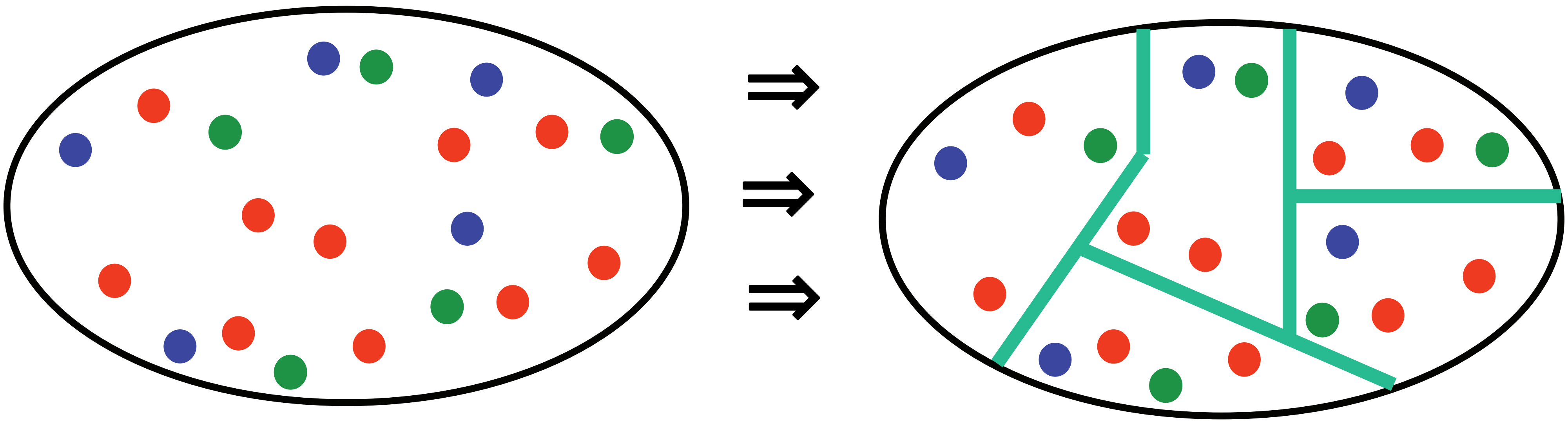
3つの資源節点集合を持つ4点連結グラフを均等分割する問題について

石井 利昌 (豊橋技術科学大学)
岩田 健吾 (マツダ)
永持 仁　(京都大学)

Background

Districting Problem
- political constituencies
- school board boundaries
- sales or delivery regions

Criteria
- equity
- contiguousness



k-bipartition Problem

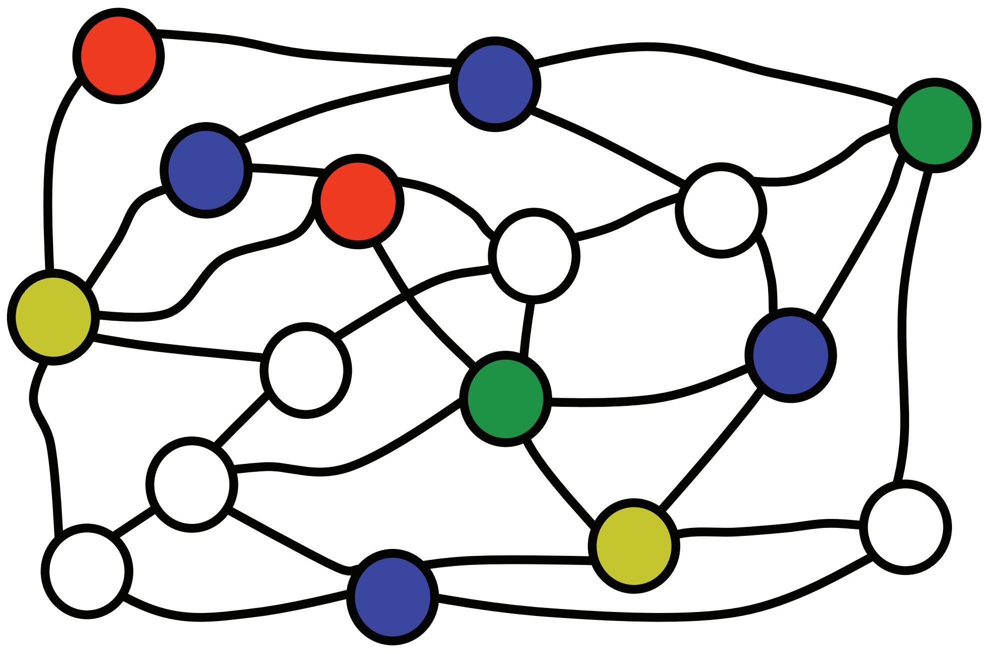
Input:

i) graph $G=(V, E)$.

ii) disjoint subsets $T_1, T_2, T_3, T_4, .., T_k \subseteq V$

(Resource sets)

($|T_i|$: even)



k-bipartition Problem

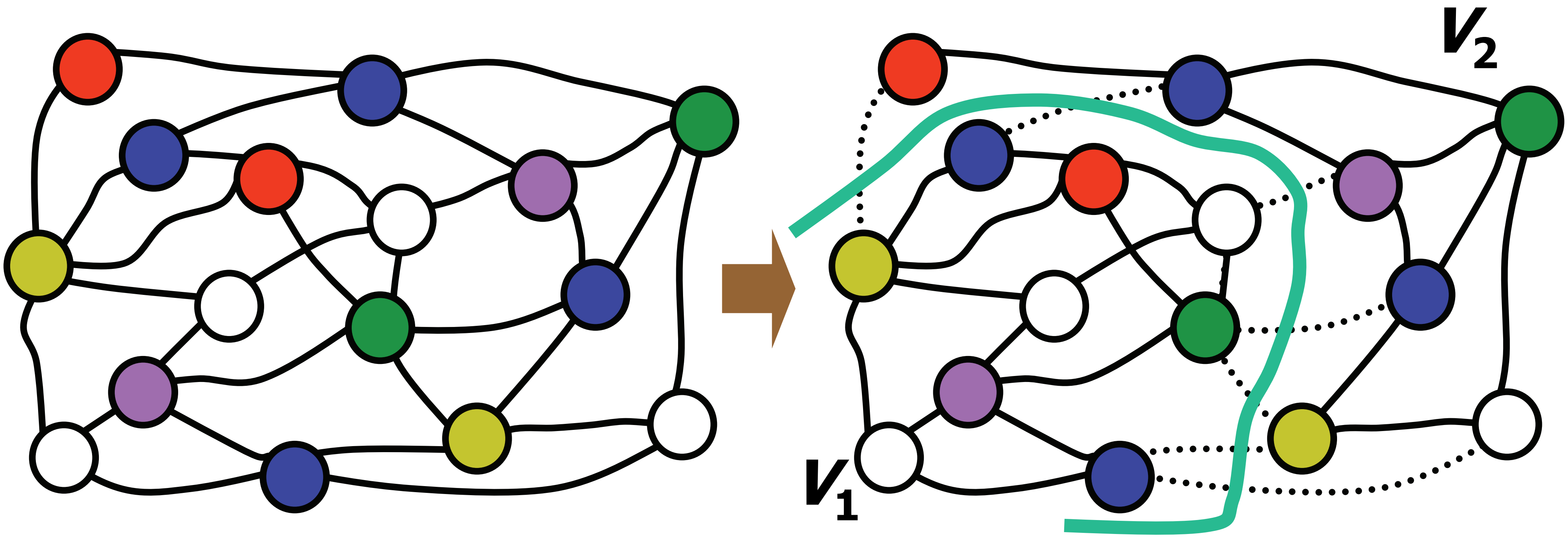
Output:

a partition $\{V_1, V_2\}$ of V s.t.

(1) $|T_i \cap V_1| = |T_i \cap V_2| = |T_i|/2$ for each i,

(2) Both of V_1 and V_2 induce connected graphs.

_k_-bipartition



Related Results

k : # of resource sets, $n=|V|$, $m=|E|$

- Testing whether a k-bipartition exists or not is NP-hard even if $k=1$ [Dyer, Frieze 85][Chleikova 99]

Related Results

k : # of resource sets, $n=|V|$, $m=|E|$

- Testing whether a k-bipartition exists or not is NP-hard even if $k=1$ [Dyer, Frieze 85][Chleikova 99]

- Sufficient condition for which a k-bipartition exists:

1-bipartition · · · 2-connectivity suffices.

$O(m)$ time [Suzuki et al.90][Wada, Kawaguchi94]

2-bipartition · · · 3-connectivity suffices.

$O(n^2 \log n)$ time [Nagamochi et al. 02]

Conjecture

Every $(k+1)$-connected graph admits a k-bipartition.

Our Recent Results

3-bipartition

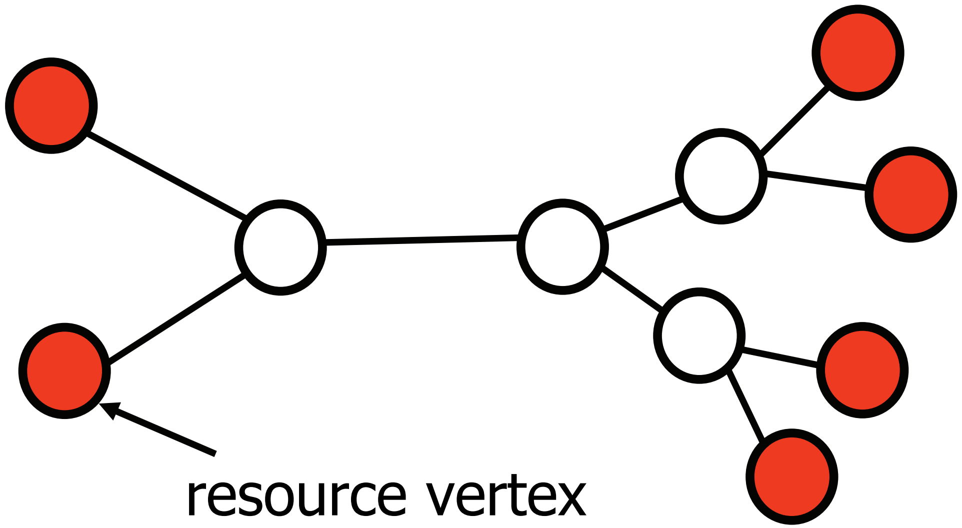
(1) 5-vertex-connectivity does not suffice!

(2) 4-vertex-connectivity suffices if K4 is contained.

(3) For the edge version of k-bipartition (k=1,2,3), (k+1)-edge-connectivity suffices.

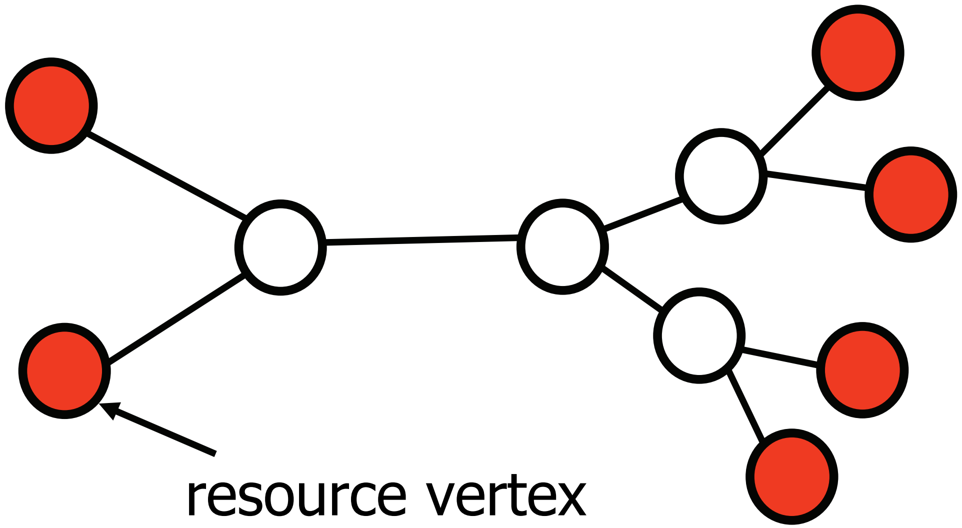
A 1-vertex-connected graph which has no 1-bipartition of V

k=1



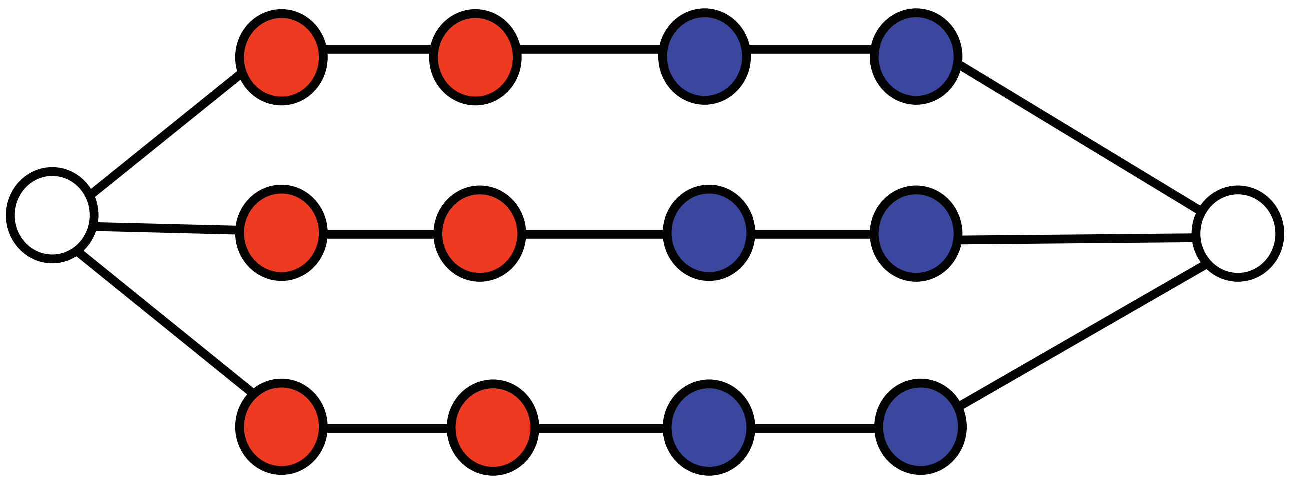
A 1-vertex-connected graph which has no 1-bipartition of V

k=1



A 2-vertex-connected graph which has no 2-bipartition of V

k=2



Our Results

3-bipartition

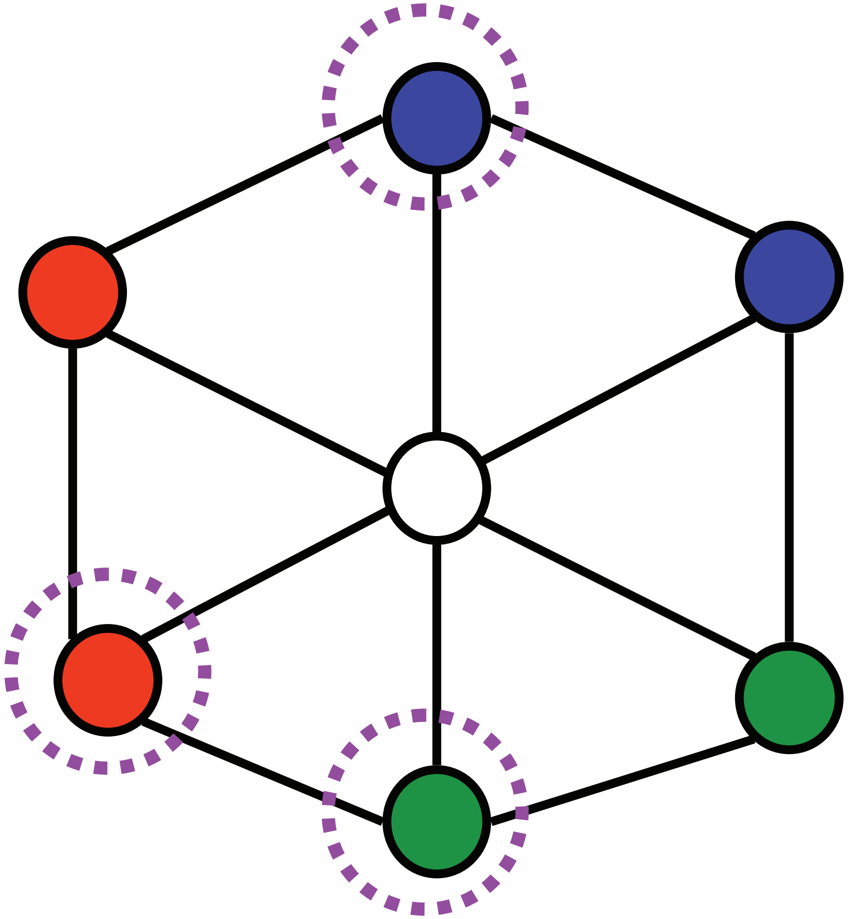
(1) 5-vertex-connectivity does not suffice!

(2) 4-vertex-connectivity suffices if K4 is contained.

(3) For the edge version of k-bipartition (k=1,2,3),
(k+1)-edge-connectivity suffices.

3-bipartition

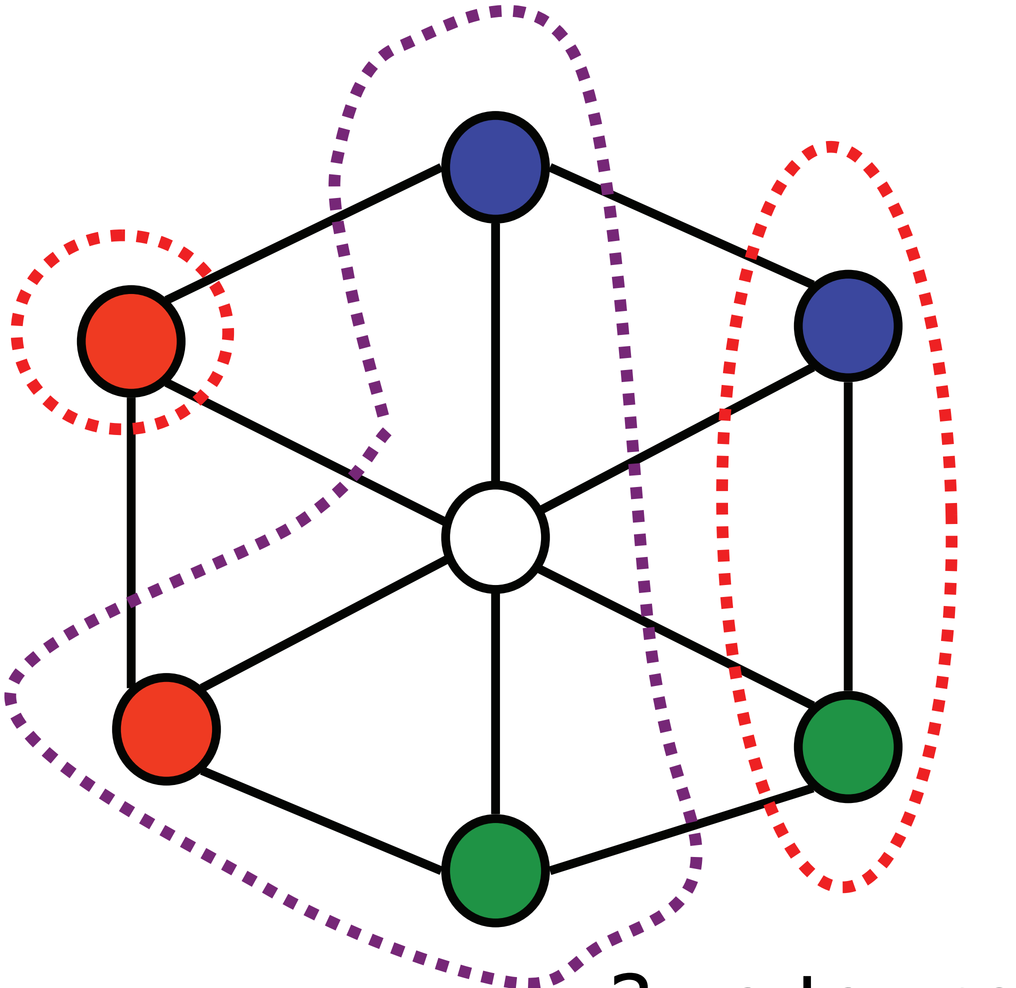
3-vertex-connectivity does not suffice.



3-vertex-connected graph

3-bipartition

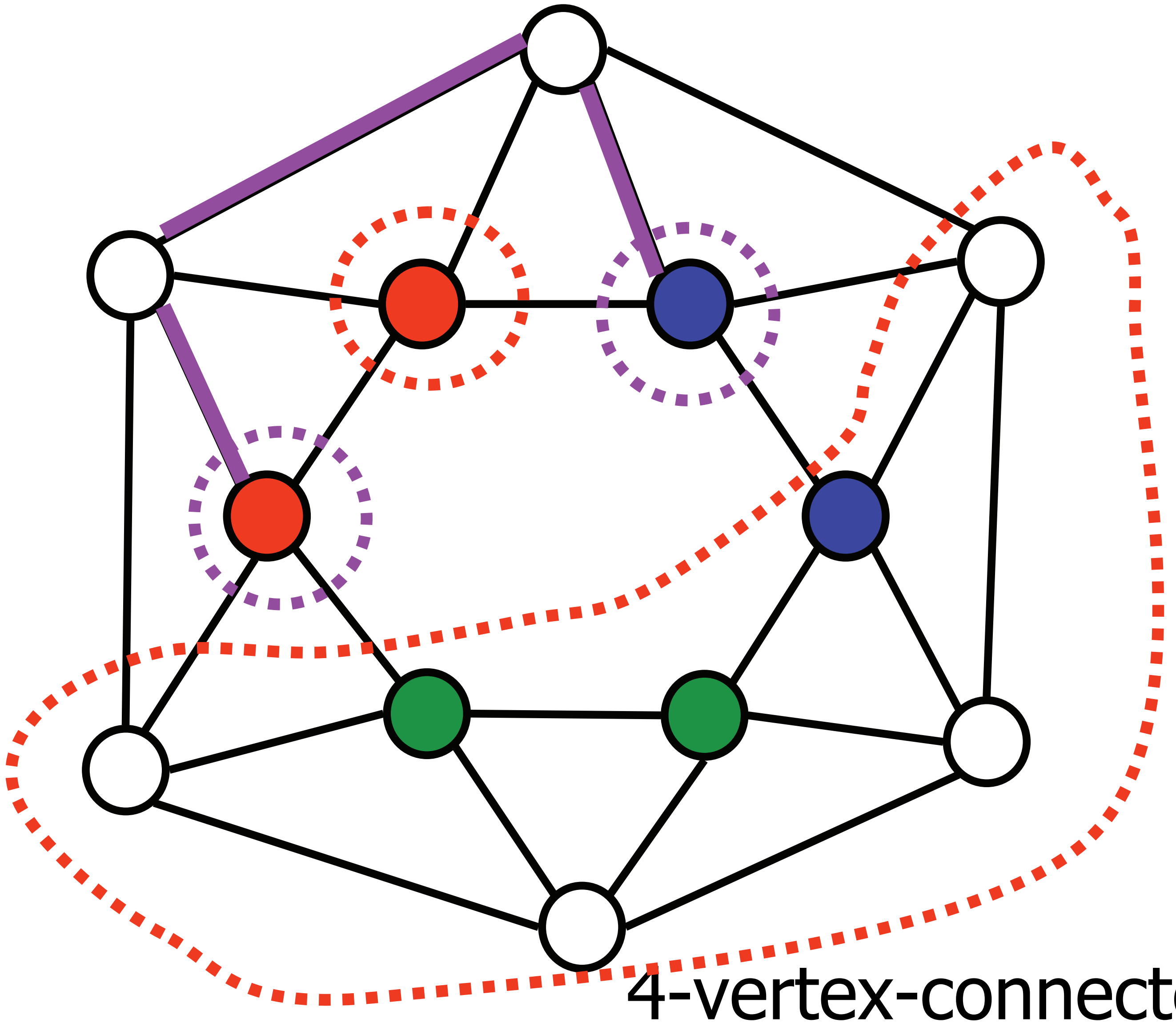
3-vertex-connectivity does not suffice.



3-vertex-connected graph

3-bipartition

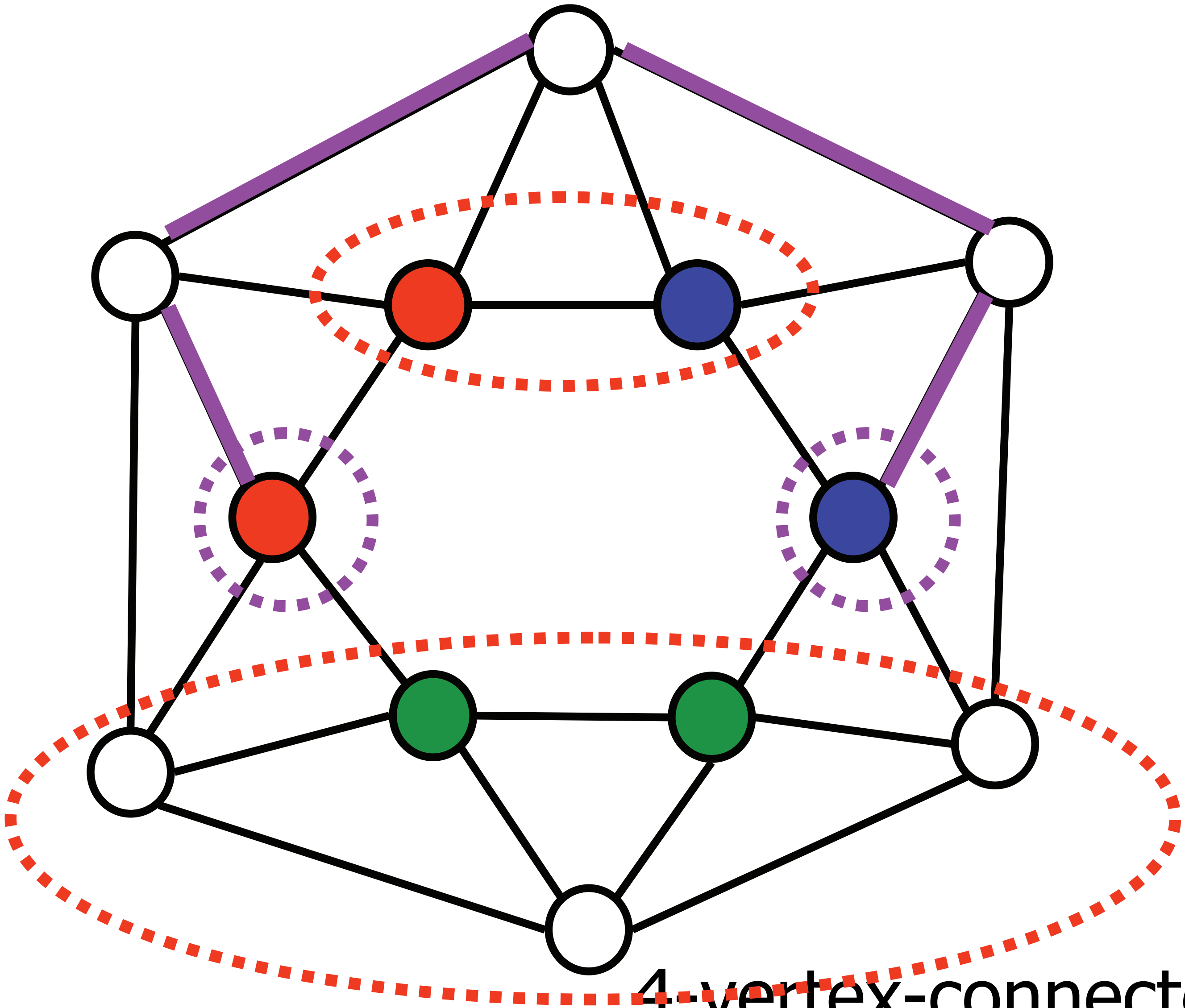
4-vertex-connectivity does not suffice.



4-vertex-connected graph

3-bipartition

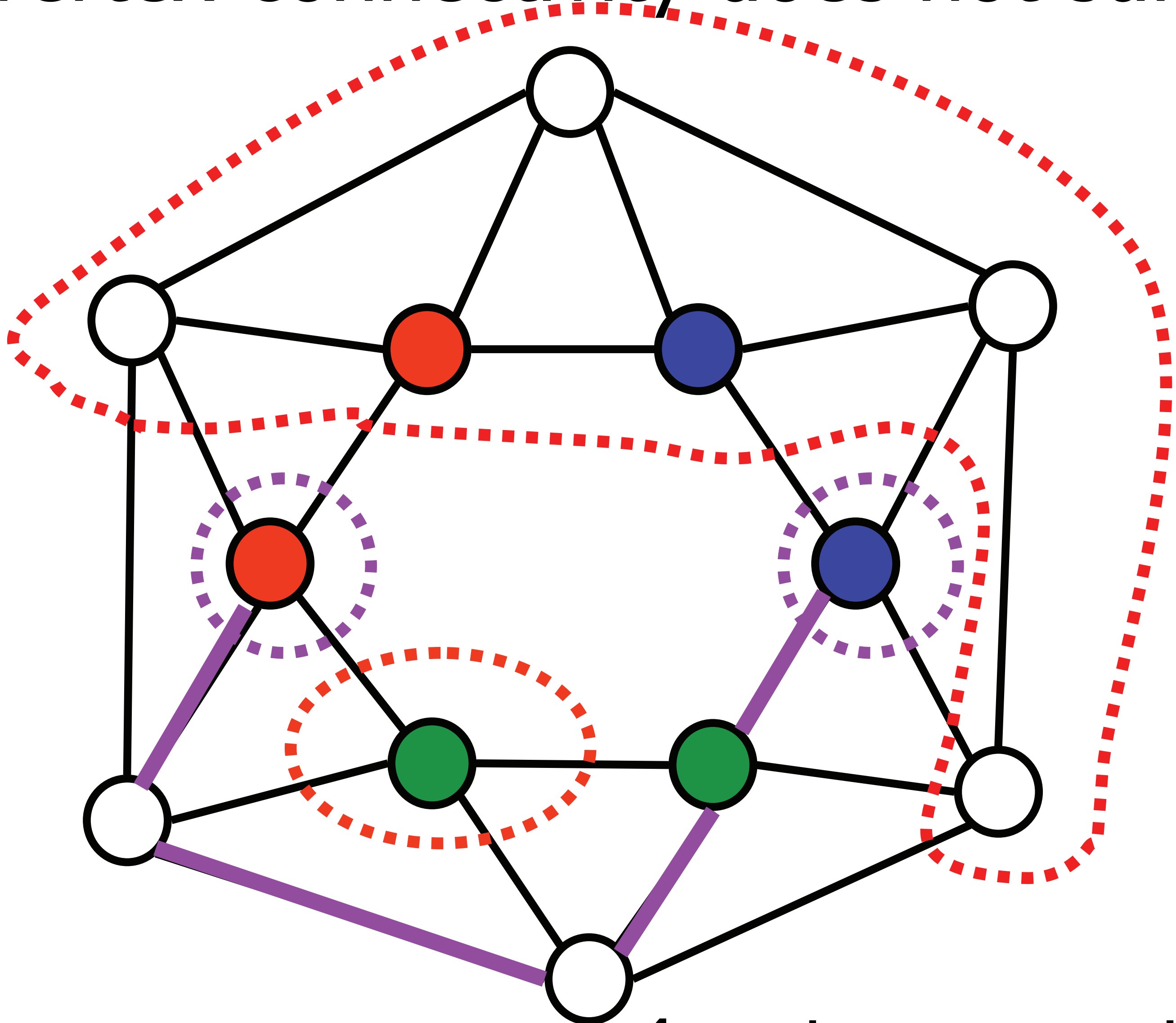
4-vertex-connectivity does not suffice.



4-vertex-connected graph

3-bipartition

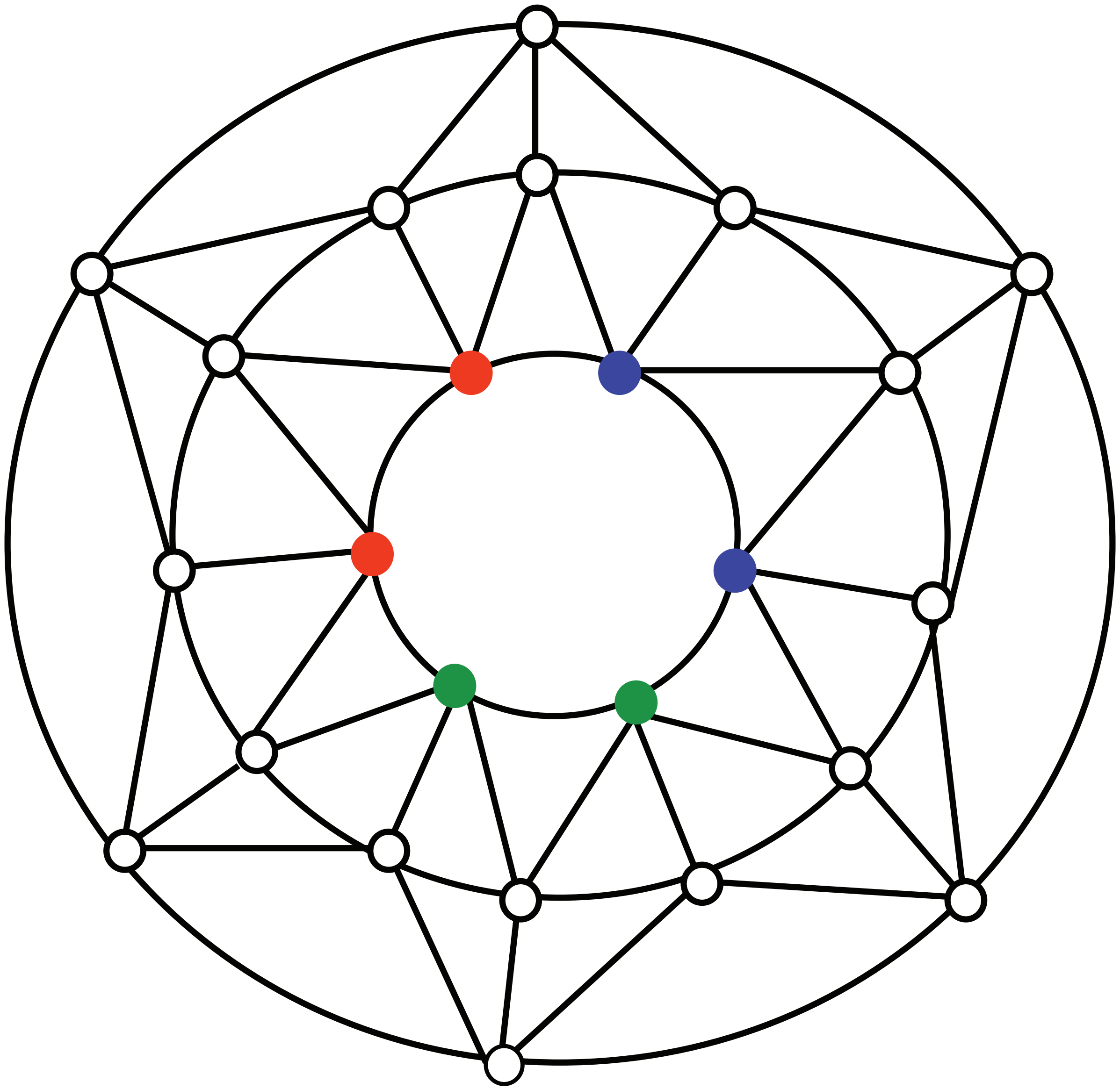
4-vertex-connectivity does not suffice.



4-vertex-connected graph

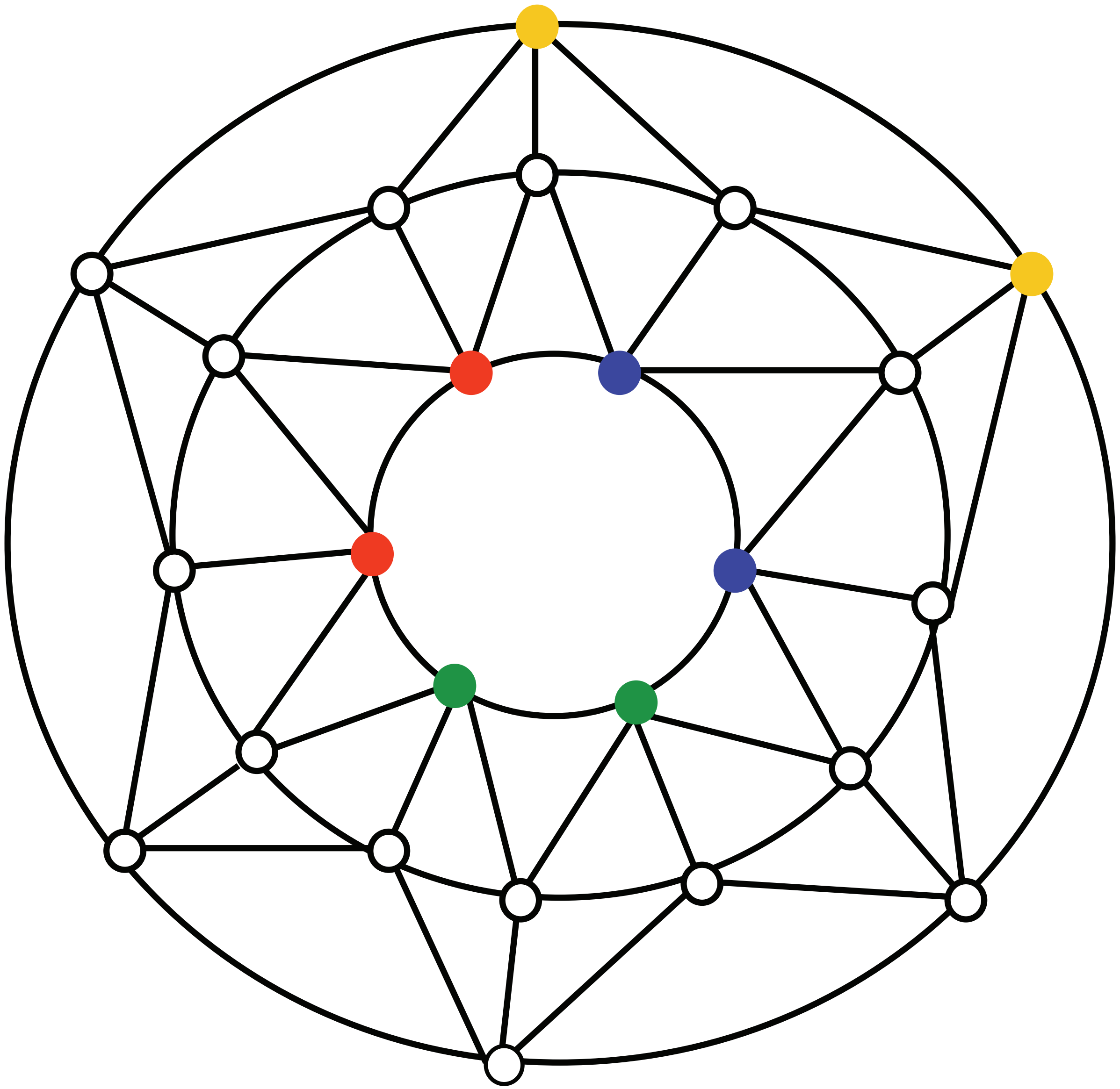
3-bipartition

5-vertex-connectivity does not suffice.



4-bipartition

5-vertex-connectivity does not suffice.



Our Results

3-bipartition

(1) 5-vertex-connectivity does not suffice!

(2) 4-vertex-connectivity suffices if K4 is contained.

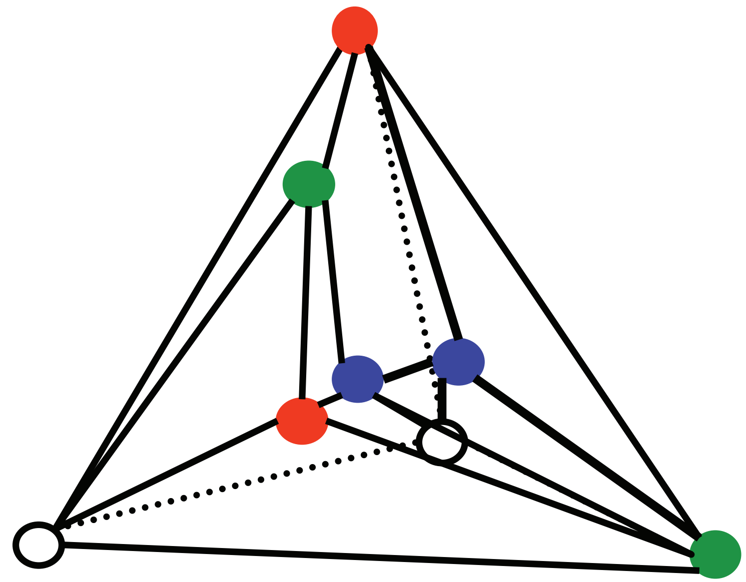
(3) For the edge version of k-bipartition (k=1,2,3), (k+1)-edge-connectivity suffices.

Theorem

If G is a 4-vertex-connected graph and contains K4, then there exists a 3-bipartition, and moreover, a 3-bipartiton can be found in $O(n^3 \log n)$ time.

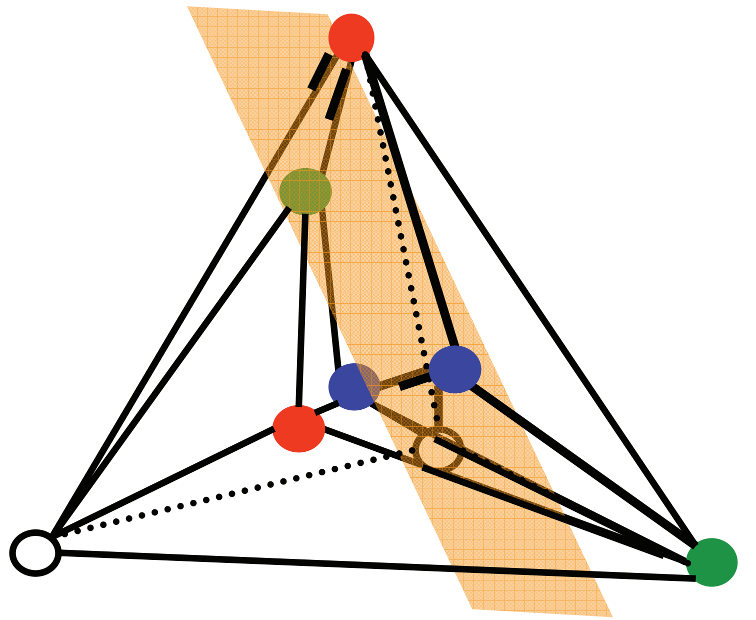
Algorithm for finding a 3-bipartition

Reduction to a geometrical problem [Nagamochi et al. 02]



Phase 1

Find an embedding of G into the 3-dimensional space R^3 called **"convex-embedding"**.



Phase 2

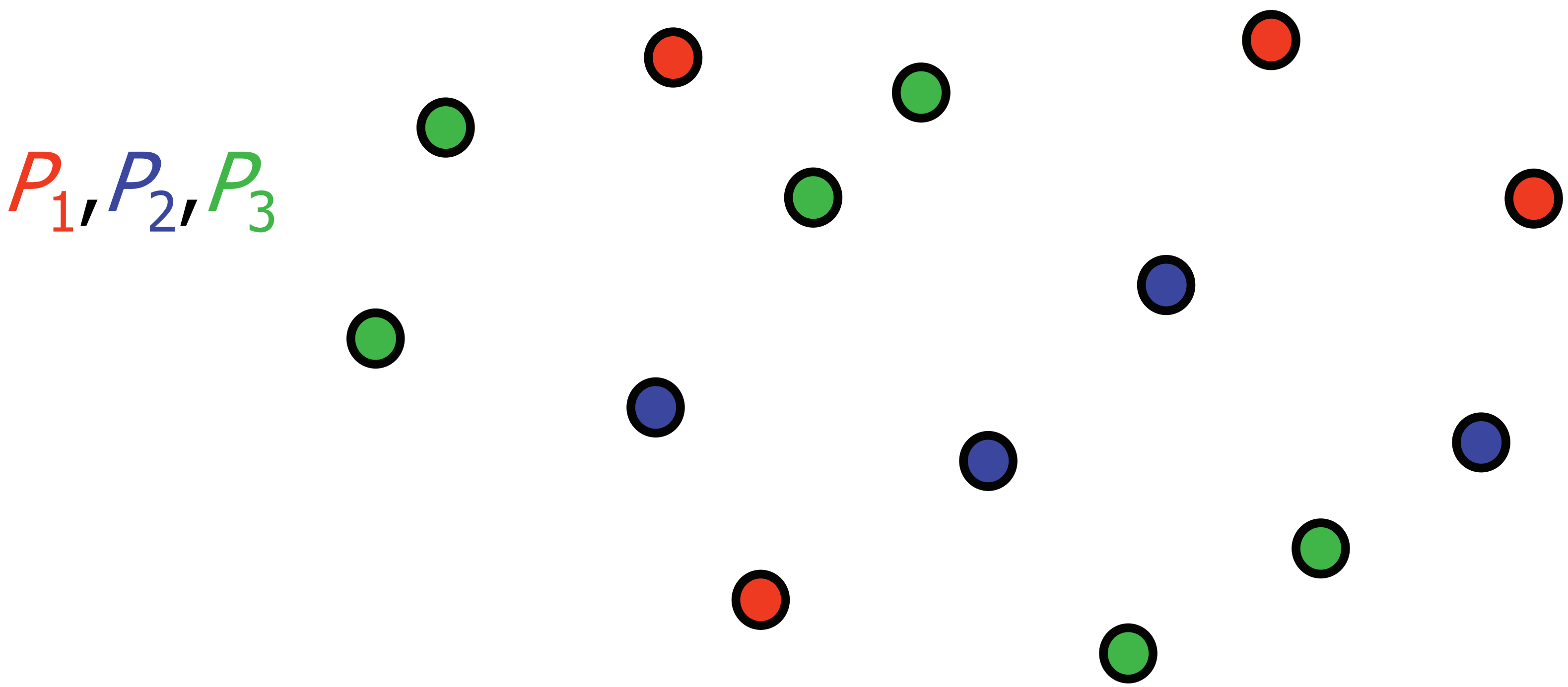
Bisect V in R^3 into $\{V_1, V_2\}$ by a plane called **"ham-sandwich cut"**.

Ham-sandwich cut

$P_1, P_2, \ldots, P_k$: k subsets of points

Ham-sandwich cut with respect to $P_1, P_2, \ldots, P_k$

$\cdots$ hyperplane bisecting each P_i

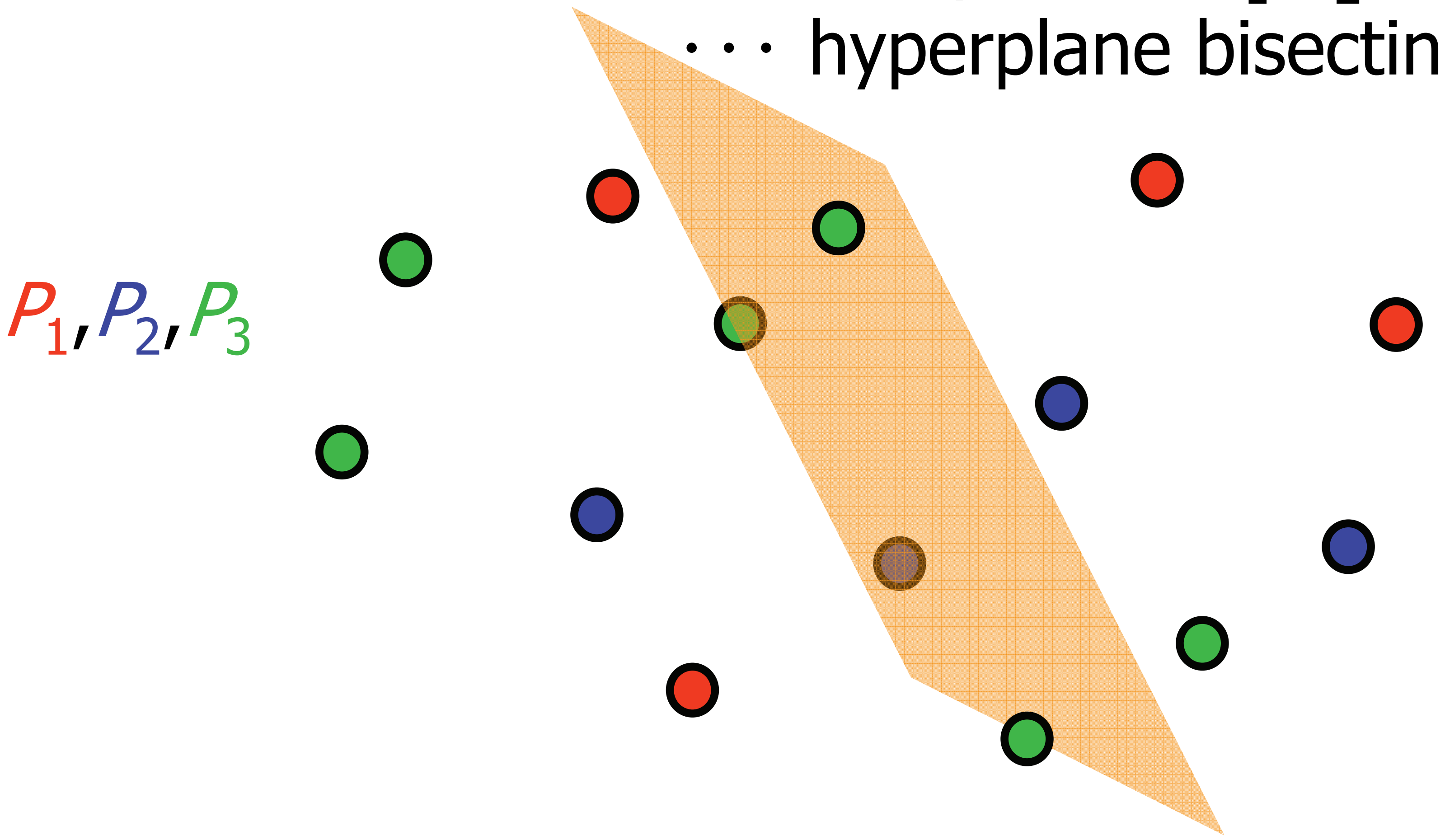


Ham-sandwich cut

$P_1, P_2, \ldots, P_k$: k subsets of points

Ham-sandwich cut with respect to $P_1, P_2, \ldots, P_k$

$\cdots$ hyperplane bisecting each P_i

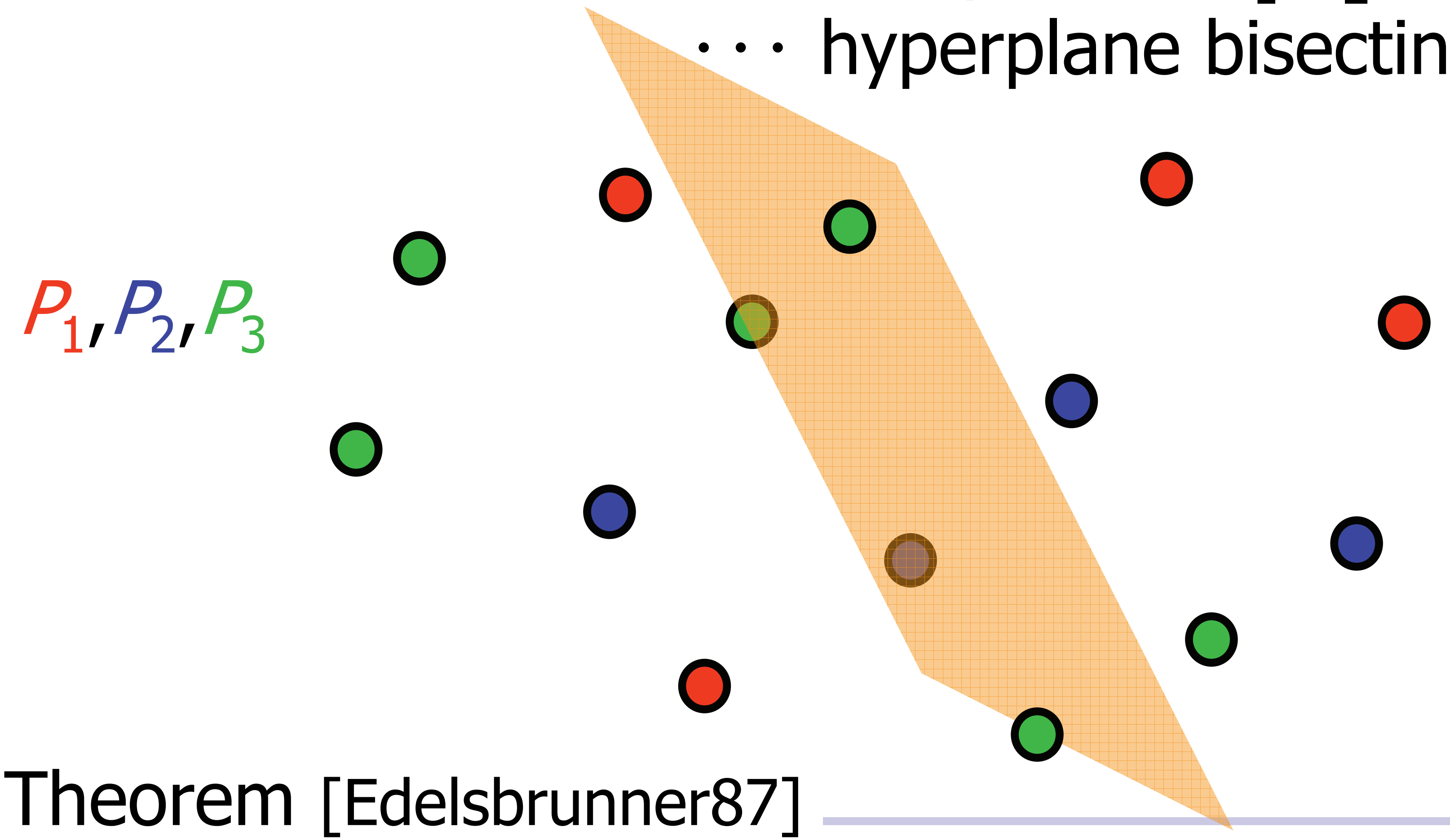


Ham-sandwich cut

$P_1, P_2, \ldots, P_k$: k subsets of points

Ham-sandwich cut with respect to $P_1, P_2, \ldots, P_k$

··· hyperplane bisecting each P_i



Theorem [Edelsbrunner87]

In R^k, a ham-sandwich cut w.r.t. $P_1, \ldots, P_k$ always exists.

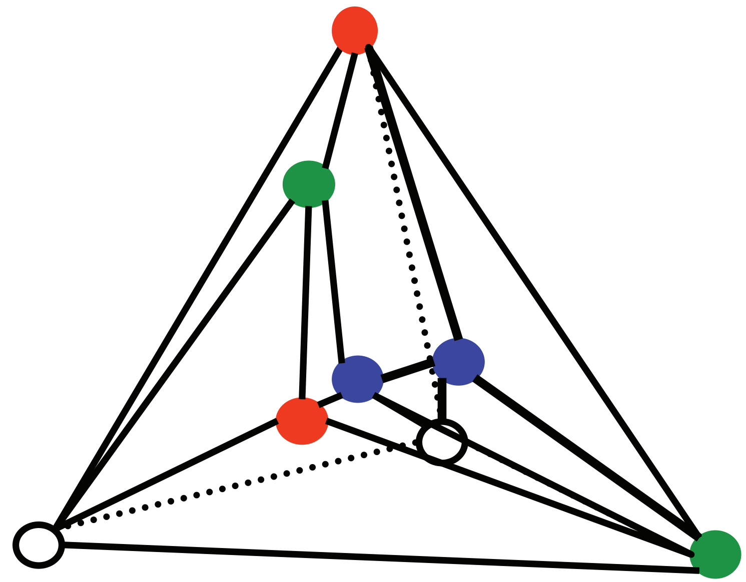
$O(n^{3/2})$ if k=3 (n : #points) [Chi-Yuan et al.94]

Algorithm for finding a 3-bipartition

Reduction to a geometrical problem [Nagamochi et al. 02]

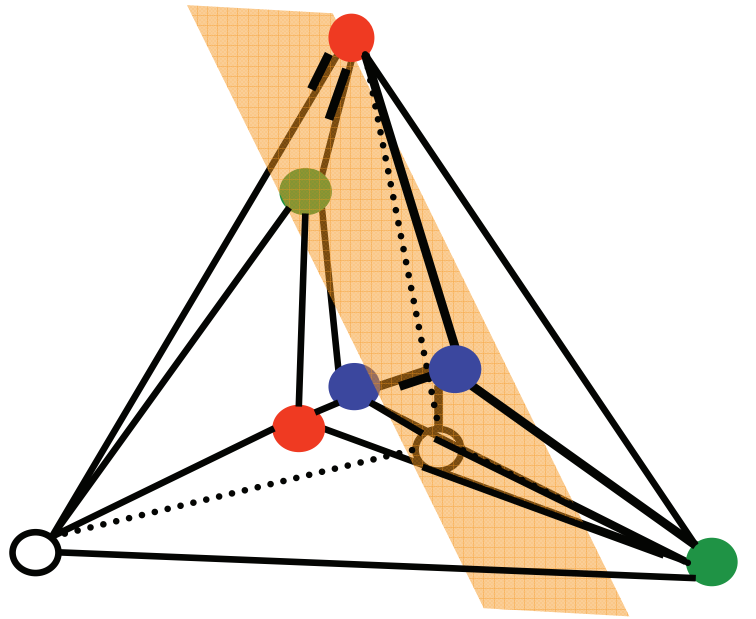
Phase 1

Find an embedding of G into the 3-dimensional space R^3 called **"convex-embedding"**.



Phase 2

Bisect V in R^3 into $\{V_1, V_2\}$ by a plane called **"ham-sandwich cut"**.



→(1) $|T_i \cap V_1| = |T_i \cap V_2| = |T_i|/2$

Convex Embedding

[Nagamochi et al.02]

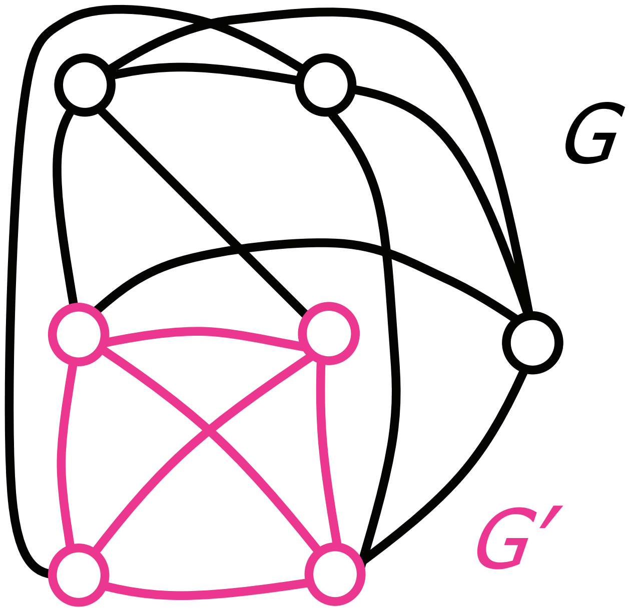
$N_G(v)$: the set of neighbors of v.

$f: V \rightarrow R^k$ is a **convex embedding** of G with boundary G' into R^k,

if (i)

(ii)

(iii)

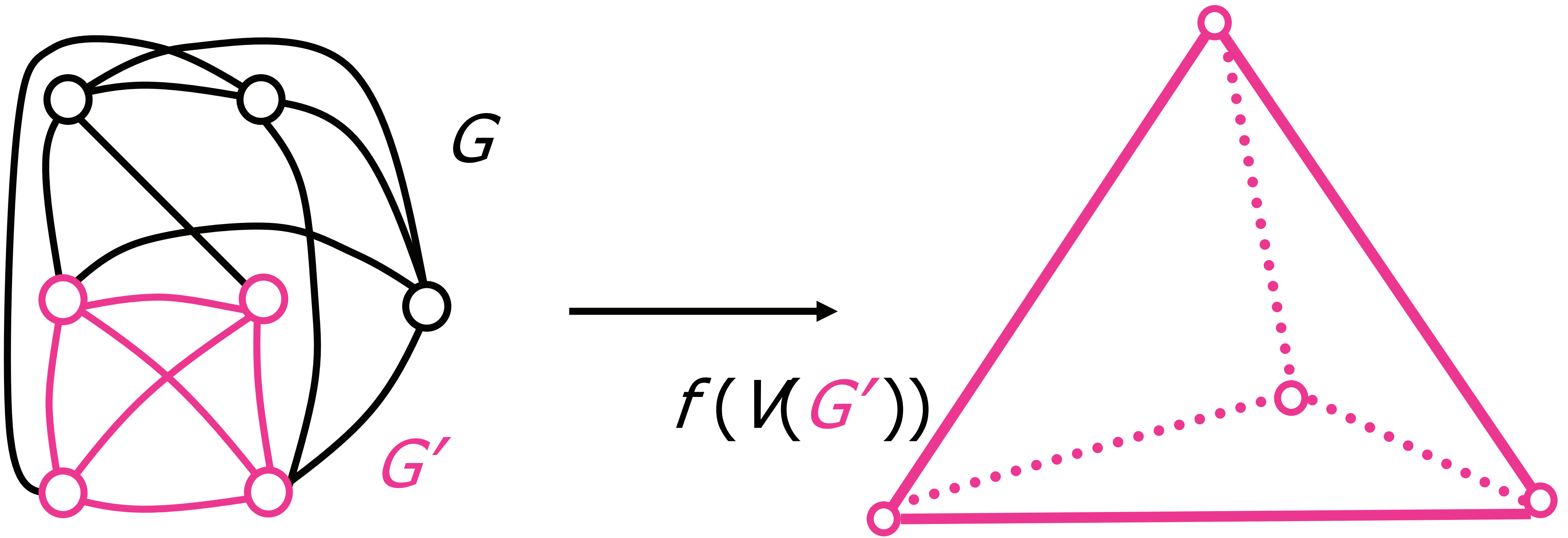


Convex Embedding

[Nagamochi et al.02]

$N_G(v)$: the set of neighbors of v.

$f : V \rightarrow R^k$ is a **convex embedding** of G with boundary G' into R^k,
if (i) the convex hull of $f(V(G'))$ is isomorphic to G',
(ii)
(iii)

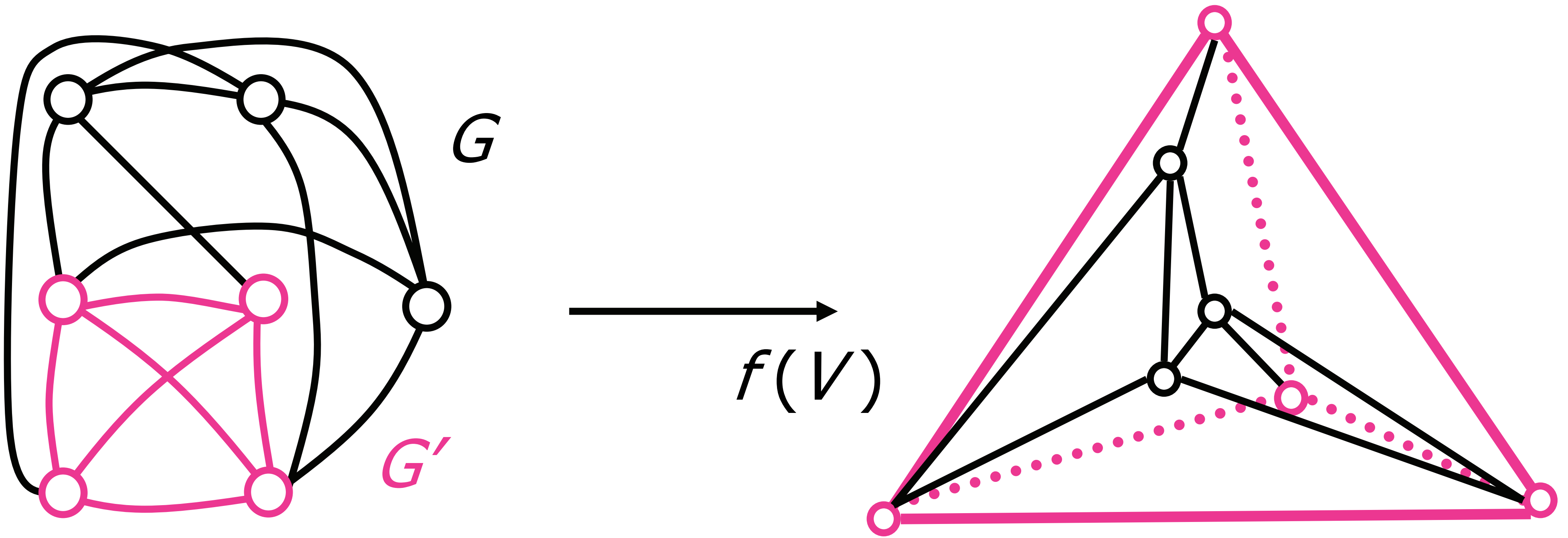


Convex Embedding

[Nagamochi et al.02]

$N_G(v)$: the set of neighbors of v.

$f: V \rightarrow R^k$ is a **convex embedding** of G with boundary G' into R^k,
if (i) the convex hull of $f(V(G'))$ is isomorphic to G',
(ii) For $\forall v \in V - V(G')$,
$f(v)$ is strictly included in the convex hull of $f(N_G(v))$.
(iii)

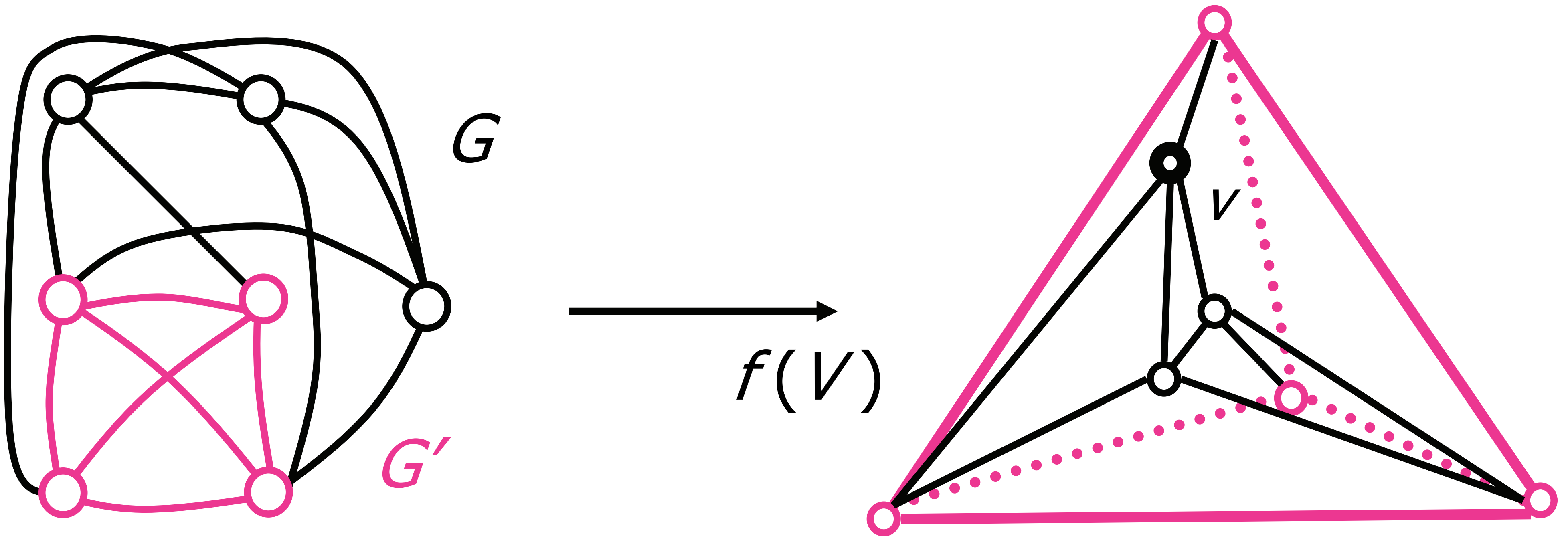


Convex Embedding

[Nagamochi et al.02]

$N_G(v)$: the set of neighbors of v.

$f: V \rightarrow R^k$ is a **convex embedding** of G with boundary G' into R^k,
if (i) the convex hull of $f(V(G'))$ is isomorphic to G',
(ii) For $\forall v \in V - V(G')$,
$f(v)$ is strictly included in the convex hull of $f(N_G(v))$.
(iii)

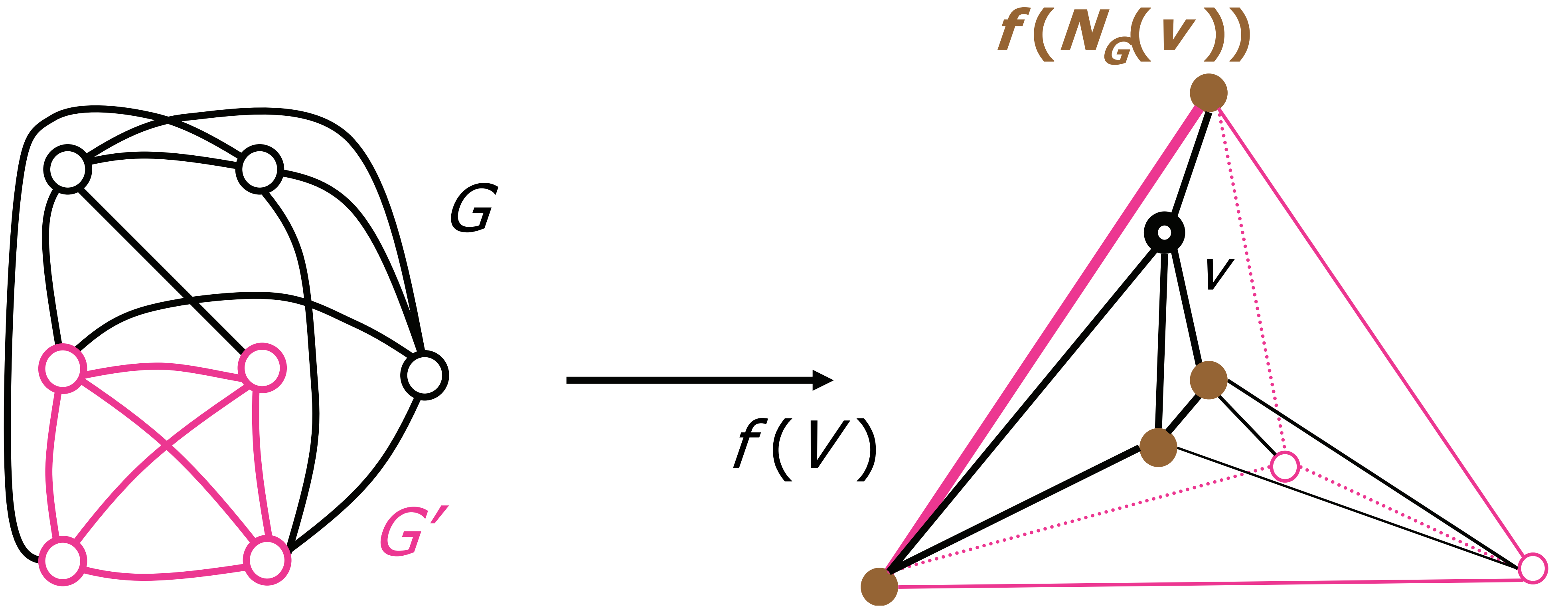


Convex Embedding

[Nagamochi et al.02]

$N_G(v)$: the set of neighbors of v.

$f: V \to R^k$ is a **convex embedding** of G with boundary G' into R^k,
if (i) the convex hull of $f(V(G'))$ is isomorphic to G',
(ii) For $\forall v \in V - V(G')$,
$f(v)$ is strictly included in the convex hull of $f(N_G(v))$.
(iii)



Convex Embedding

[Nagamochi et al.02]

$N_G(v)$: the set of neighbors of v.

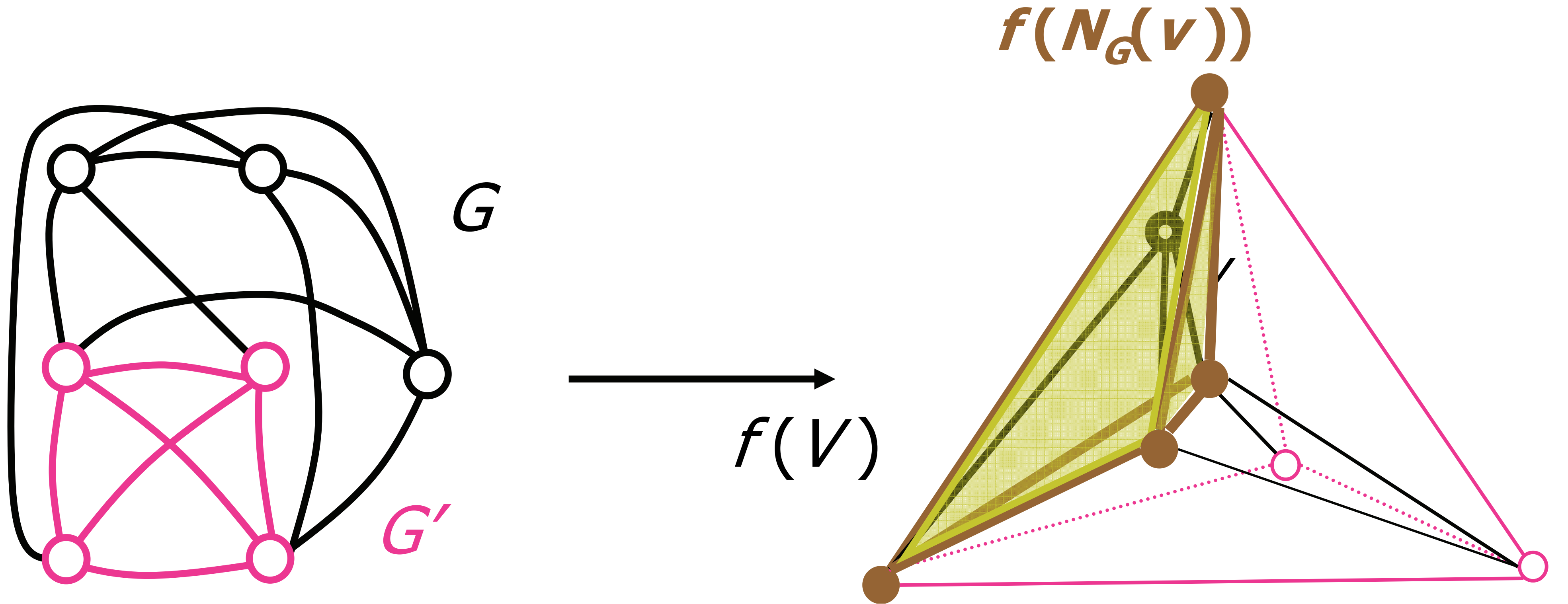
$f: V \rightarrow R^k$ is a **convex embedding** of G with boundary G' into R^k,

if (i) the convex hull of $f(V(G'))$ is isomorphic to G',

(ii) For $\forall v \in V - V(G')$,

$f(v)$ is strictly included in the convex hull of $f(N_G(v))$.

(iii)



Convex Embedding

[Nagamochi et al.02]

$N_G(v)$: the set of neighbors of v.

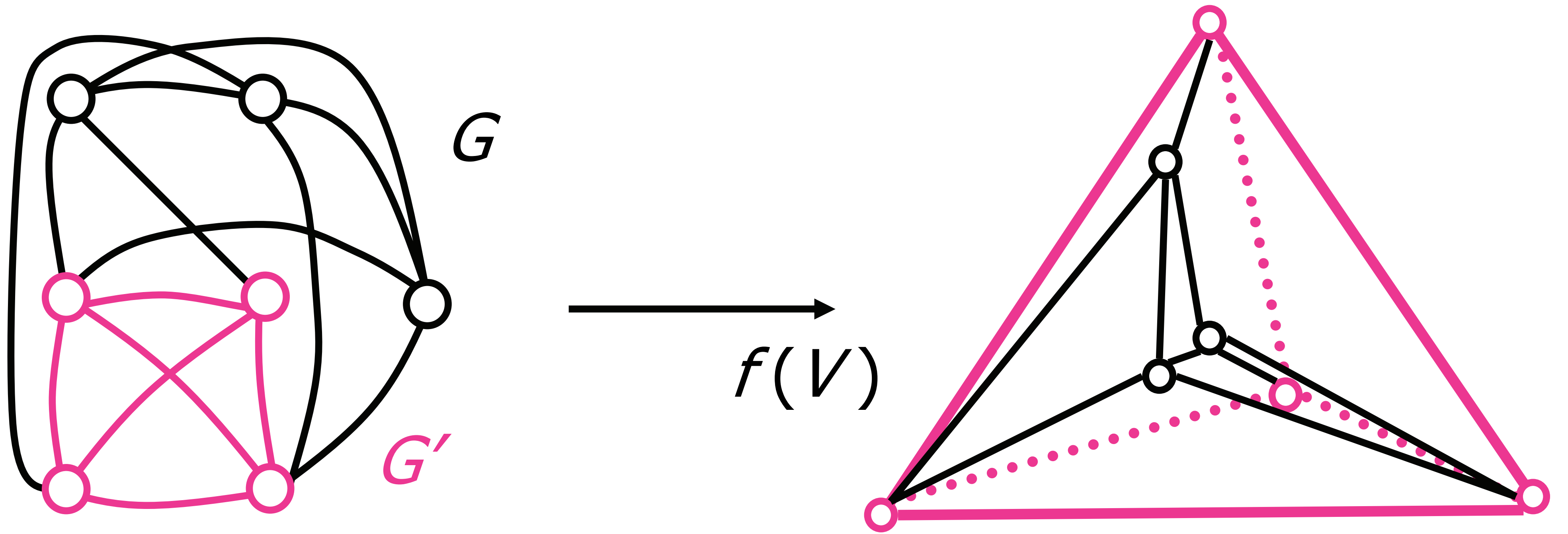
$f: V \rightarrow R^k$ is a **convex embedding** of G with boundary G' into R^k,

if (i) the convex hull of $f(V(G'))$ is isomorphic to G'.

(ii) For $\forall v \in V - V(G')$,

$f(v)$ is strictly included in the convex hull of $f(N_G(v))$.

(iii) Points of $\{f(v) \mid v \in V\}$ are in general position.



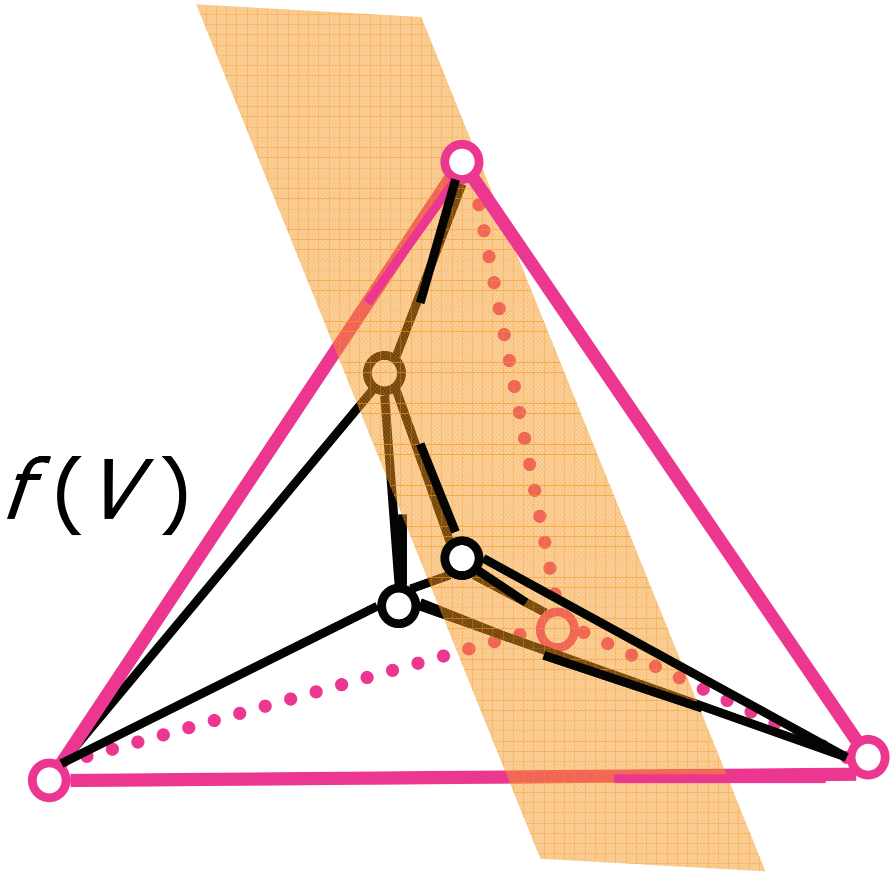
Convex Embedding

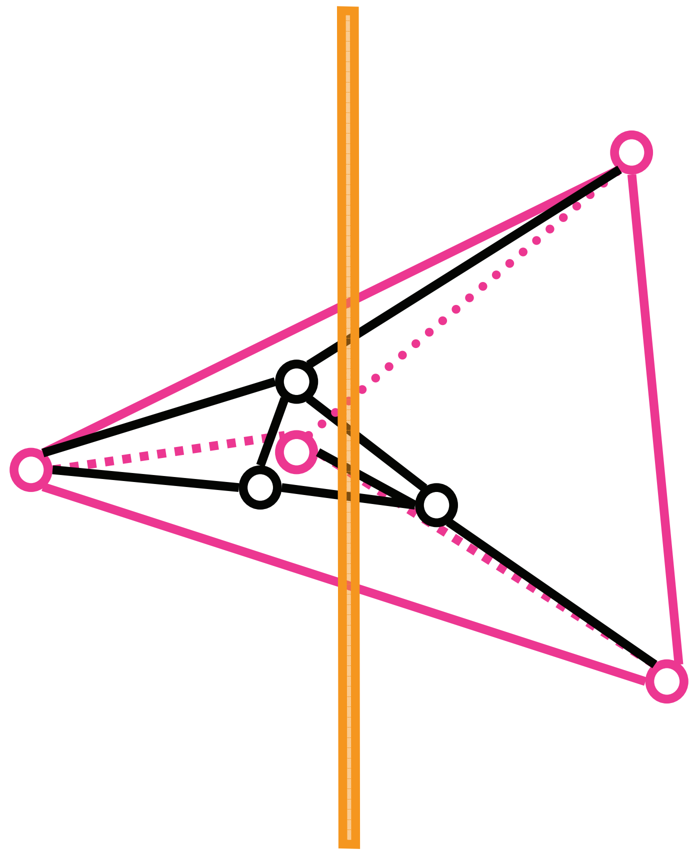
Lemma [Nagamochi et al.02]

$f: V \rightarrow R^k$: a convex embedding of G with boundary G' into R^k.

$\{V_1, V_2\}$: a parition of V obtained by separating $f(V)$ with **an arbitrary hyperplane**.

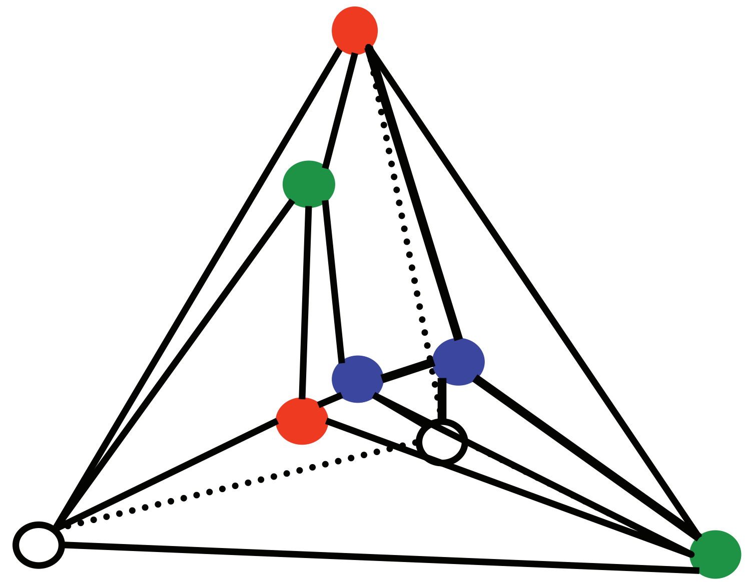
$\Rightarrow$ Both of V_1 and V_2 induce connected graphs.

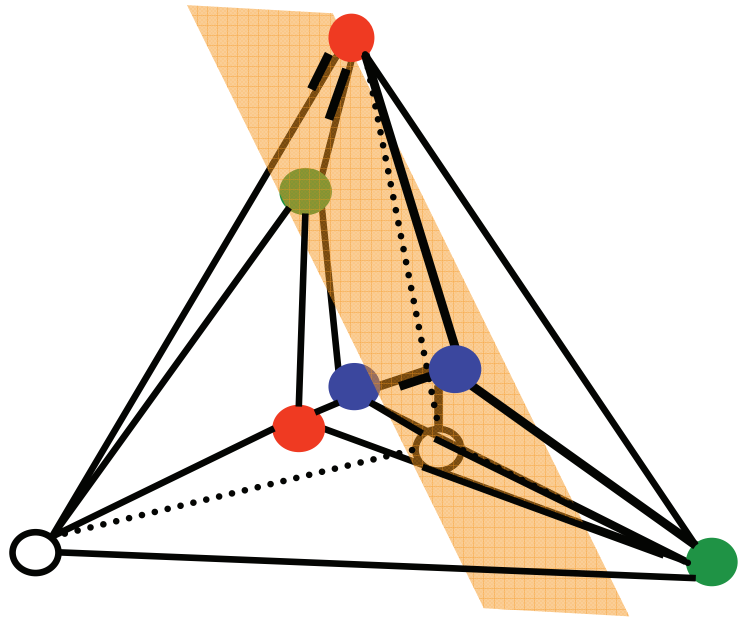




Algorithm for finding a 3-bipartition

Reduction to a geometrical problem [Nagamochi et al. 02]





Phase 1

Find an embedding of G into the 3-dimensional space R^3 called **"convex-embedding"**.

Phase 2

Bisect V in R^3 into $\{V_1, V_2\}$ by a plane called **"ham-sandwich cut"**.

→(1) $|T_i \cap V_1| = |T_i \cap V_2| = |T_i|/2$

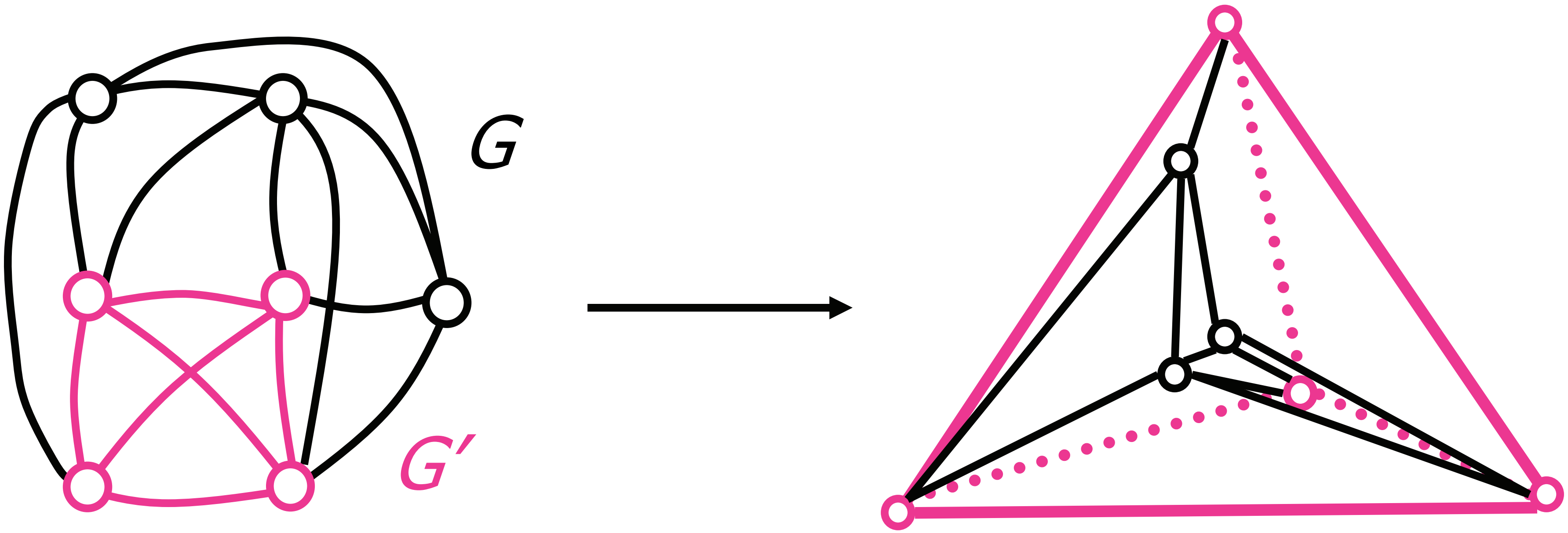
(2) V_1 and V_2 induce connected graphs

3-bipartition

Theorem

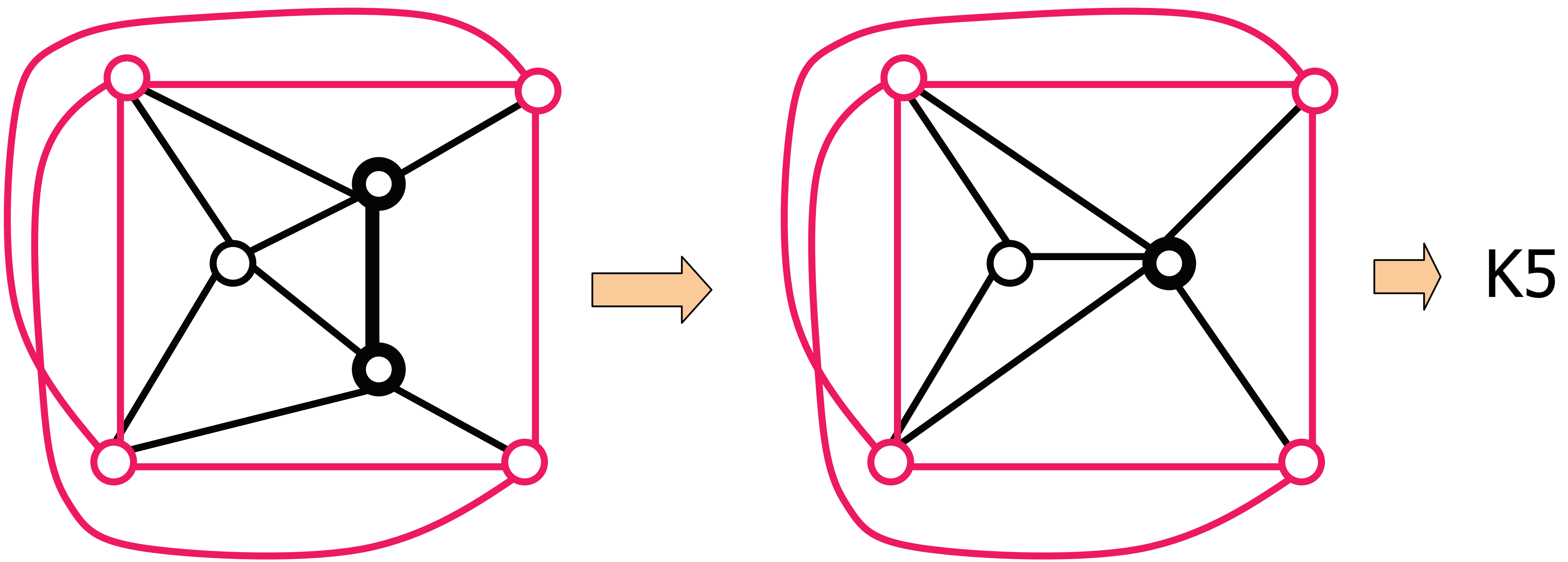
G : a 4-connected graph which includes K4 (denoted by G').

$\Rightarrow G$ has a convex embedding into R^3 with boundary G'



Key Lemma

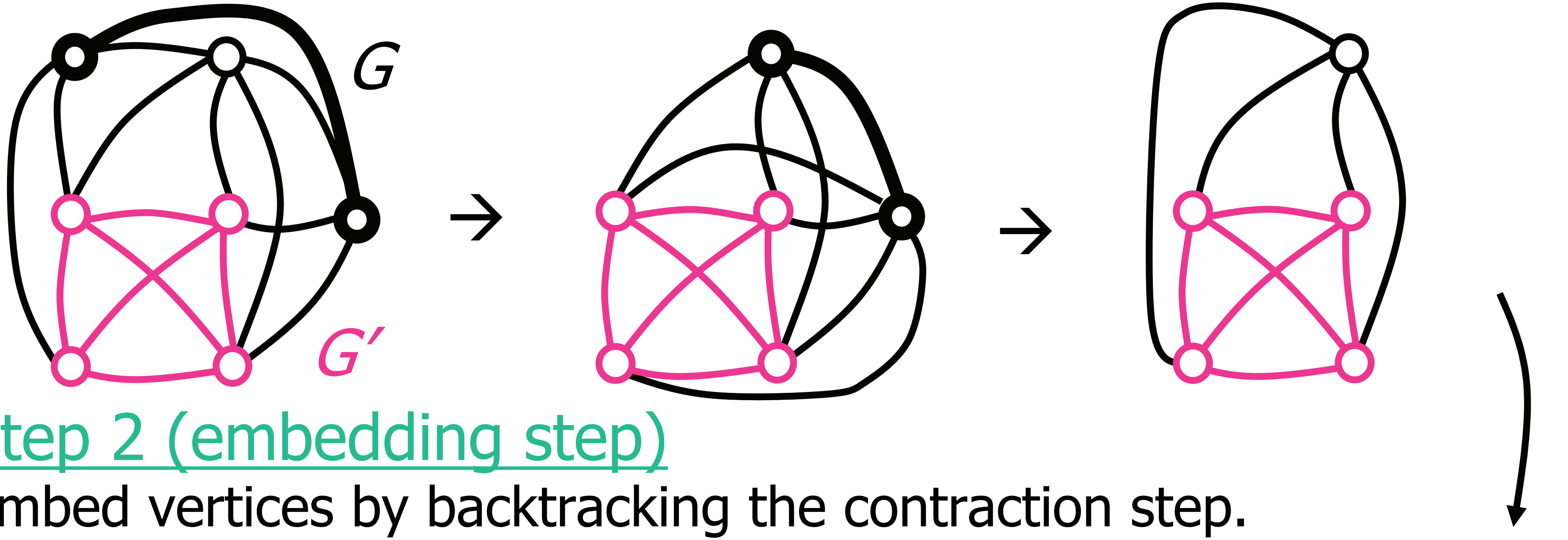
Lemma Let G be a 4-vertex-connected graph G ($\neq$K5), and H be a subgraph of G with H=K4.
Then G has a contractible edge in $E(G)$-$E(H)$ in such a sense that its contraction preserves 4-vertex-connectivity.



Algorithm for finding a convex embedding into R^3

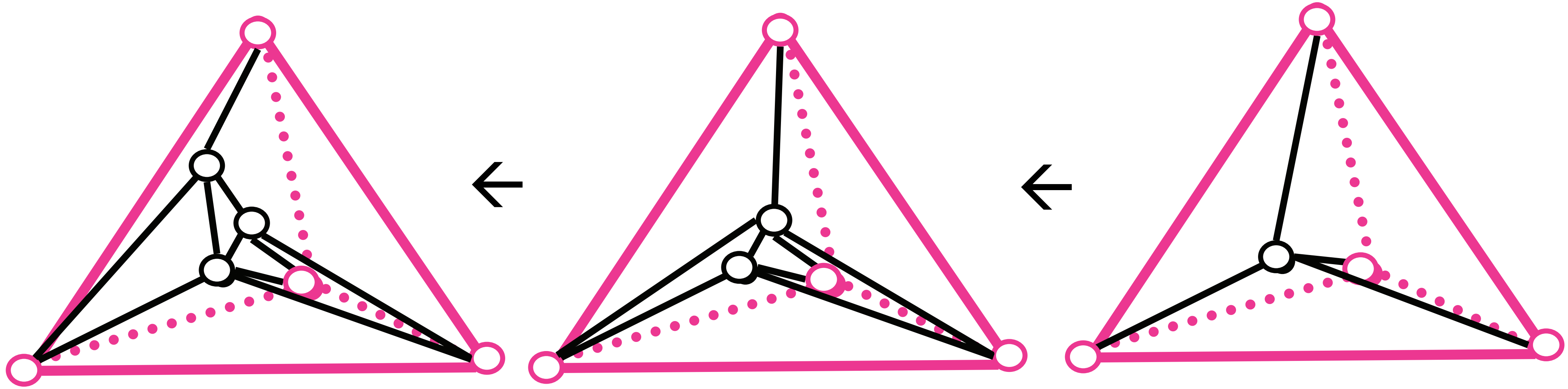
Step 1 (contraction step)

Contract edges not contained in G' while preserving 4-connectivity



Step 2 (embedding step)

Embed vertices by backtracking the contraction step.



Embedding Step

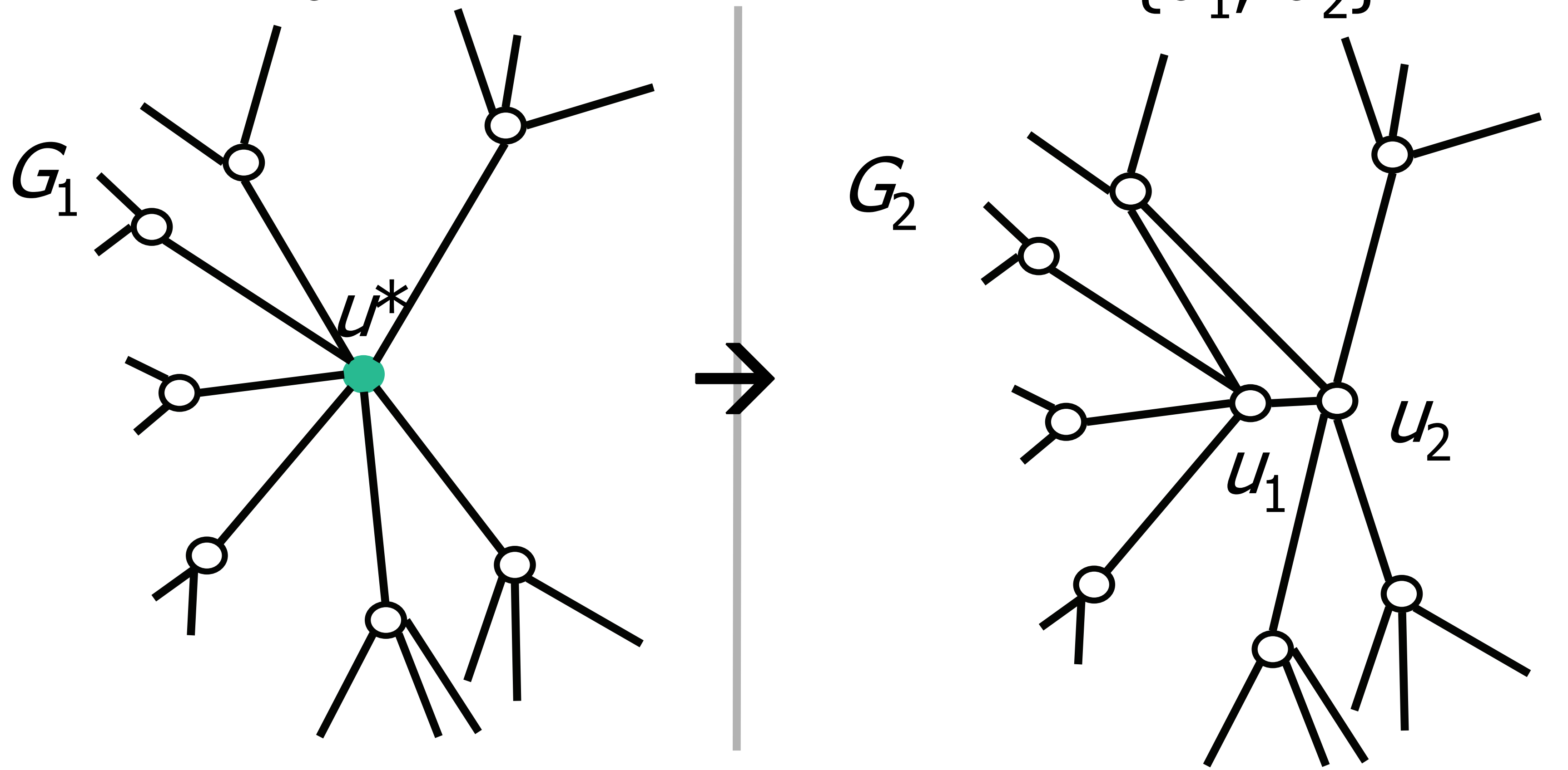
Given:

G_1: graph obtained from G_2 by contracting u_1 and u_2 into u^* such that $(u_1, u_2) \in E$, $|N_G(u_i)| \geqq 4$

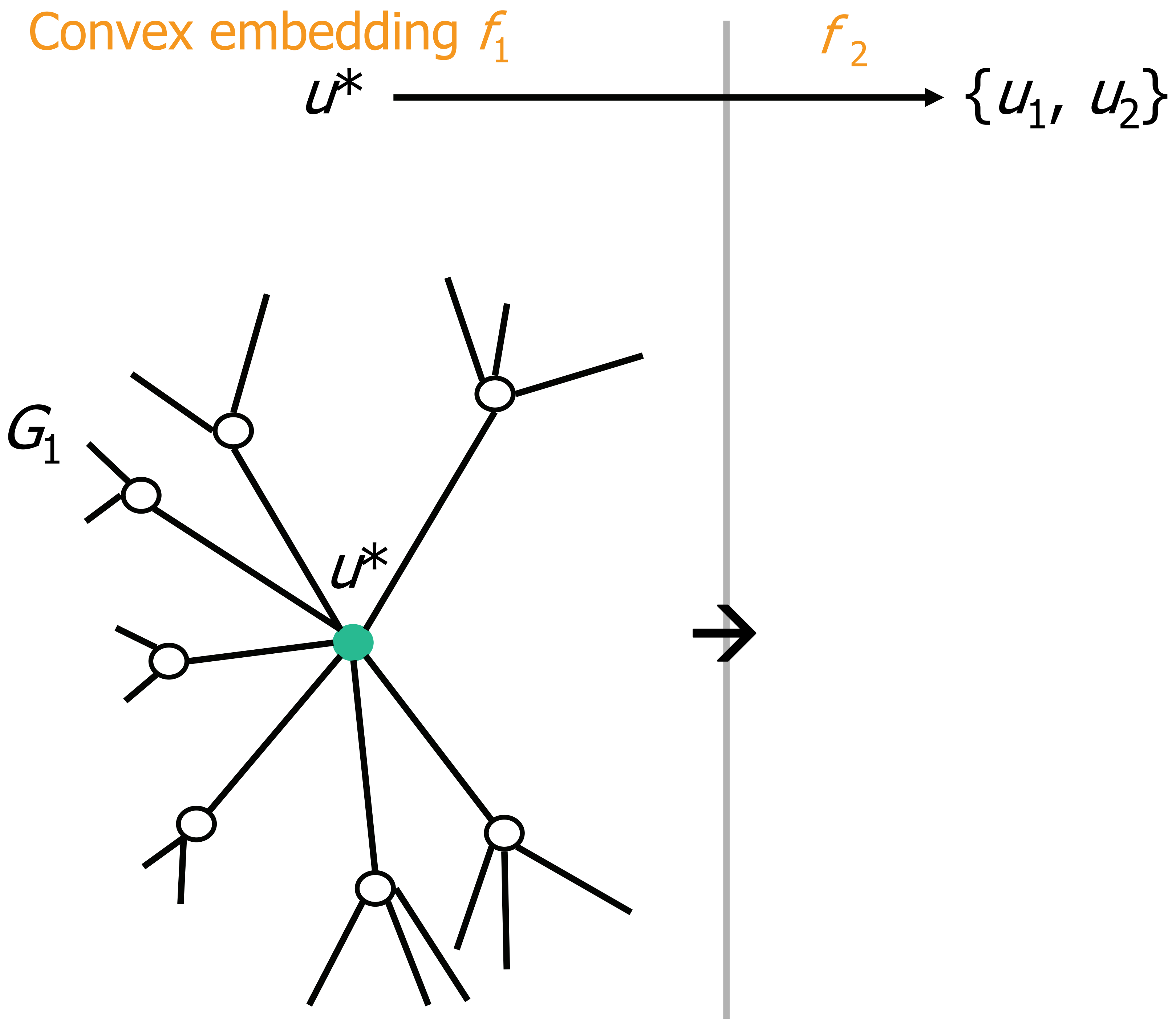
f_1: convex embedding of G_1

Convex embedding f_1 — Convex embedding of G_2

$u^* \longrightarrow \{u_1, u_2\}$



Embedding Step

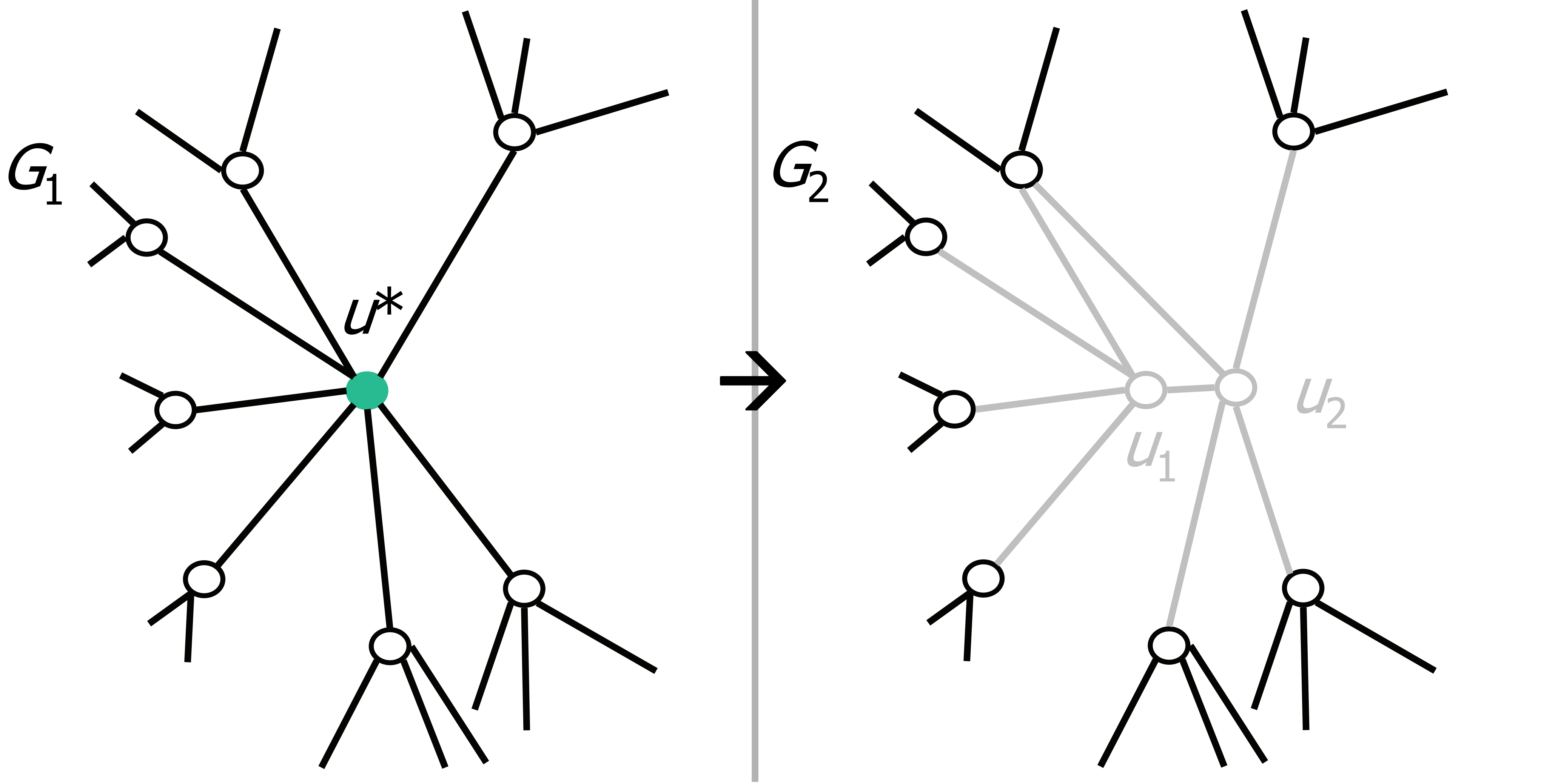


Embedding Step

Convex embedding f_1 $\quad$ f_2

$u^* \longrightarrow \{u_1, u_2\}$

i) $f_2(u) = f_1(u)$ for $\forall\ u \neq u_1, u_2$



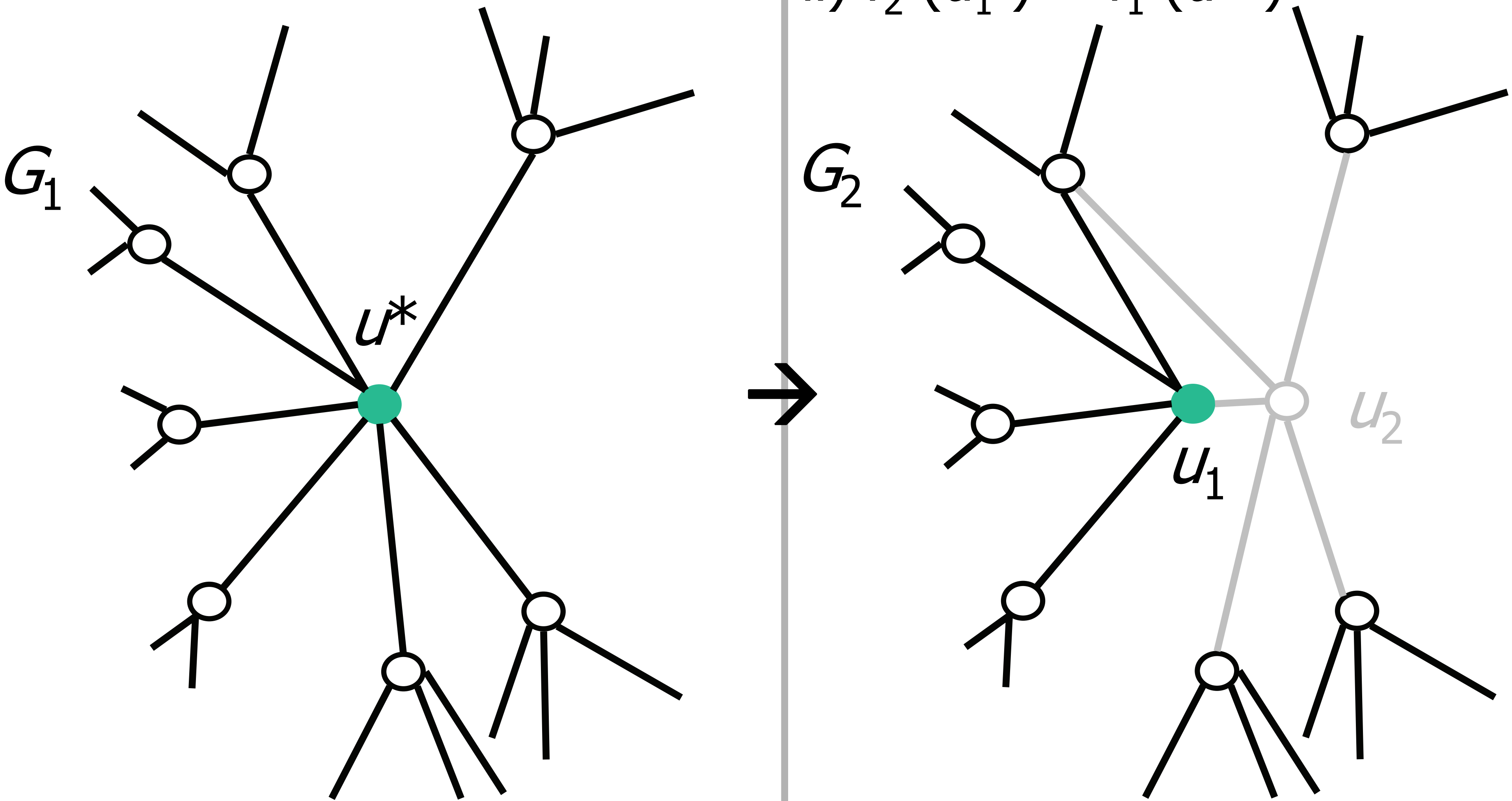
Embedding Step

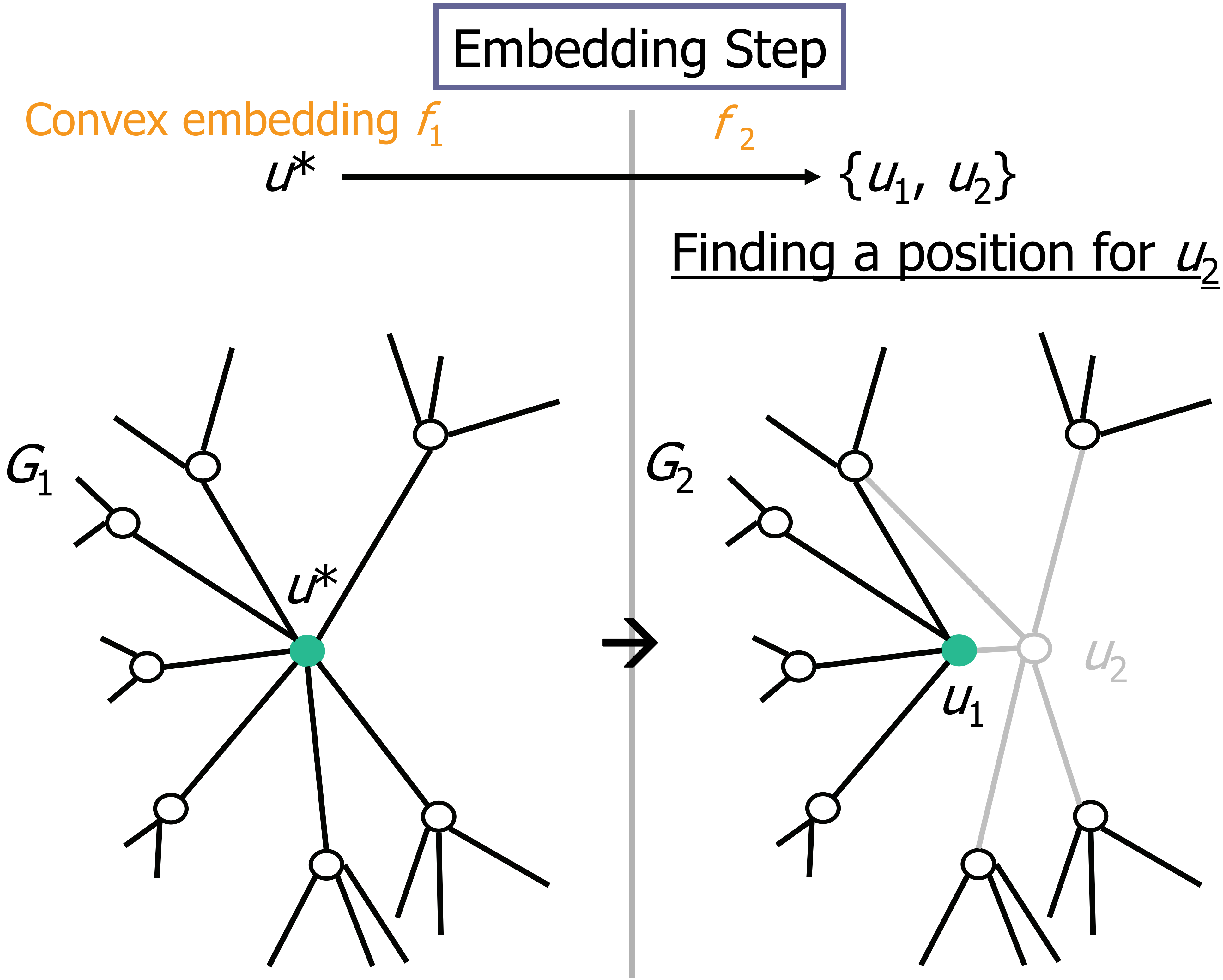
Convex embedding f_1 $\quad$ f_2

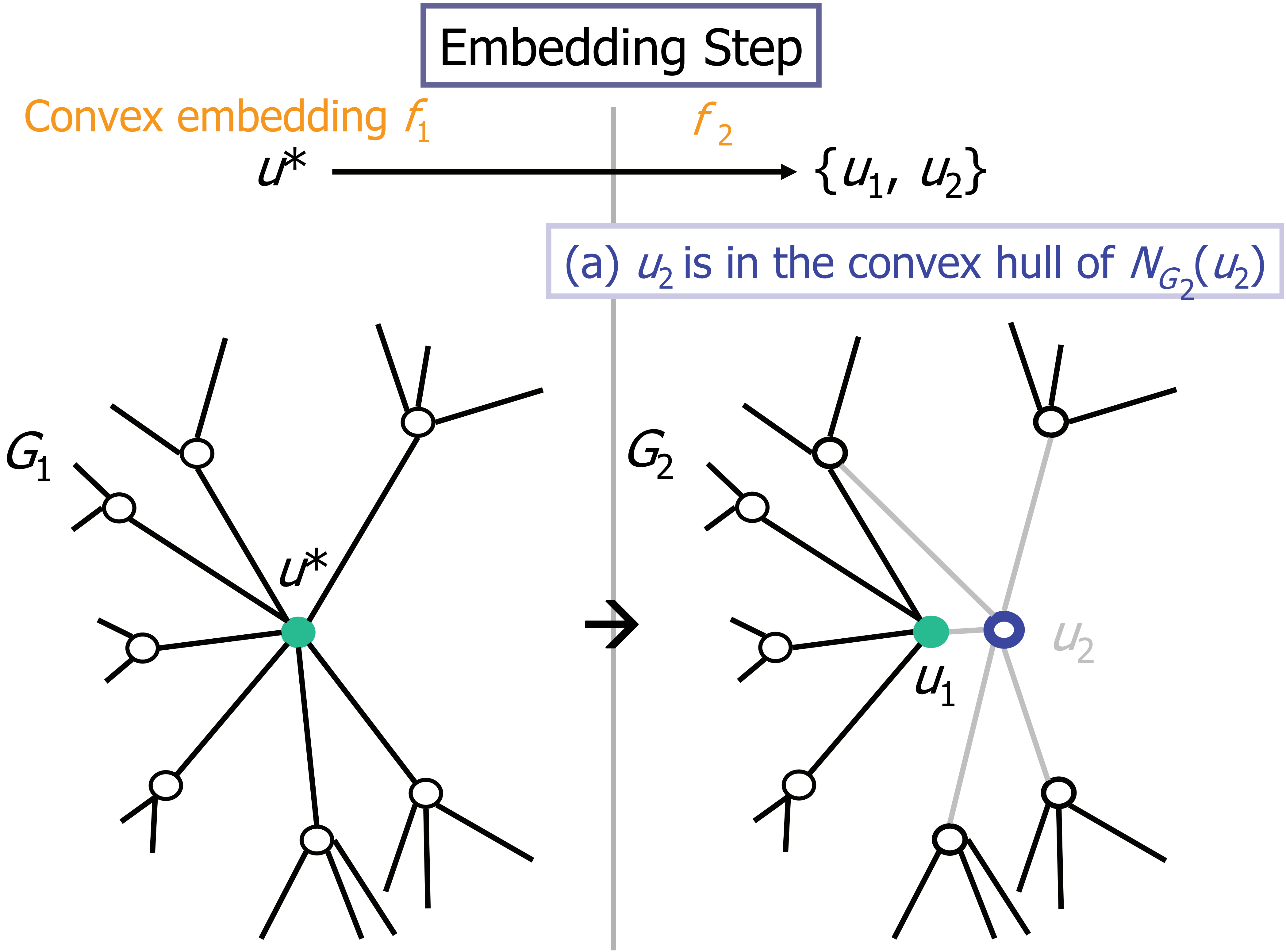
$u^* \longrightarrow \{u_1, u_2\}$

i) $f_2(u) = f_1(u)$ for $\forall\ u \neq u_1, u_2$

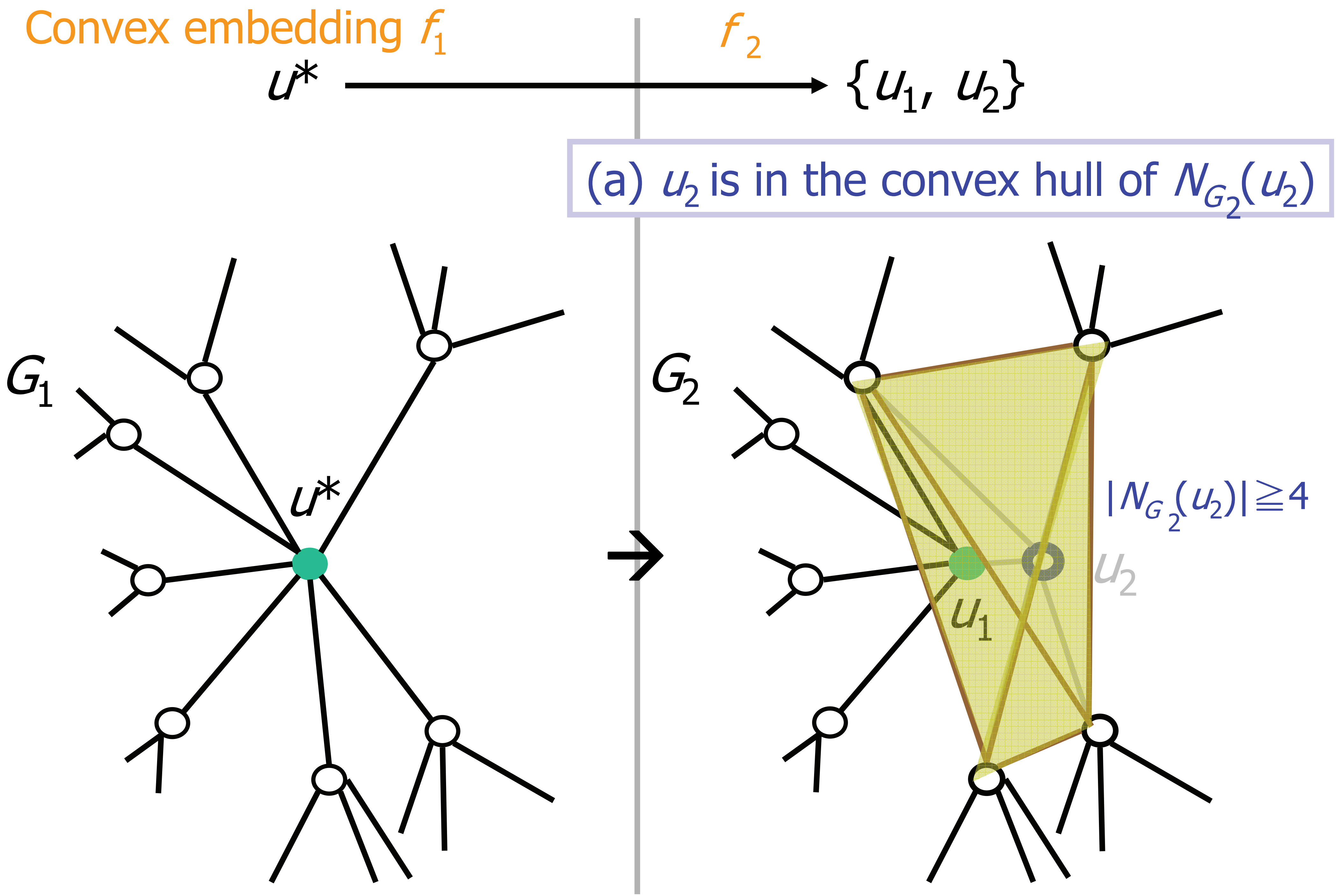
ii) $f_2(u_1) = f_1(u^*)$

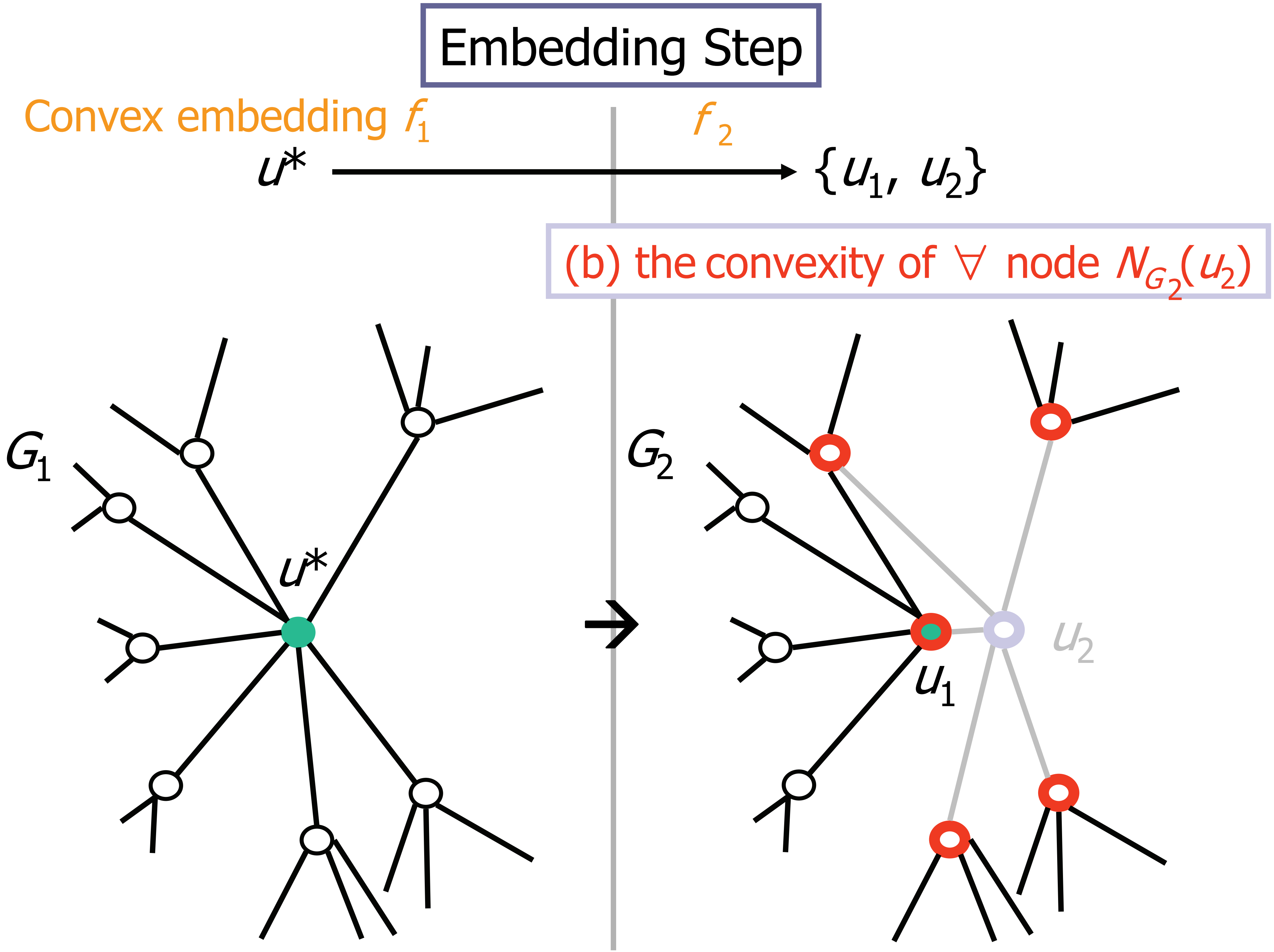



Embedding Step
Convex embedding f_1
f_2
u^*
$\{u_1, u_2\}$
Finding a position for u_2
G_1
u^*
G_2
u_1
u_2


Embedding Step
Convex embedding f_1
f_2
u^*
$\{u_1, u_2\}$
(a) u_2 is in the convex hull of $N_{G_2}(u_2)$
G_1
u^*
G_2
u_1
u_2

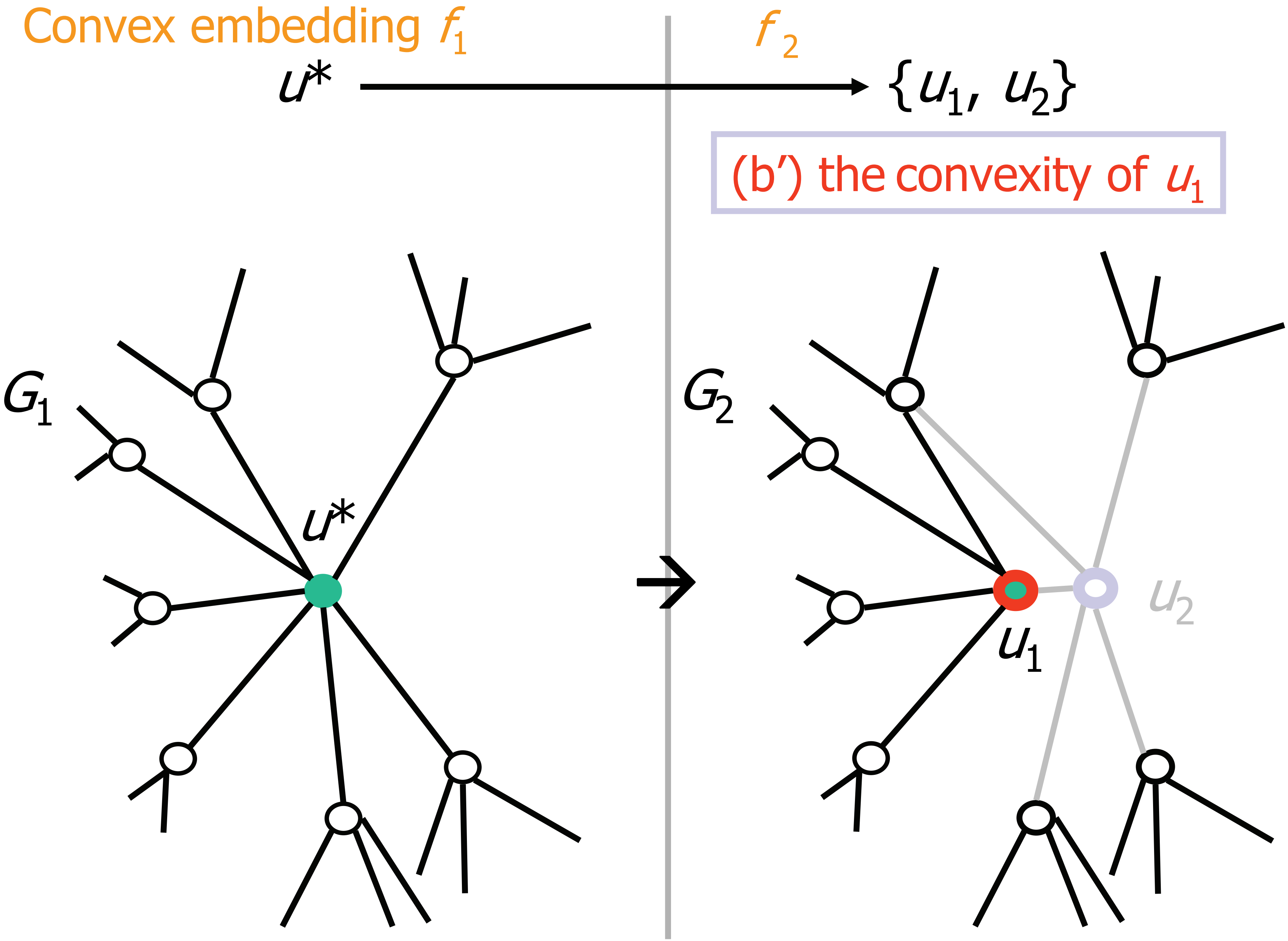
Embedding Step

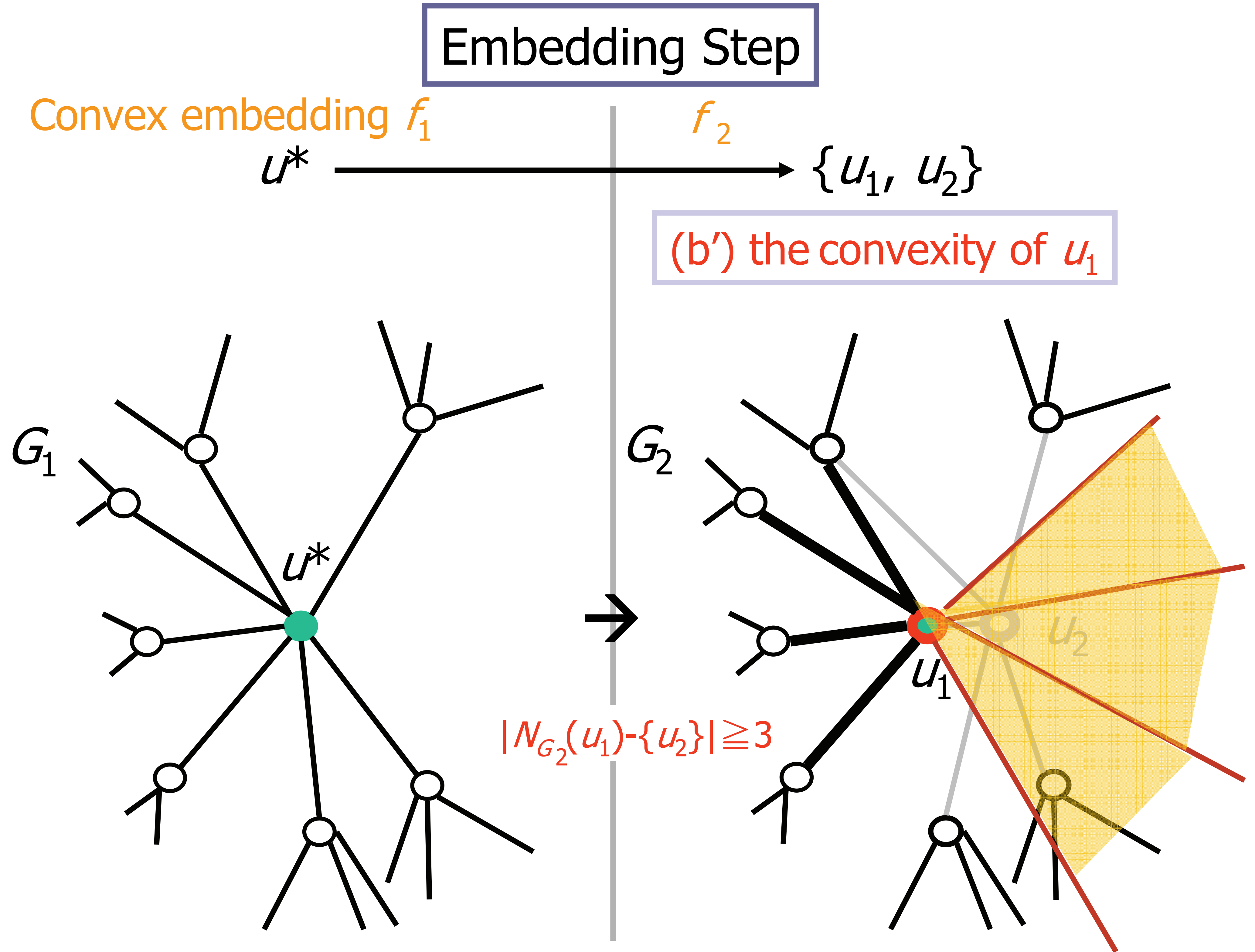


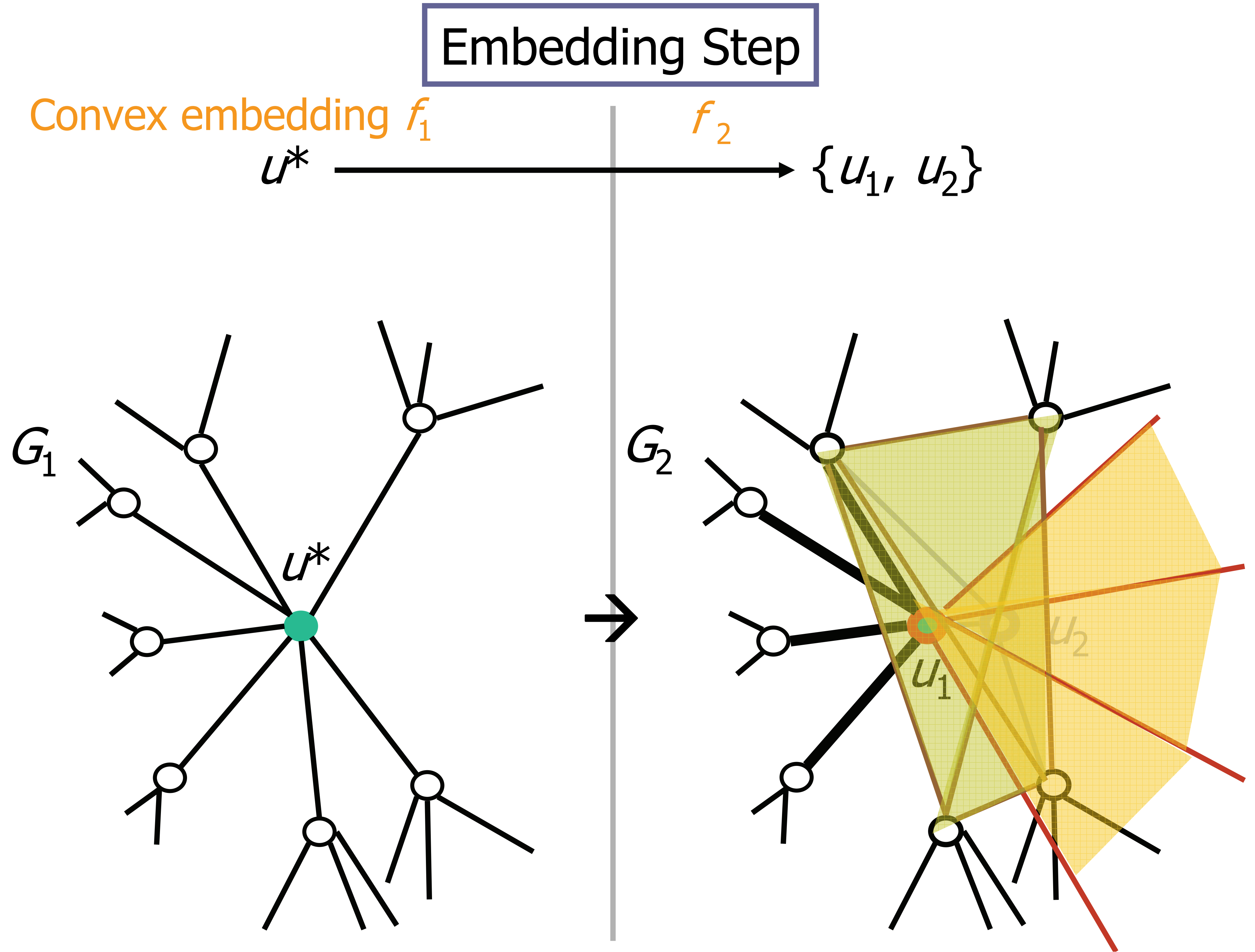


Embedding Step
Convex embedding f_1
f_2
u^*
$\{u_1, u_2\}$
(b) the convexity of ∀ node $N_{G_2}(u_2)$
G_1
u^*
G_2
u_1
u_2

Embedding Step



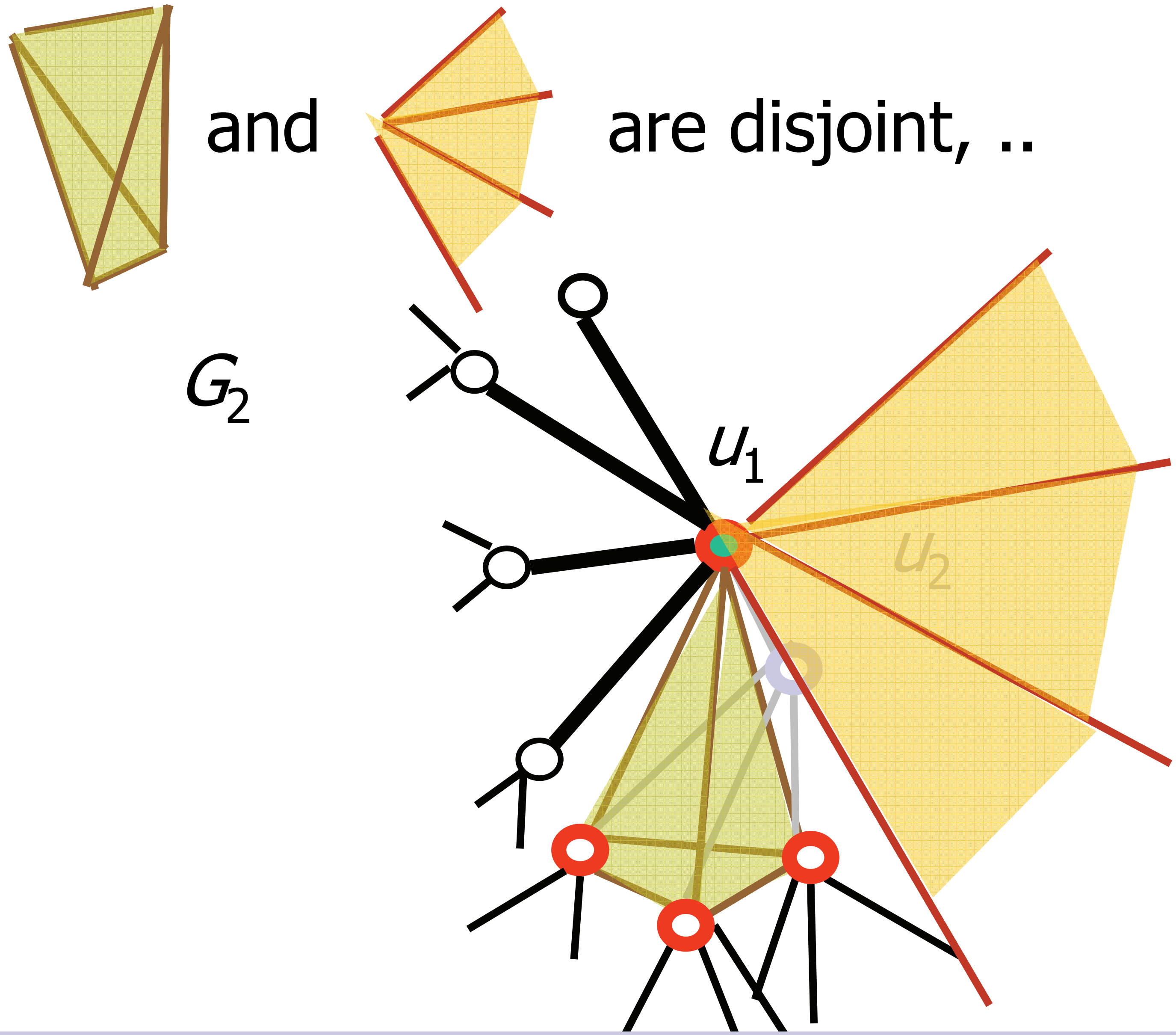

Embedding Step
Convex embedding f_1
f_2
u^*
$\{u_1, u_2\}$
(b') the convexity of u_1
G_1
u^*
G_2
u_1
u_2
$|N_{G_2}(u_1)-\{u_2\}| \geqq 3$



Embedding Step
Convex embedding f_1
f_2
u^*
$\{u_1, u_2\}$
G_1
u^*
G_2
u_1
u_2

Embedding Step

If [figure] and [figure] are disjoint, ..



⇒ In G_1, u^* cannot be included in the convex hull of $N_{G_1}(u^*)$.

⇒ contradicting that f_1 is a convex-embedding.

Our Results

3-bipartition

(1) 5-vertex-connectivity does not suffice!

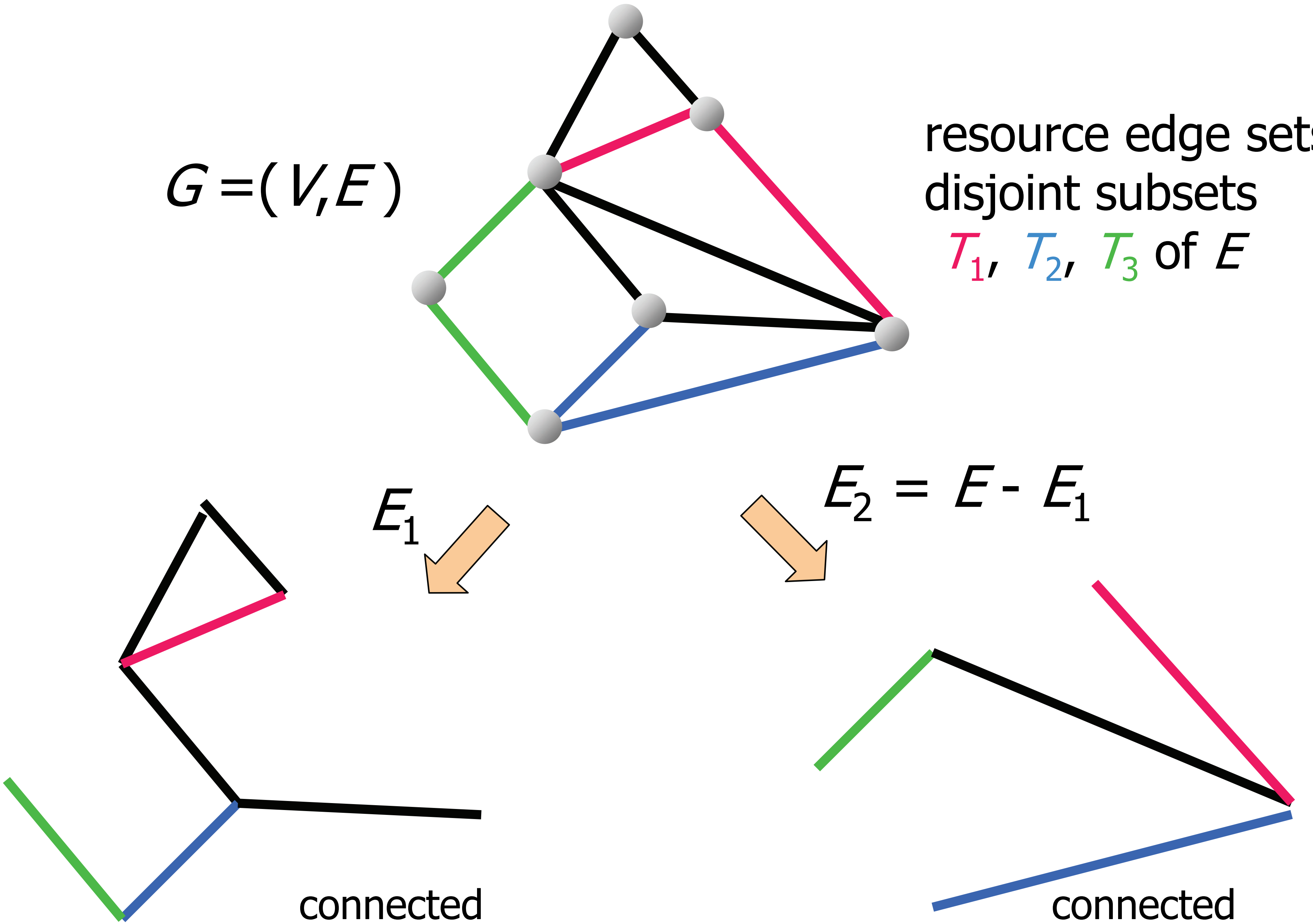
(2) 4-vertex-connectivity suffices if K4 is contained.

(3) For the edge version of k-bipartition (k=1,2,3), (k+1)-edge-connectivity suffices.

Edge-Version

$G = (V, E)$

resource edge sets:
disjoint subsets
T_1, T_2, T_3 of E



Edge-Version

Input: a graph and subsets T_i of resource edge sets
Output: a bipartition $\{E_1, E_2\}$ of E
s.t. $|E_1 \cap T_i| = |E_2 \cap T_i|$
E_1 and E_2 induce connected graphs.

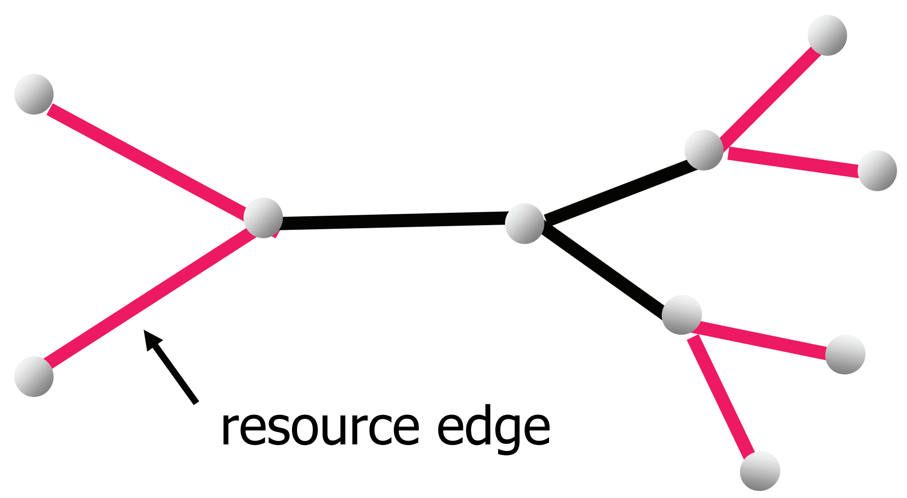
For the edge version of k-bipartition (k=1,2,3), (k+1)-edge-connectivity suffices.

G --> Line graph $L(G)$

(k+1)-edge-connected --> (k+1)-vertex-connected & K_{k+1}

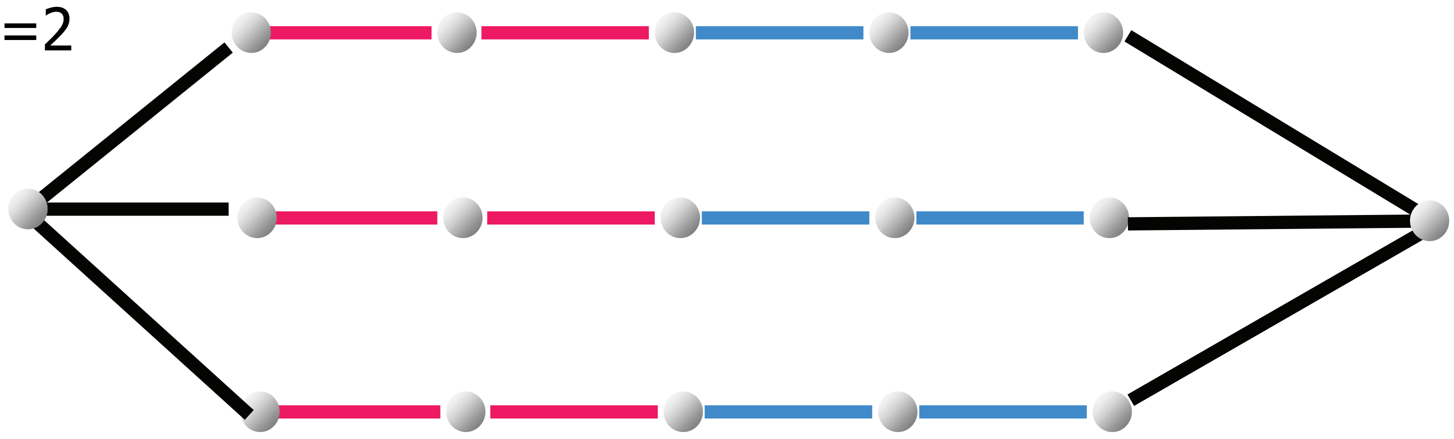
A 1-edge-connected graph which has no 1-bipartition of E

$k=1,\ \lambda=1$

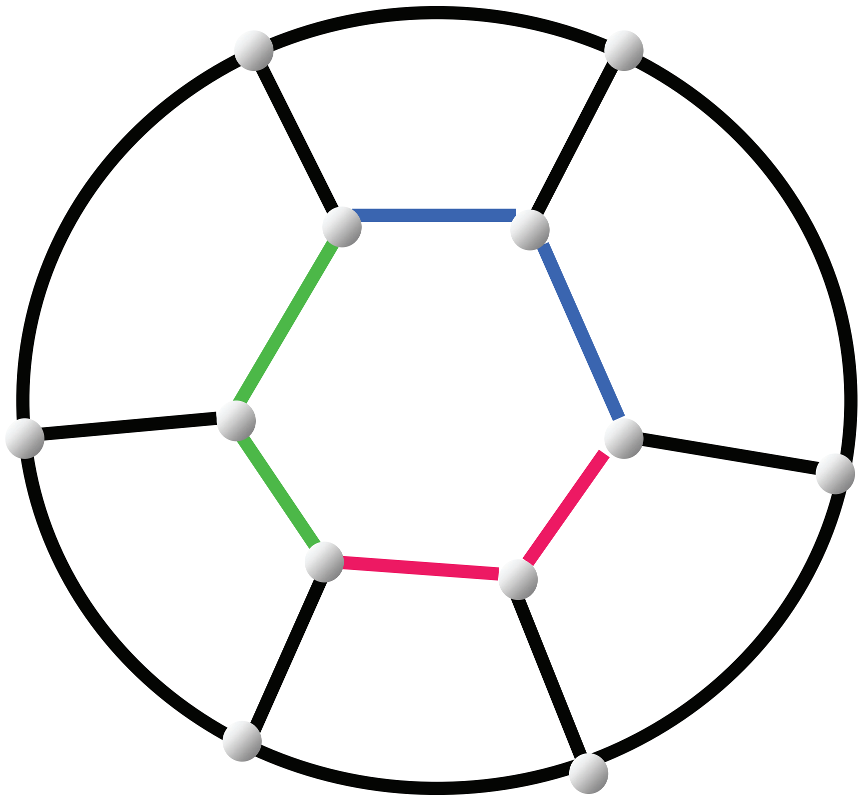


A 2-edge-connected graph which has no 2-bipartition of E

$k=2,\ \lambda=2$



A 3-edge-connected graph which has no 3-bipartition of E



24 vertices
deg=5
κ =5

What we have done is ...

Every 4-vertex-connected graph G admits a 3-bipartition if G has a K_4

5-vertex-connecitivity does not suffice for 3-bipartition
5-vertex-connecitivity does not suffice for 4-bipartition
5-vertex-connecitivity does not suffice for 5-bipartition

The vertex version implies the edge version.

Every $(k+1)$-edge-connected graph G admits a k-bipartition of E $(k=1,2,3)$.

Open Problems

·Sufficient condition for which a k-bipartition exists

Conjecture

Every (k+1)-vertex-connected graph with K_{k+1} admits a k-bipartition.

the edge version

Conjecture

Every (k+1)-edge-connected graph admits a k-bipartition.

Open Problem

Define $f(k)$ be the smallest p such that every p-vertex-connected graph admits a k-bipartition.

$f(1)=2$, $f(2)=3$

For $k>5$, prove $f(k) \geqq k+1$.

$f(3) \geqq 6$, $f(4) \geqq 6$, $f(5) \geqq 6$

For $k>3$, bound $f(k)$ from above by k+constant.

$f(k) = O(\Sigma |T_i|)$

The same questions for the edge version.