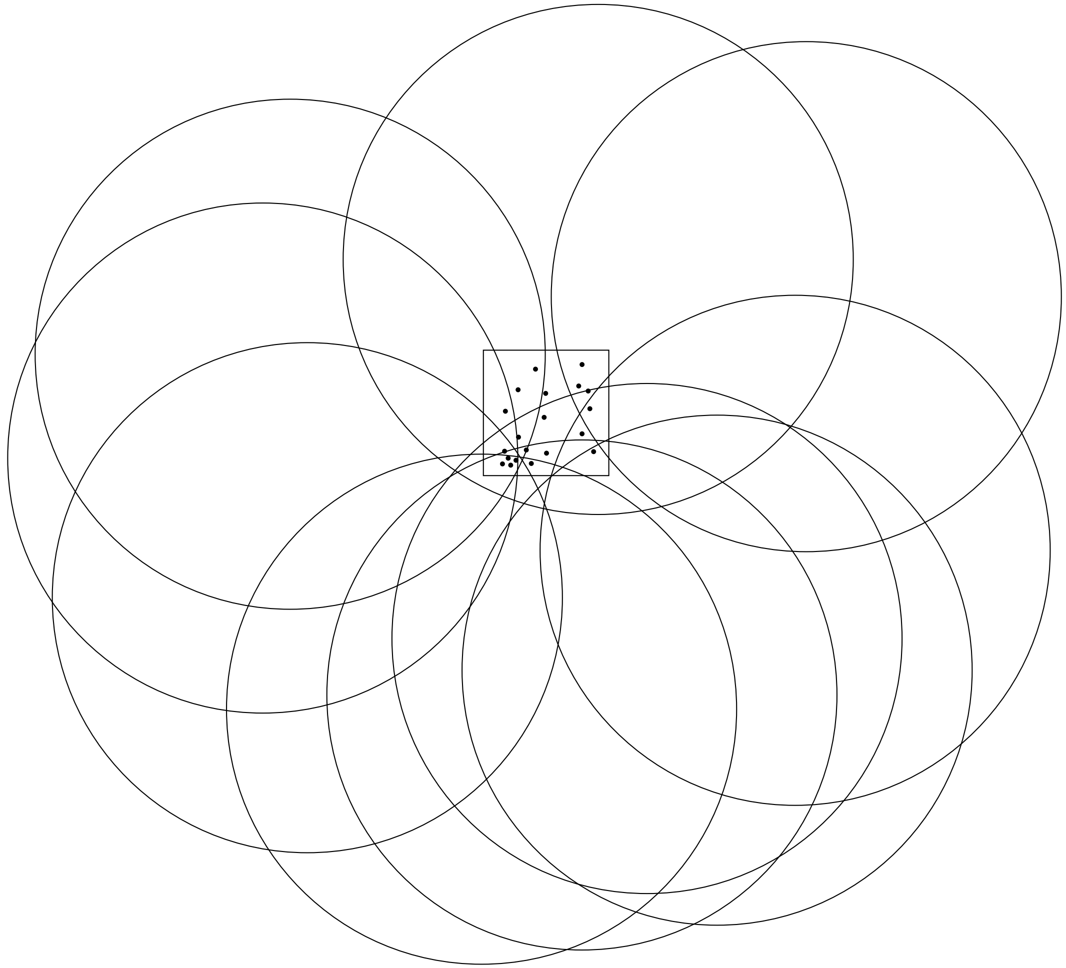
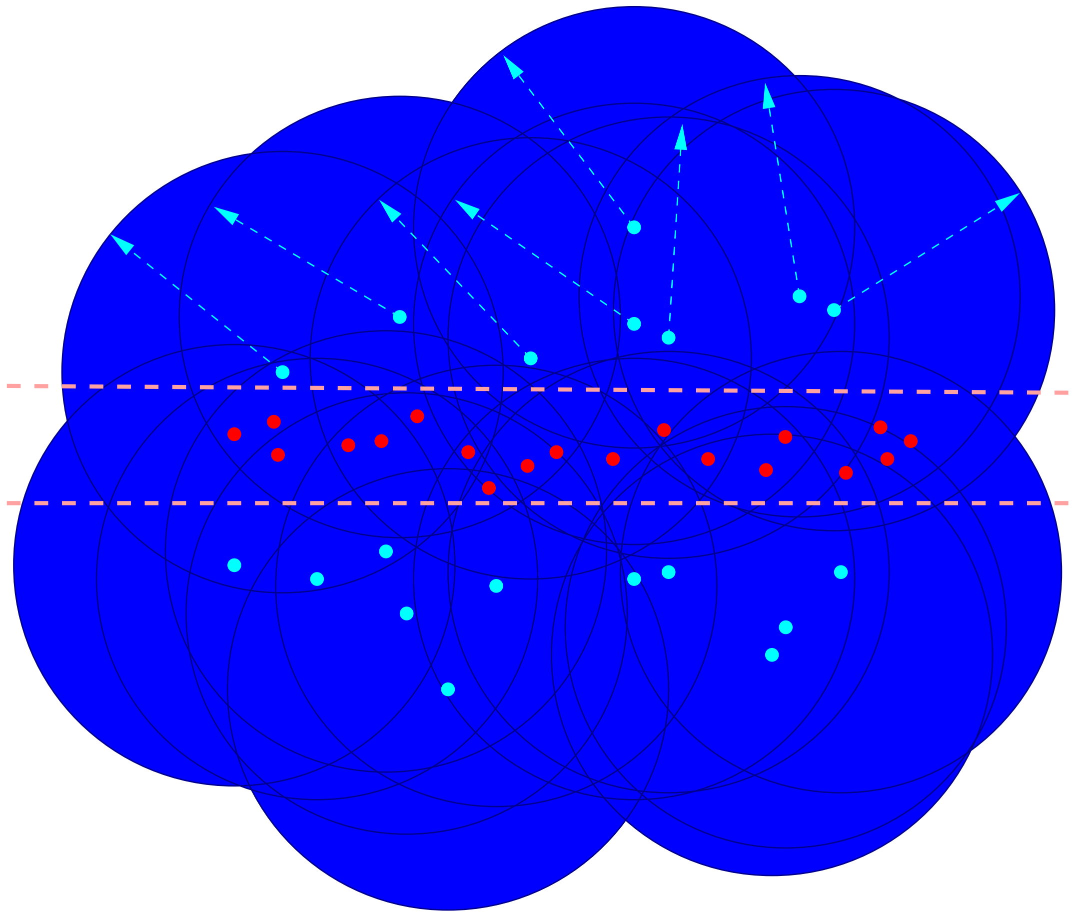


Remark. $O(1)$-approximation algorithms are known for unweighted disk cover [Brönninmann and Goodrich, 1995].

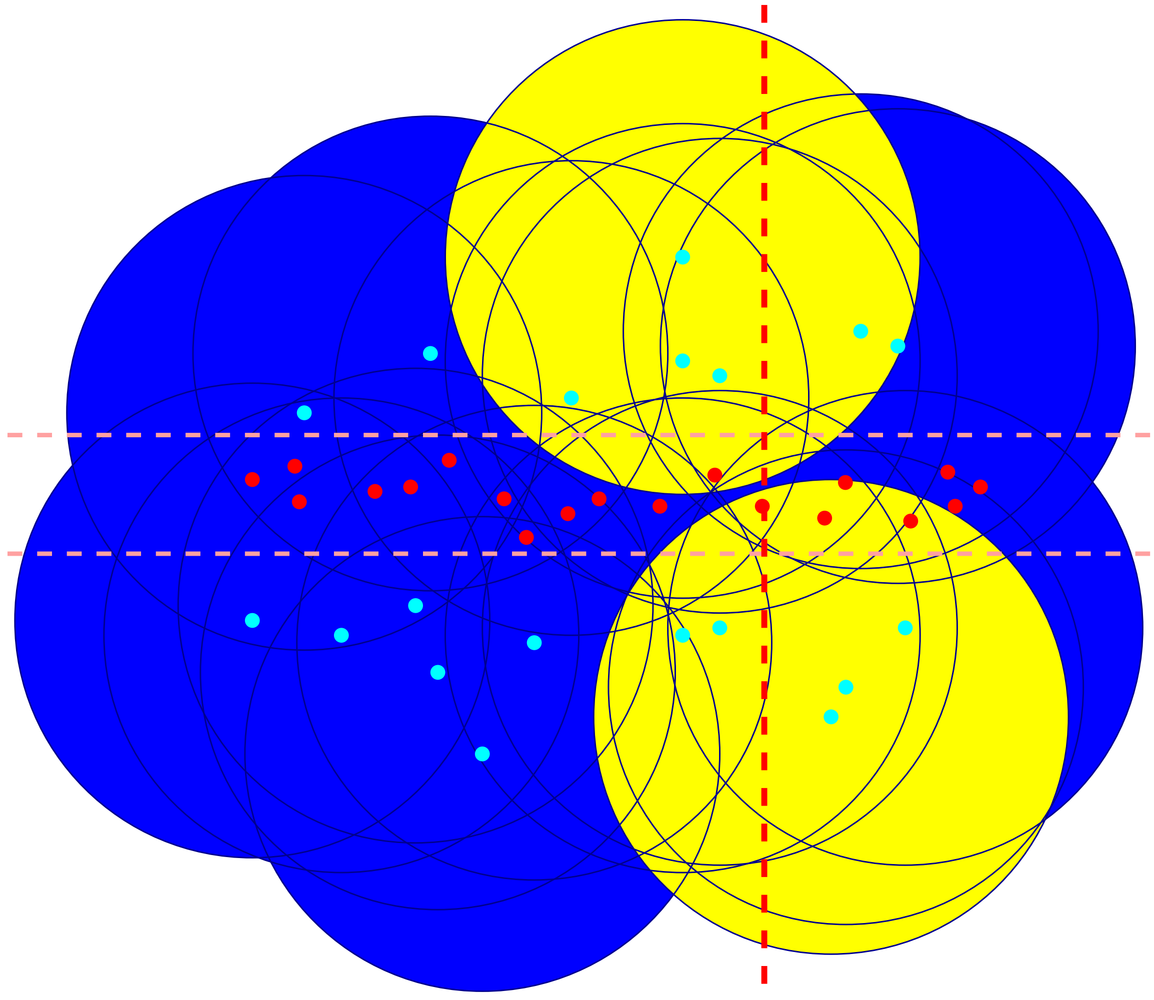
Polynomial-time solvable subproblem

- Given a set of points in a strip, and a set of weighted unit disks with centers outside the strip, compute a minimum weight set of disks covering the points.



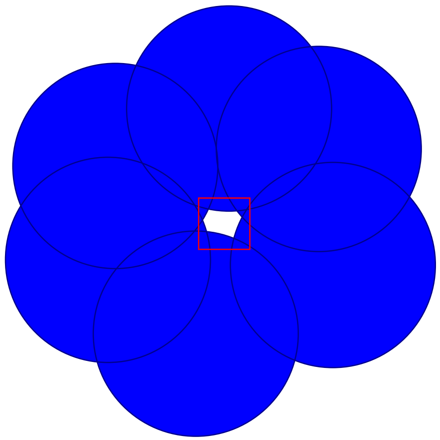
Dynamic programming

- Vertical sweepline, table entry for every pair of disks that could be on the lower and upper envelope:

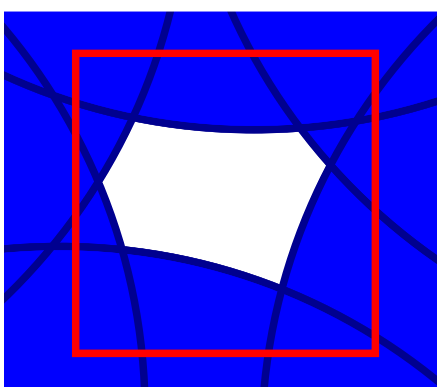


Main cases: One hole or many holes

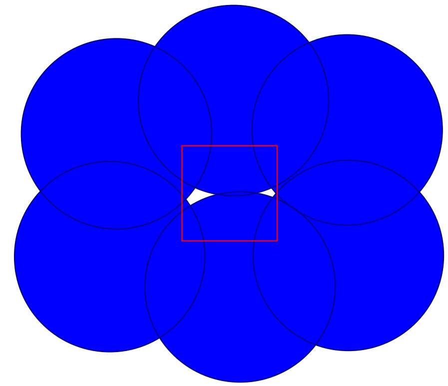
One-hole case:



Enlarged:



Many-holes case:



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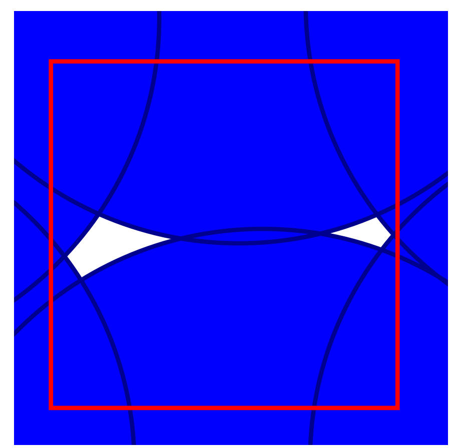
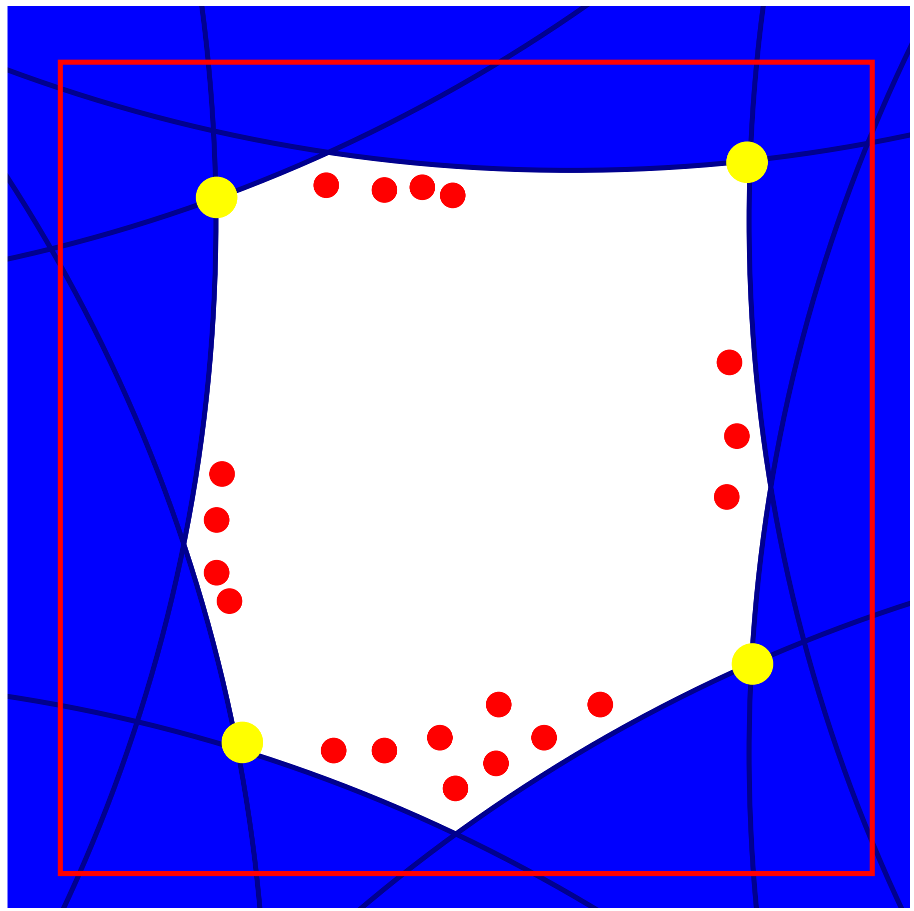


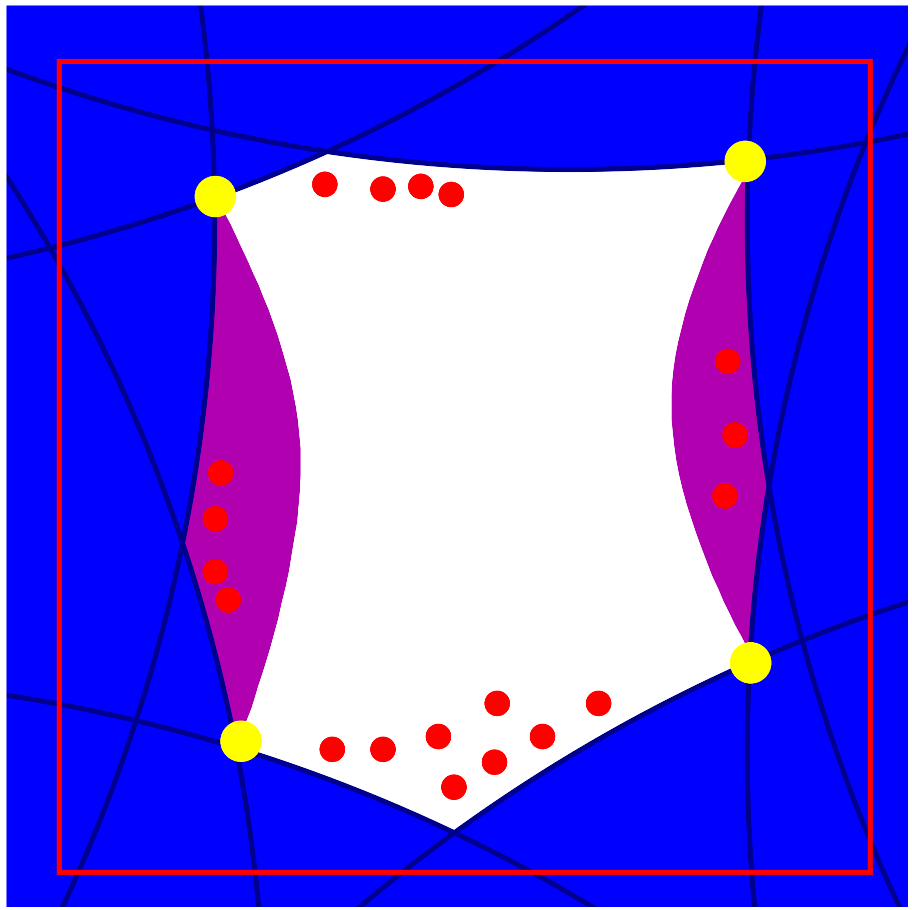
Sketch of the one-hole case

Step 1: Guess the four “corner points” of the optimal solution (each of them is defined by two disks).



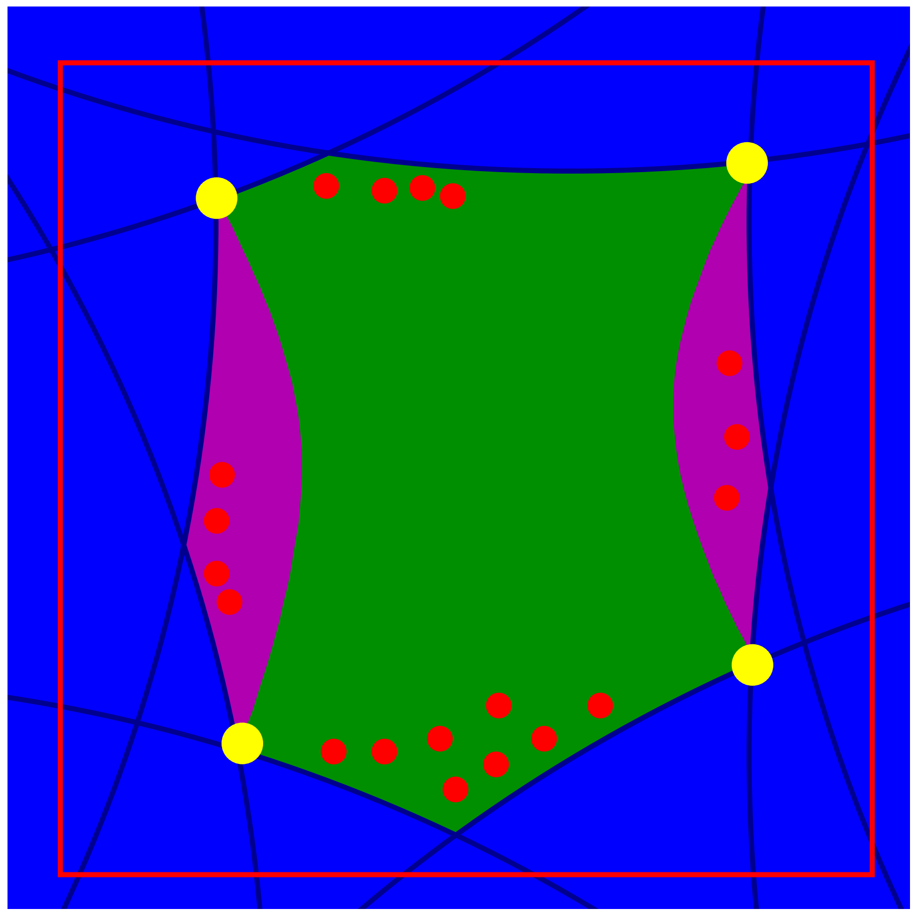
Sketch of the one-hole case

Step 2: Two regions that can only be covered with disks whose centers are to the left or right of the square.



Sketch of the one-hole case

Step 3: Remaining area can only be covered with disks whose centers are above or below the square.



Summary: MWDS in unit disk graphs

- Partition the plane into unit squares and solve the problem for each square separately. (We lose a constant factor compared to OPT.)
- For each square, reduce the weighted dominating set problem to a weighted disk cover problem.
- Distinguish one-hole case and many-holes case.
- In each case, we have a 2-approximation or optimal algorithm for covering points in the square with weighted unit disks.
- This implies the constant-factor approximation algorithm for MWDS in unit disk graphs.

Weighted Connected Dominating Sets

Theorem. There is a constant-factor approximation algorithm for MWCDS in unit disk graphs.

Algorithm Sketch:

- First, compute an $O(1)$-approximate MWDS D.
- Build auxiliary graph H with a vertex for each component of D, and weighted edges corresponding to paths with at most two internal vertices.
- Compute a minimum spanning tree of H and add the disks corresponding to its edges to D.

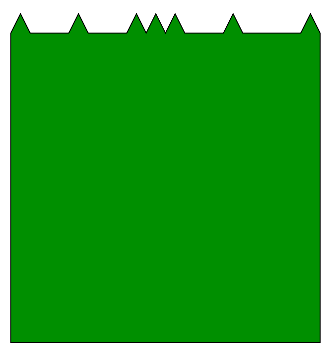
We can show: The total weight of the disks added to D is at most $17 \cdot \mathrm{OPT}$, where OPT is the weight of a minimum weight connected dominating set. The overall approximation ratio is then $72 + 17 = 89$.

Further results on MDS and MWDS

Theorem. [E, van Leeuwen 2006] For disk graphs with bounded ply, there is a $(3 + \varepsilon)$-approximation algorithm for MWDS.

Theorem. [E, van Leeuwen 2006] For rectangle intersection graphs, MDS is APX-hard.

Theorem. [E, van Leeuwen 2006] For intersection graphs of “squares with bumps” (or even for similar, convex objects), MDS cannot be approximated with ratio $o(\log n)$ unless $P = NP$.



Open Problems

Disk graphs

- Improve running-time and/or approximation ratio for MWDS in unit disk graphs.

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  - Is there an $O(1)$-approximation algorithm or even a PTAS?
  - Is the problem APX-hard?
- What is the complexity of the maximum clique problem in disk graphs?
  (polynomial for unit disk graphs [Clark et al., 1990], NP-hard for ellipses [Ambühl, Wagner 2002])

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- Can we achieve approximation ratio $o(\log n)$ for MDS and MWDS?

Rectangle intersection graphs

- What is the best possible approximation ratio for maximum independent set?
  - Known: For every $c > 0$, there is an approximation algorithm with ratio $1 + \frac{1}{c} \log n$. [Berman et al., 2001]
  - Known: If all rectangles have the same height, there is a PTAS. [Agarwal et al., 1998]
- Can we achieve approximation ratio $o(\log n)$ for MDS and MWDS?
- Can rectangle intersection graphs be **colored** with $O(\omega)$ colors, where ω is the clique number?
  (best known upper bound: $O(\omega^2)$ colors [Asplund and Grünbaum, 1960])

Thank you!