Approximate Distance Oracles
and
Spanners with sublinear surplus

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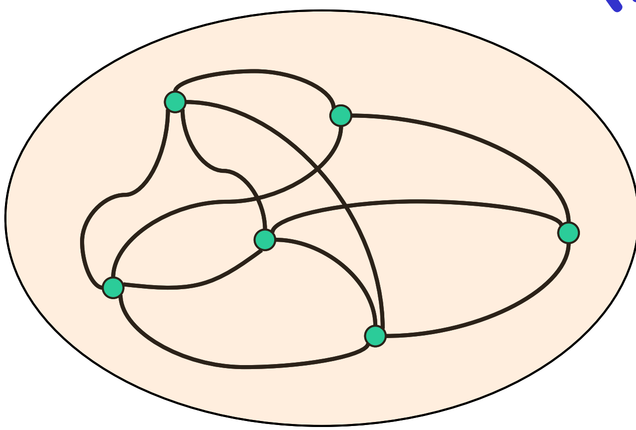
Approximate Distance Oracles (TZ'01)



Weighted undirected graph

APSP algorithm

mn time

n^2 space

n by n distance matrix

$mn^{1/k}$ time

$n^{1+1/k}$ space

Compact data structure

u,v

$O(1)$ query time

stretch $2k-1$

$\delta'(u,v)$

Stretch-Space tradeoff is essentially optimal!

Approximate Shortest Paths

Let $\delta(u,v)$ be the distance from u to v.

Multiplicative error

An estimated distance $\delta'(u,v)$ is of **stretch** t iff

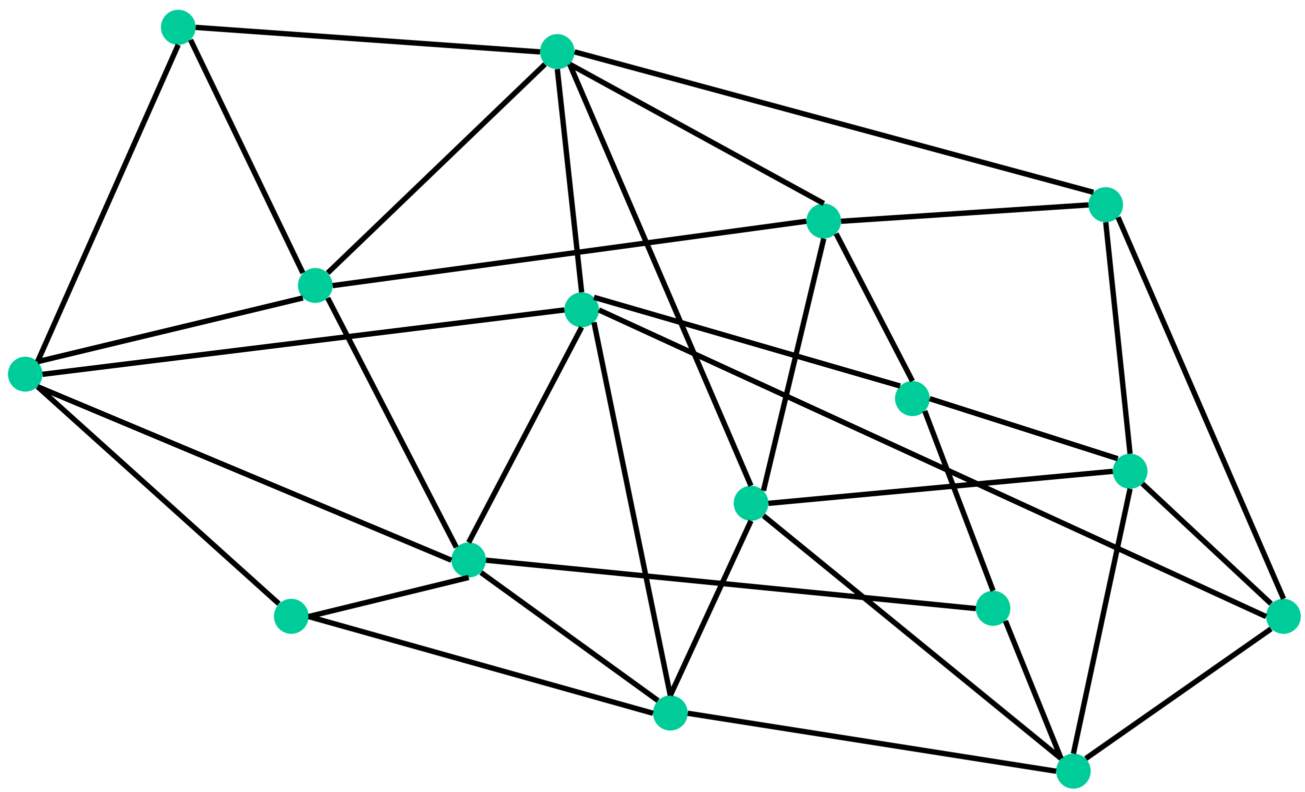
$$\delta(u,v) \le \delta'(u,v) \le t \cdot \delta(u,v)$$

Additive error

An estimated distance $\delta'(u,v)$ is of **surplus** t iff

$$\delta(u,v) \le \delta'(u,v) \le \delta(u,v) + t$$

Spanners



Given an **arbitrary** dense graph, can we always find a relatively **sparse subgraph** that approximates **all** distances fairly well?

Spanners [PU'89,PS'89]

Let $G=(V,E)$ be a weighted undirected graph.

A subgraph $G'=(V,E')$ of G is said to be a t-spanner of G iff $\delta_{G'}(u,v) \leq t\,\delta_G(u,v)$ for every u,v in V.

Theorem:

Every weighted undirected graph has a $(2k-1)$-spanner of size $O(n^{1+1/k})$. [ADDJS '93]

Furthermore, such spanners can be constructed deterministically in linear time. [BS '04] [TZ '04]

The size-stretch trade-off is essentially optimal.

(Assuming there are graphs with $\Omega(n^{1+1/k})$ edges of girth $2k+2$, as conjectured by Erdös and others.)

Additive Spanners

Let $G=(V,E)$ be a unweighted undirected graph.

A subgraph $G'=(V,E')$ of G is said to be an additive t-spanner if G iff $\delta_{G'}(u,v) \le \delta_G(u,v) + t$ for every $u,v \in V$.

Theorem: Every unweighted undirected graph has an additive 2-spanner of size $O(n^{3/2})$. [ACIM '96] [DHZ '96]

Theorem: Every unweighted undirected graph has an additive 6-spanner of size $O(n^{4/3})$. [BKMP '04]

Major open problem

Do all graphs have additive spanners with only $O(n^{1+\varepsilon})$ edges, for every $\varepsilon>0$?

Spanners with sublinear surplus

Theorem:

For every $k>1$, every undirected graph $G=(V,E)$ on n vertices has a subgraph $G'=(V,E')$ with $O(n^{1+1/k})$ edges such that for every $u,v\in V$, if $\delta_G(u,v)=d$, then $\delta_{G'}(u,v)=d+O(d^{1-1/(k-1)})$.

$$d \Rightarrow d+O(d^{1-1/(k-1)})$$

Extends and simplifies a result of Elkin and Peleg (2001)

All sorts of spanners

A subgraph $G'=(V,E')$ of G is said to be a functional f-spanner if G iff $\delta_{G'}(u,v) \leq f(\delta_G(u,v))$ for every $u,v \in V$.

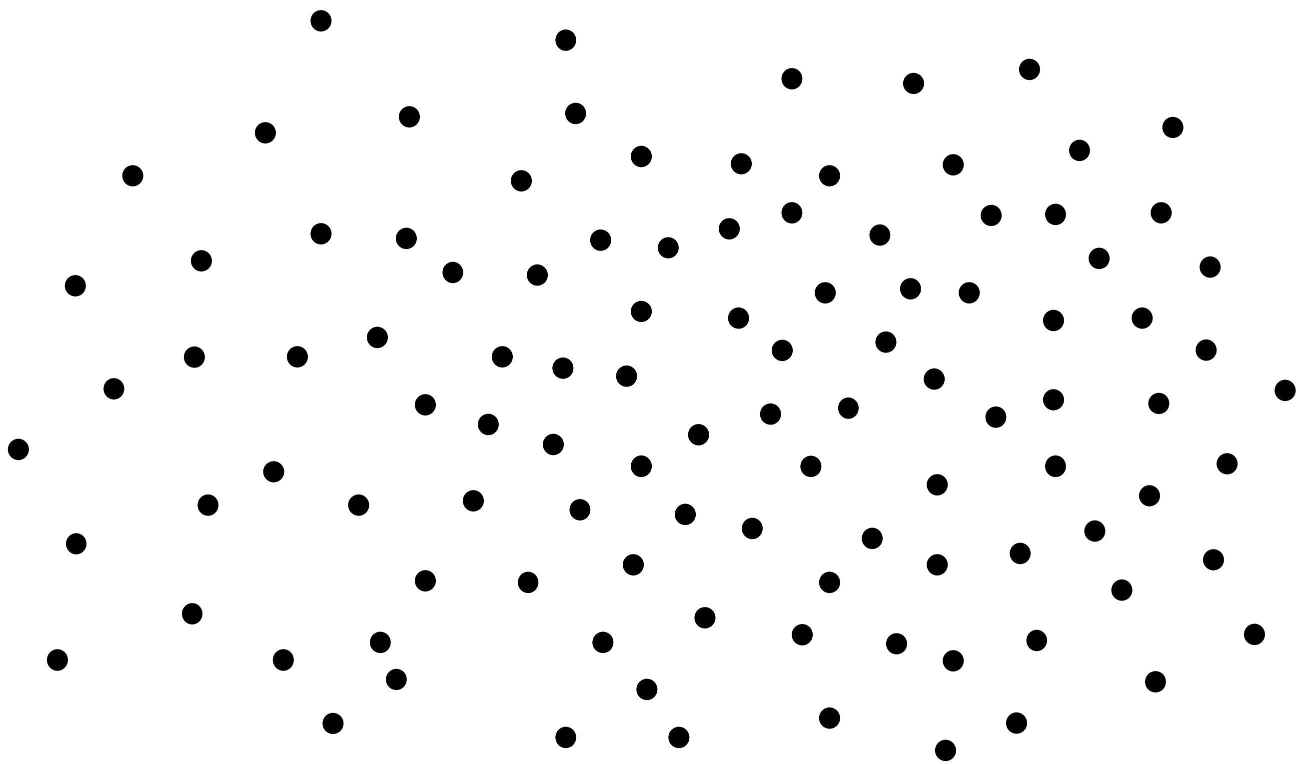
size	$f(d)$	reference
$n^{1+1/k}$	$(2k-1)d$	[ADDJS '93]
$n^{3/2}$	$d+2$	[ACIM '96] [DHZ '96]
$n^{4/3}$	$d+6$	[BKMP '04]
$\beta n^{1+\delta}$	$(1+\varepsilon)d+\beta(\varepsilon,\delta)$	[EP '01]
$n^{1+1/k}$	$d+O(d^{1-1/(k-1)})$	[TZ '05]

Part I

Approximate Distance Oracles

Approximate Distance Oracles [TZ'01]

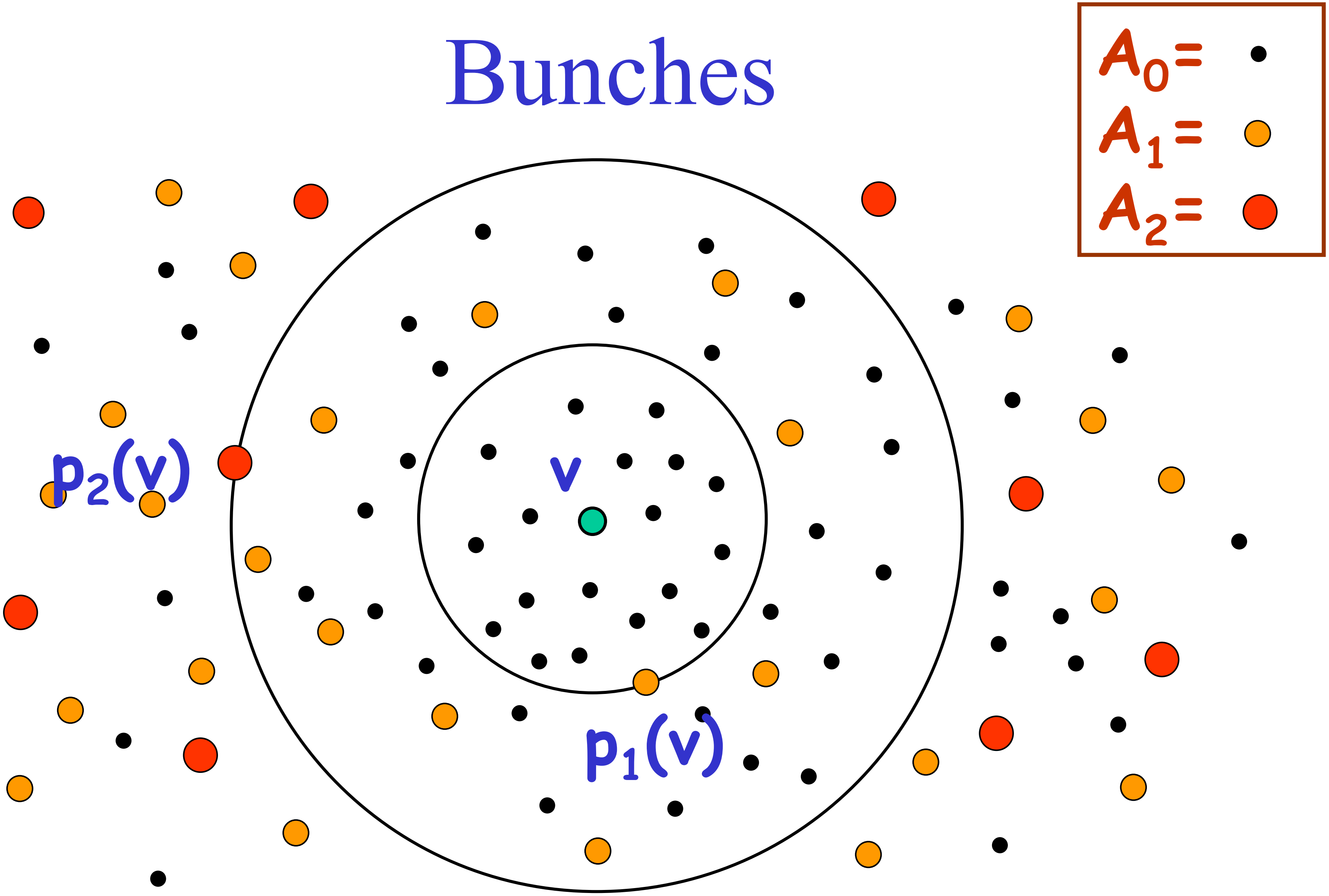
A hierarchy of centers



$A_0 \leftarrow V$; $A_k \leftarrow \emptyset$;

$A_i \leftarrow \text{sample}(A_{i-1}, n^{-1/k})$;

Bunches



$$B(v) \leftarrow \bigcup_i \{ w \in A_i - A_{i+1} \mid \delta(w,v) < \delta(A_{i+1},v) \}$$

Lemma: $\mathrm{E}[|B(v)|] \le kn^{1/k}$

Proof: $|B(v) \cap A_i|$ is stochastically dominated by a geometric random variable with parameter $p=n^{-1/k}$.

The data structure

Keep for every vertex $v \in V$:

- The centers $p_1(v), p_2(v), \ldots, p_{k-1}(v)$
- A **hash table** holding $B(v)$

For every $w \in V$, we can check, in **constant time**, whether $w \in B(v)$, and if so, what is $\delta(v,w)$.

Query answering algorithm

Algorithm $\text{dist}_k(u,v)$

$w \leftarrow u$, $i \leftarrow 0$

while $w \notin B(v)$

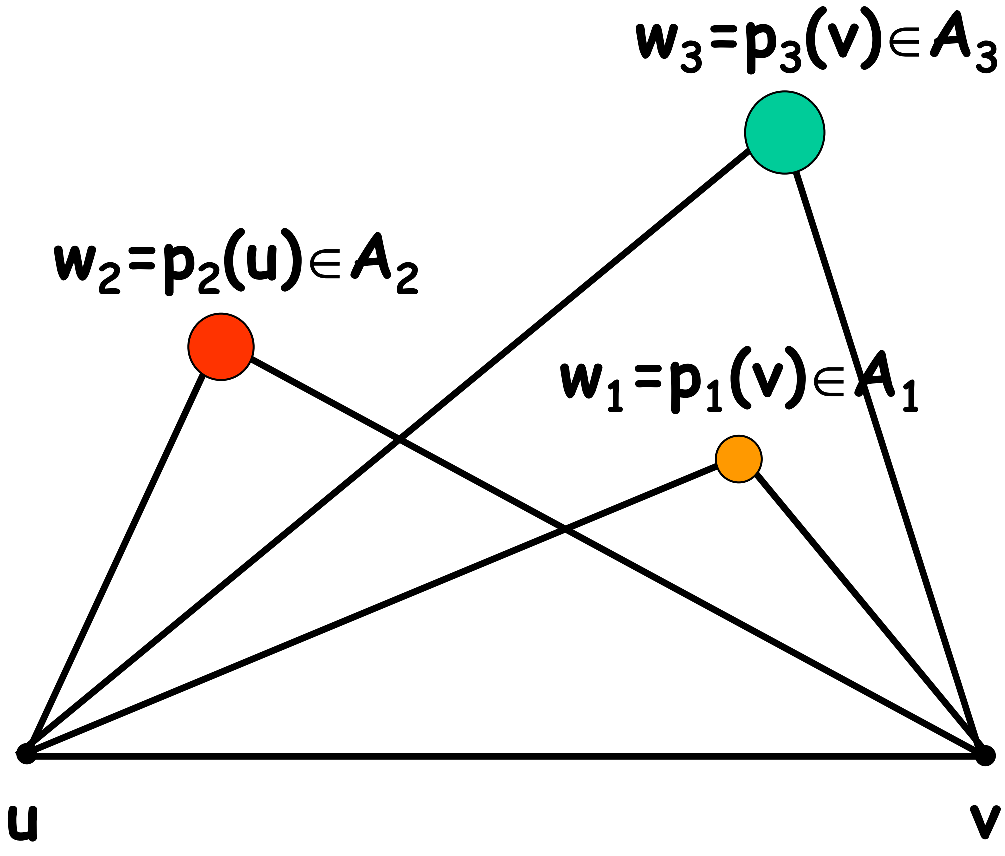
{ $i \leftarrow i+1$

$(u,v) \leftarrow (v,u)$

$w \leftarrow p_i(u)$ }

return $\delta(u,w) + \delta(w,v)$

Query answering algorithm



Analysis

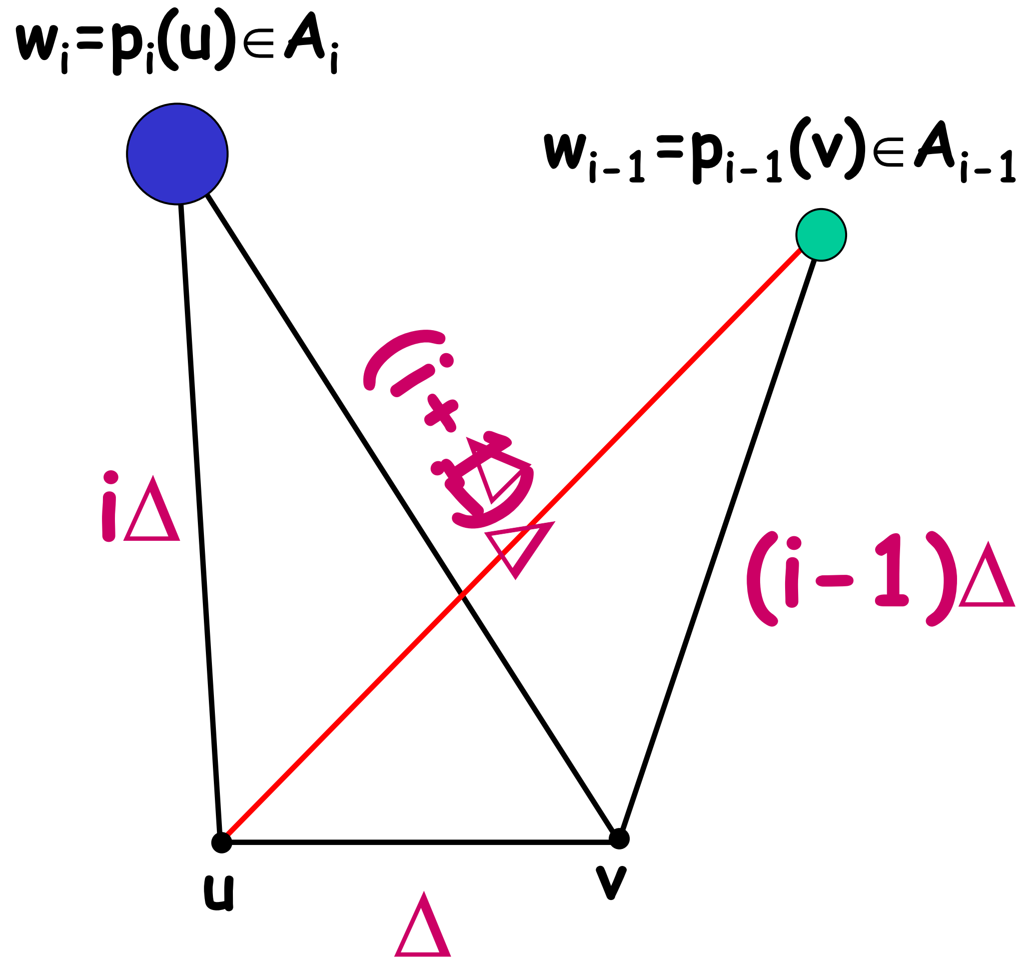
<u>Claim 1:</u>

$\delta(u,w_i) \le i\Delta$, i even

$\delta(v,w_i) \le i\Delta$, i odd

<u>Claim 2:</u>

$$\delta(u,w_i) + \delta(w_i,v) \le (2i+1)\Delta \le (2k-1)\Delta$$

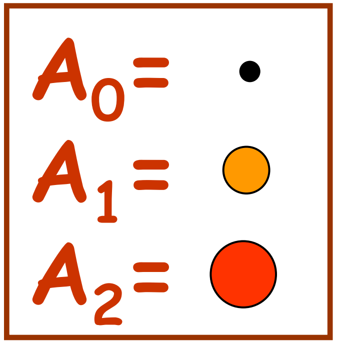


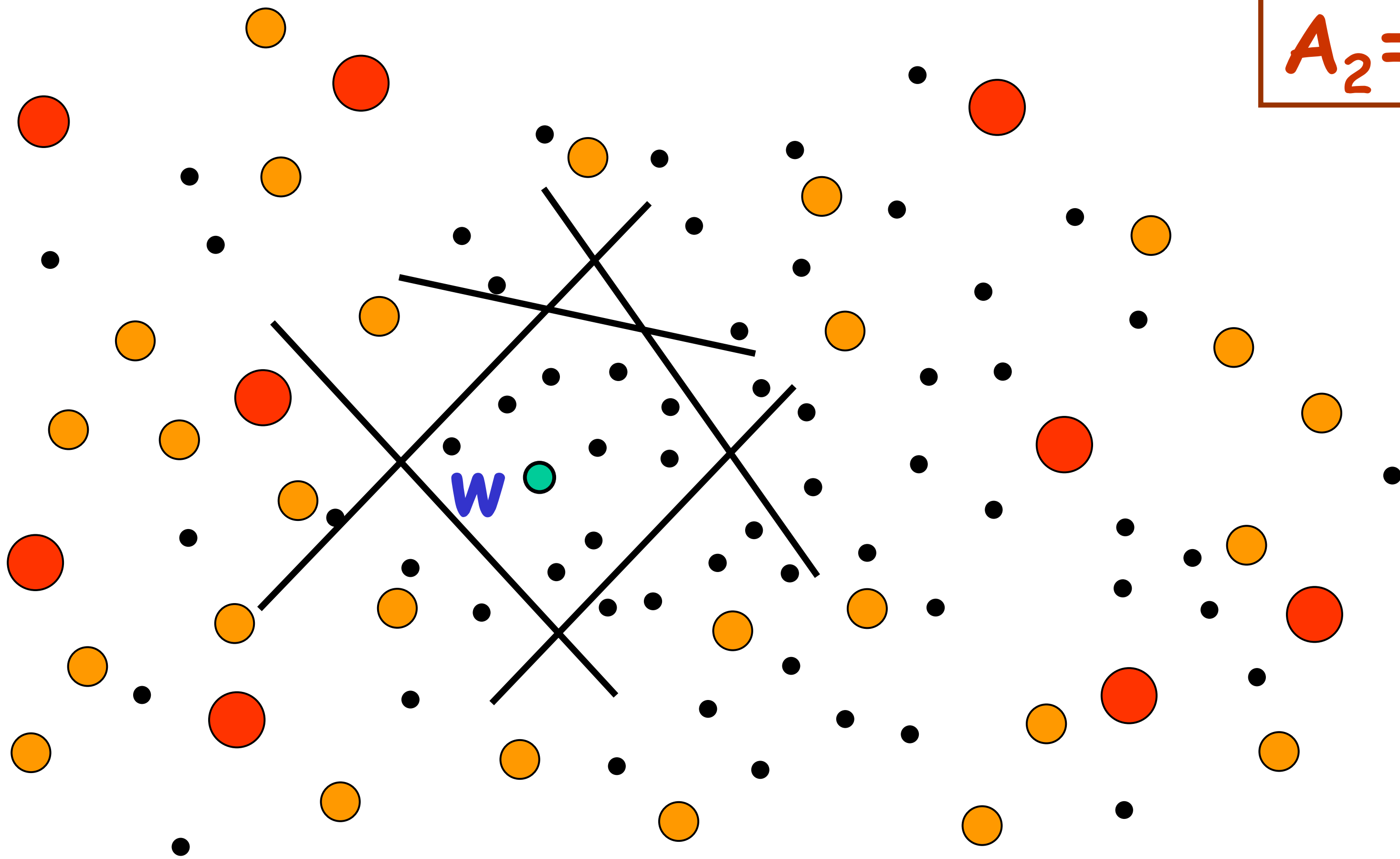
Where are the spanners?

Define clusters, the "dual" of bunches.

For every $u \in V$, include in the spanner a tree of shortest paths from u to all the vertices in the cluster of u.

Clusters





$$C(w) \leftarrow \{v \in V \mid \delta(w,v) < \delta(A_{i+1},v)\} \quad , \quad w \in A_i - A_{i+1}$$

Bunches and clusters

$$w \in B(v) \quad \Leftrightarrow \quad v \in C(w)$$

$$C(w) \leftarrow \{v \in V \mid \delta(w,v) < \delta(A_{i+1},v)\} \quad ,$$

$$if \;\; w \in A_i - A_{i+1}$$

$$B(v) \leftarrow \bigcup_i \{w \in A_i - A_{i+1} \mid \delta(w,v) < \delta(A_{i+1},v)\}$$

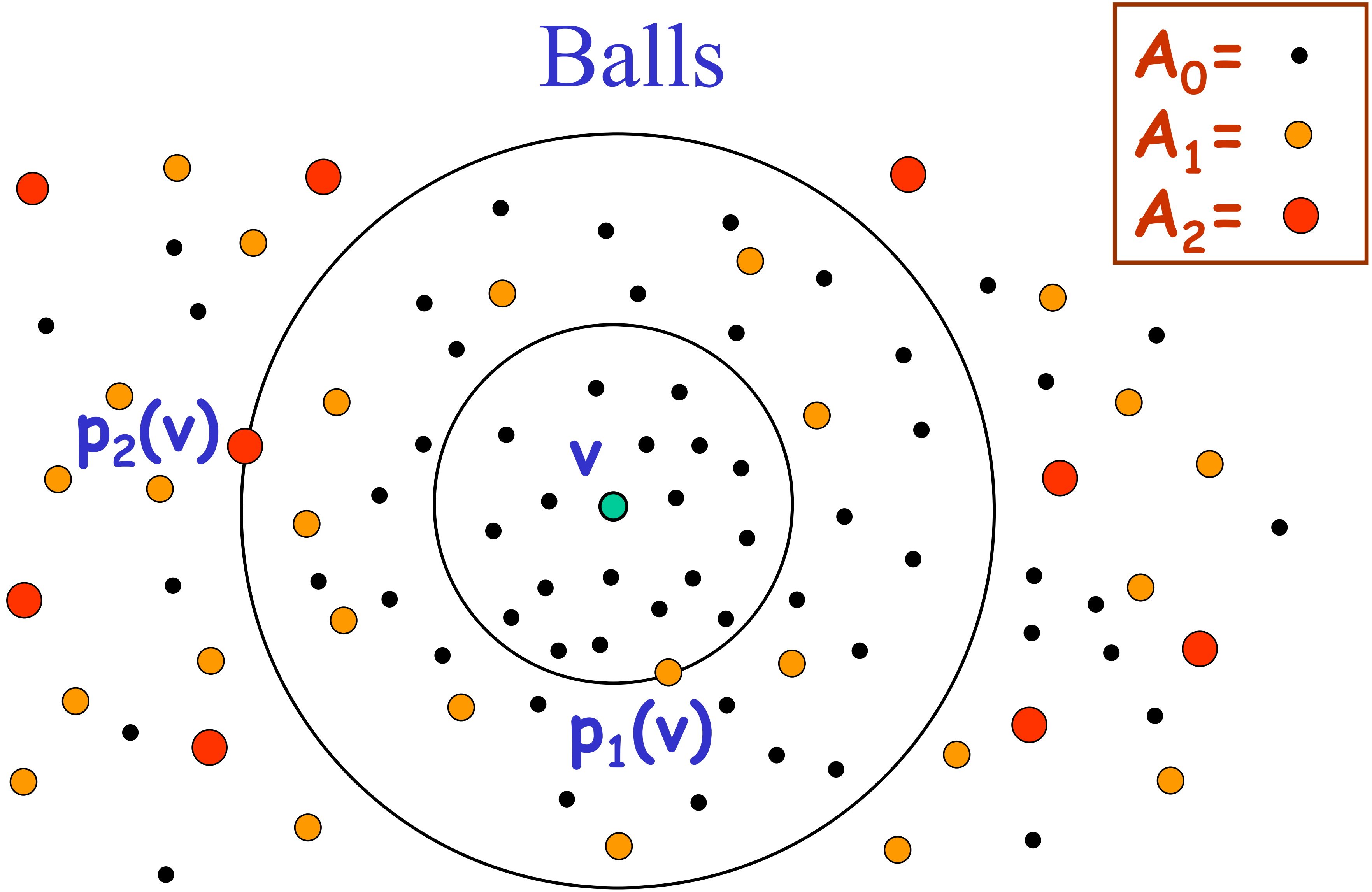
Part II

Spanners with sublinear surplus

The construction used above, when applied to unweighted graphs, produces spanners with sublinear surplus!

We present a slightly modified construction with a slightly simpler analysis.

Balls



$$Ball(u) = \{v \in V \mid \delta(u,v) < \delta(u, A_{i+1})\} \ \ , \ \ u \in A_i - A_{i+1}$$

$$Ball[u] = Ball(u) \cup \{p_{i+1}(u)\} \ \ , \ \ u \in A_i - A_{i+1}$$

The original construction

Select a hierarchy of centers $A_0 \supset A_1 \supset \ldots \supset A_{k-1}$.

For every $u \in V$, add to the spanner a shortest paths tree of $Clust(u)$.

The modified construction

Select a hierarchy of centers $A_0 \supset A_1 \supset \ldots \supset A_{k-1}$.

For every $u \in V$, add to the spanner a shortest paths tree of $Ball(u)$.

Spanners with sublinear surplus

Select a hierarchy of centers $A_0 \supset A_1 \supset \ldots \supset A_{k-1}$.

For every $u \in V$, add to the spanner a shortest paths tree of $Ball(u)$.

The path-finding strategy

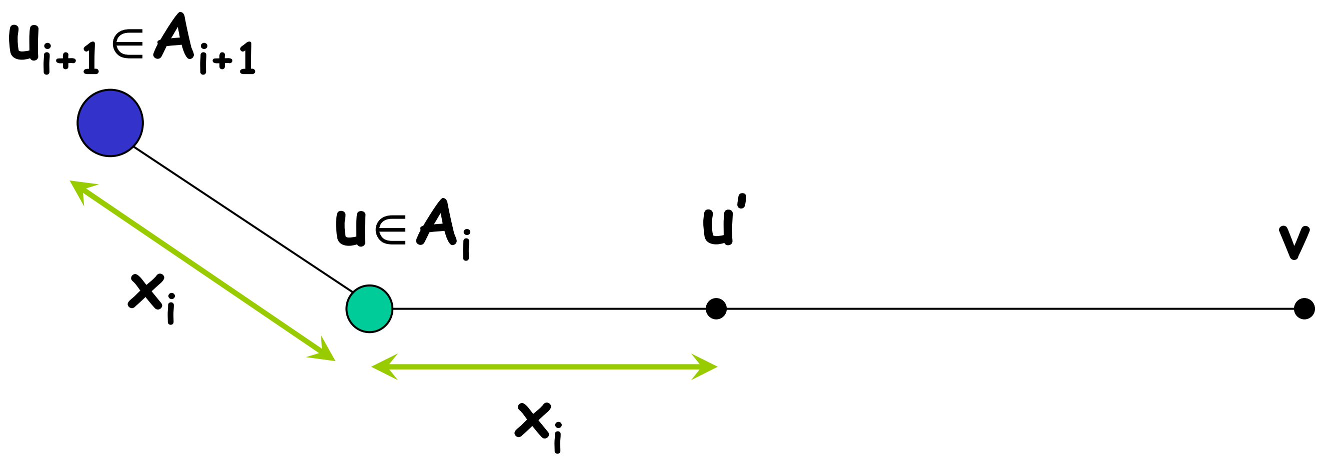
Suppose we are at $u \in A_i$ and want to go to v.

Let Δ be an integer parameter.

If the first $x_i = \Delta^i - \Delta^{i-1}$ edges of a shortest path from u to v are in the spanner, then use them.

Otherwise, head for the $(i+1)$-center u_{i+1} nearest to u.

► The distance to u_{i+1} is at most x_i. (As $u' \notin \mathrm{Ball}(u)$.)

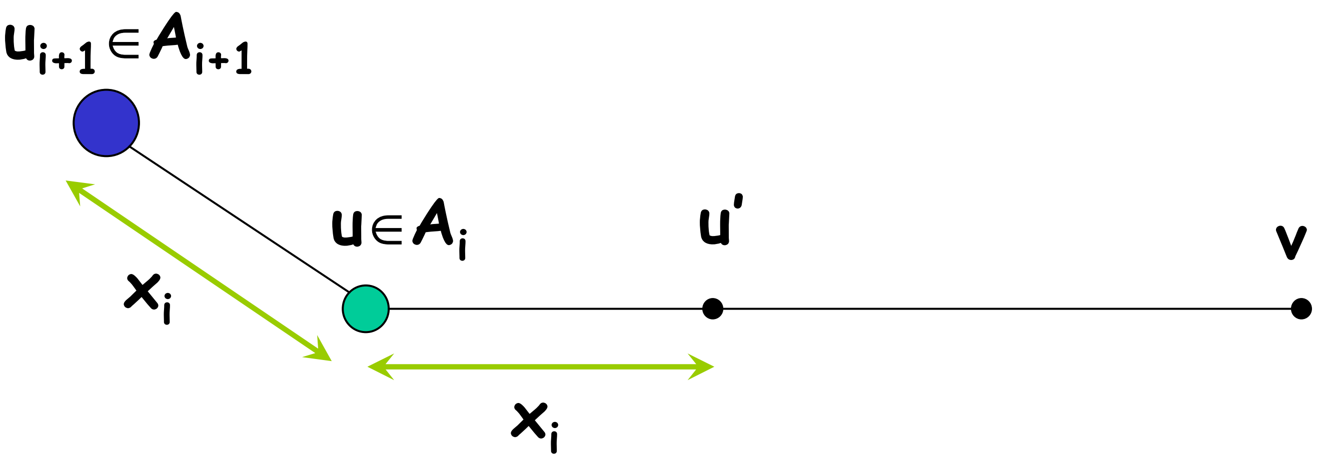


The path-finding strategy

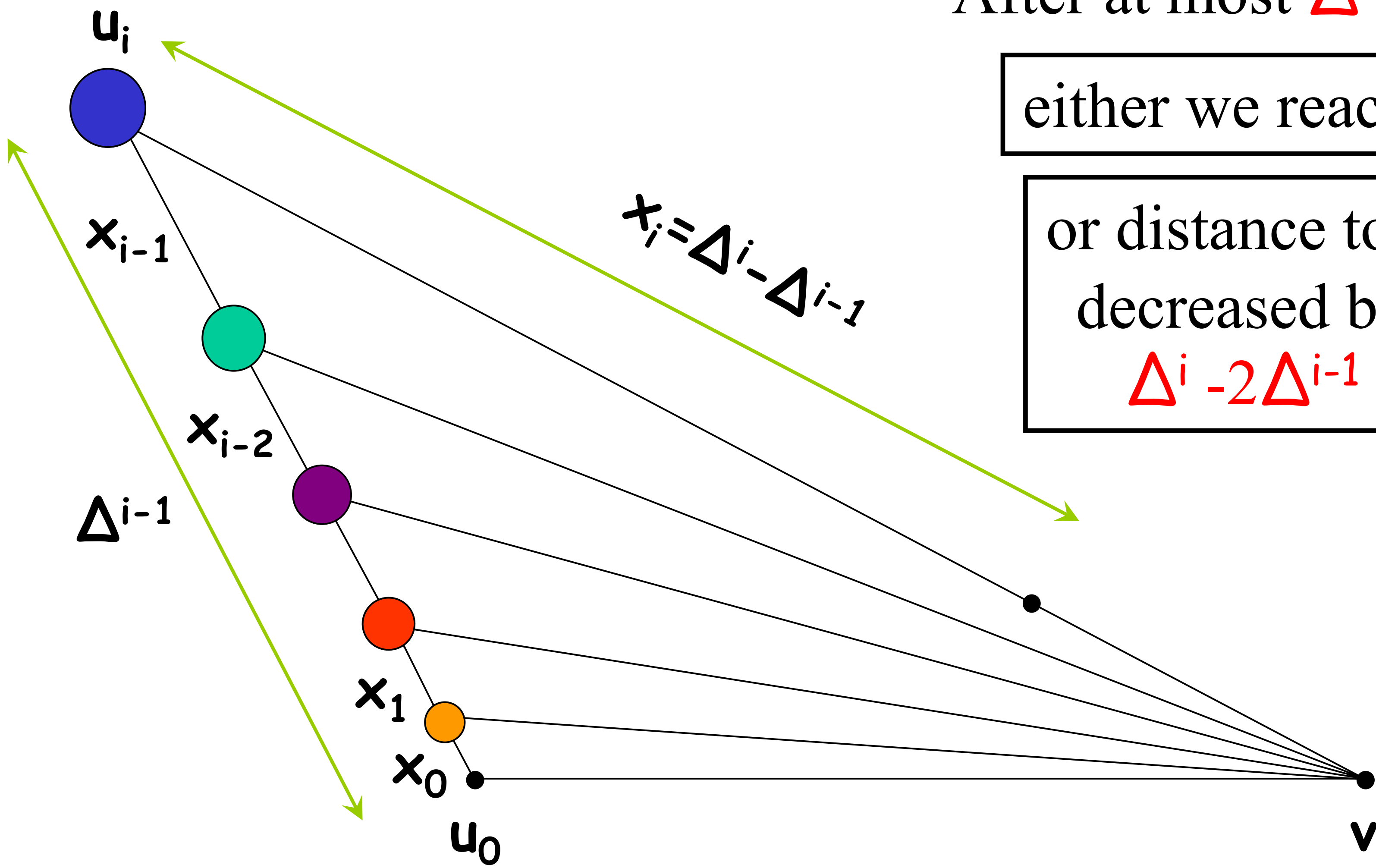
We either reach v, or at least make $x_i=\Delta^i-\Delta^{i-1}$ steps in the right direction.

Or, make at most $x_i=\Delta^i-\Delta^{i-1}$ steps, possibly in a wrong direction, but reach a center of level $i+1$.

If $i=k-1$, we will be able to reach v.



The path-finding strategy



After at most Δ^i steps:

either we reach v

or distance to v decreased by $\Delta^i - 2\Delta^{i-1}$

The path-finding strategy

After at most Δ^i steps:

either we reach v	$\Rightarrow$	Surplus $2\Delta^{i-1}$
or distance to v decreased by $\Delta^i - 2\Delta^{i-1}$	$\Rightarrow$	Stretch $\dfrac{\Delta^i}{\Delta^i - 2\Delta^{i-1}} = 1 + \dfrac{2}{\Delta - 2}$

The surplus is incurred only once!

$$\delta'(u,v) \leq \left(1 + \frac{2}{\Delta - 2}\right) \cdot \delta(u,v) + 2\Delta^{k-2}$$

Sublinear surplus

$$\delta'(u,v) \le \left(1+\frac{2}{\Delta-2}\right)\cdot\delta(u,v) + 2\Delta^{k-2}$$

$$\delta(u,v) = d \quad , \quad \Delta = \left\lceil d^{1/(k-1)} + 2 \right\rceil$$

$$\Downarrow$$

$$\delta'(u,v) \le d + O(d^{1-\frac{1}{k-1}})$$

Open problems

Arbitrarily sparse additive spanners?

Distance oracles with sublinear surplus?