Efficient Algorithms for the Longest Path Problem

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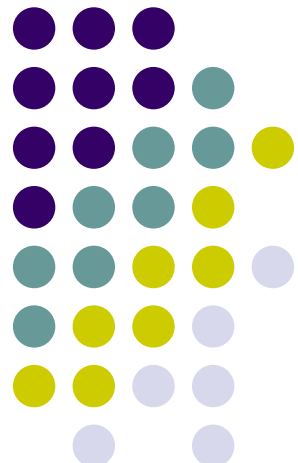


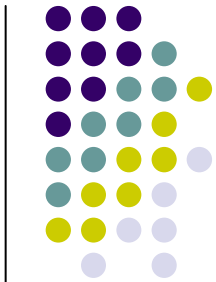
2005/3/1 NHC@Kyoto

http://www.jaist.ac.jp/~uehara/ps/longest.pdf

The Longest Path Problem

- Finding a longest (vertex disjoint) path in a given graph
- Motivation (comparing to Hamiltonian path):
  - … Approx. Algorithm, Parameterized Complexity
  - … More practical/natural
  - … More difficult(?)

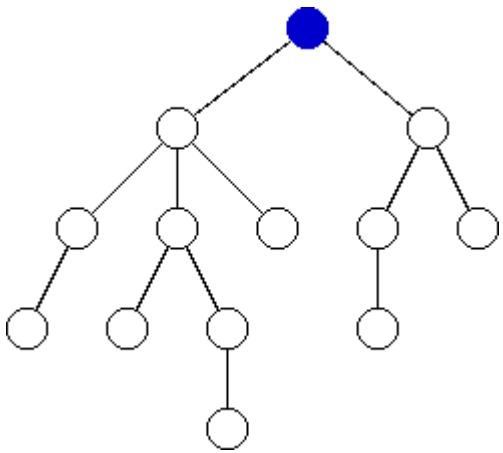
The Longest Path Problem



- Known (hardness) results;
  - We cannot find a path of length n-n^{ε} in a given Hamiltonian graph in poly-time unless P=NP [Karger, Motwani, Ramkumar; 1997]
  - We can find $O(\log n)$ length path [Alon, Yuster, Zwick;1995] ($\Rightarrow O((\log n/\mathrm{loglog}\, n)^2)$ [Björklund, Husfeldt; 2003])
  - Approx. Alg. achieves $O(n/\log n)$ [AYZ95] ($\Rightarrow O(n(\mathrm{loglog}\, n/\log n)^2)$[BH03])
  - Exponential algorithm [Monien 1985]

The Longest Path Problem

- Known polynomial time algorithm;
  - Dijkstra's Alg.(196?)：Linear alg. for finding a longest path in a tree;



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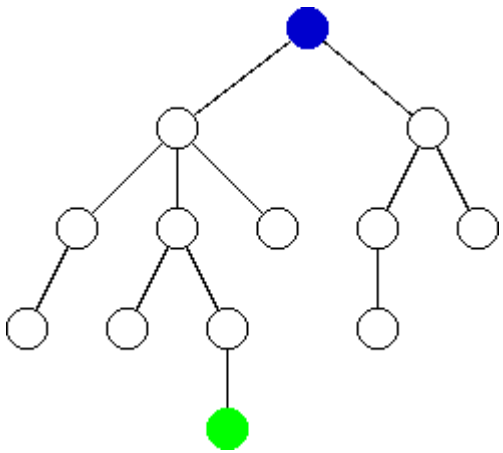

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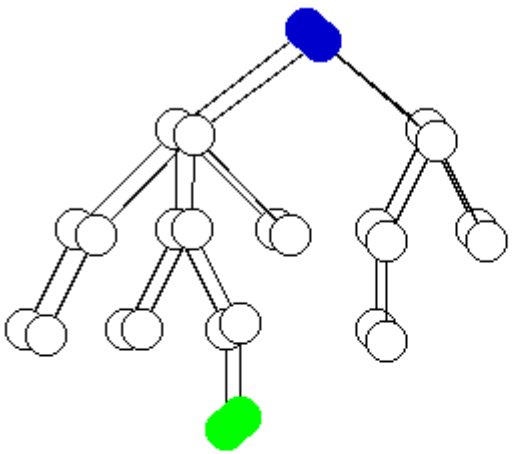

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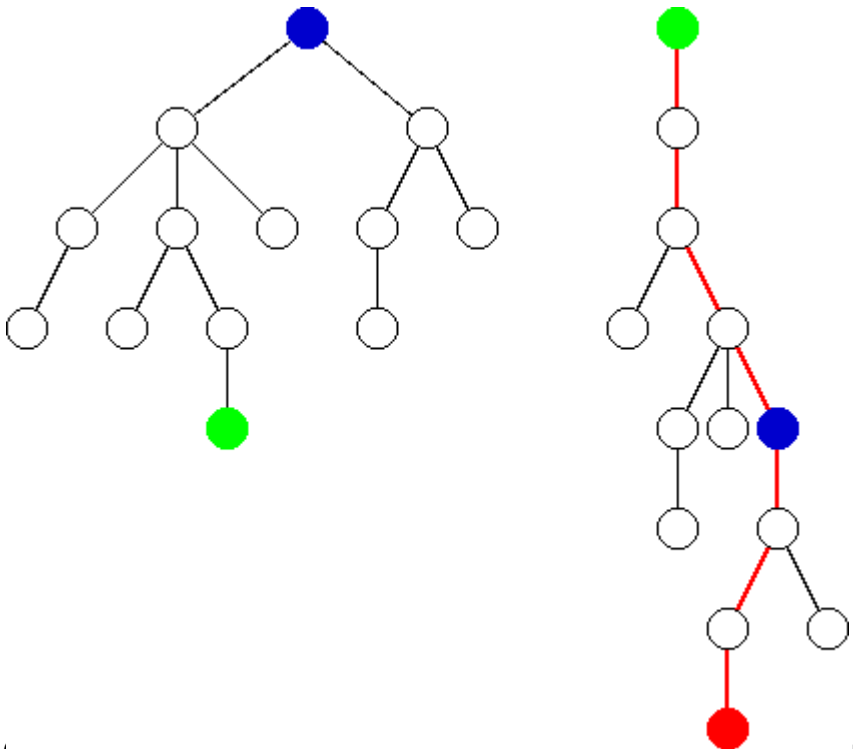
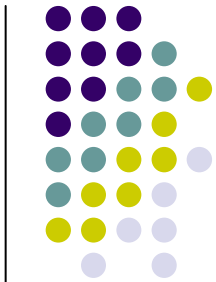


Approaches to the Efficient Algs to Longest Path Problem



1. Extension of the Dijkstra's algorithm
   - Weighted trees (linear), block graphs (linear), cacti (O(n^2)).

   (ISAAC 2004)
2. Graph classes s.t. Hamiltonian Path can be found in poly time
   - Some graph classes having interval representations (bipartite permutation, interval biconvex graphs)

   (ISAAC 2004)
3. Dynamic programming to the graph classes that have tree representations (on going)
   - Cacti(linear), …

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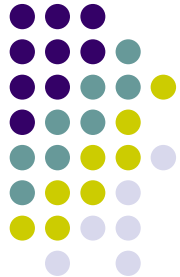

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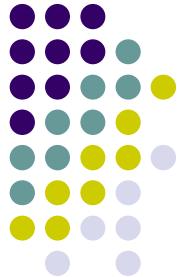
1. Ex of Dijkstra's Alg

Bulterman et.al. (*IPL*,2002) showed that the correctness of Dijkstra's alg stands for;

1. For each *u,v,*
   length of the shortest path between *u* and *v*
   = length of the longest path between *u and v*
2. For each *u,v,w*,
   $d(u,v) \leqq d(u,w) + d(w,v)$
3. For each *u,v,w,*
   $d(u,v) = d(u,w) + d(w,v)$ if and only if
   w is on the unique path between *u* and *v*

1. Ex of Dijkstra's Alg



Construct $G'=(V',E')$ from $G=(V,E)$ s.t.:

- $V \subseteq V'$
- For each $u,v \in V$,

  length of the shortest path between u,v on G'

  = length of the longest path between u,v on G
- For each $u,v \in V$,

  the shortest path between u,v on G' is unique

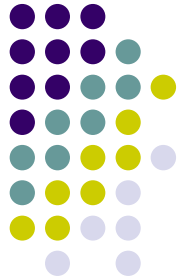
1. Ex of Dijkstra's Alg

Theorem: ExDijkstra finds a longest path if G and G' satisfy the conditions.

ExDijkstra: $G=(V,E)$ and $G'=(V',E')$

1. pick any vertex w in V;
2. find $x \in V$ with $\max\{d(w,x)\}$ on G';
3. find $y \in V$ with $\max\{d(x,y)\}$ on G';
4. x and y are the endpoints of the longest path in G, and $d(x,y)$ on G' is its length.

1. Ex of Dijkstra's Alg (Summary)



Theorem: Vertex/edge weighted tree (linear)

Theorem: Block graph ($O(|V|+|E|)$)

Theorem: Cactus ($O(|V|^2)$)

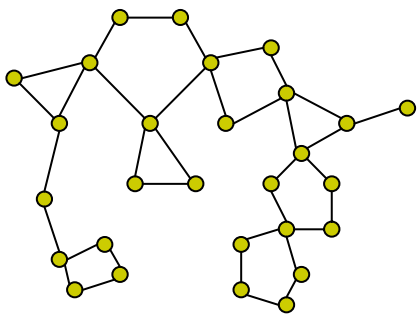
1. Ex of Dijkstra's Alg (Cacti)

Cactus:

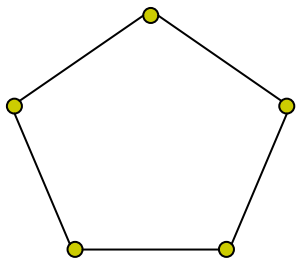
Each block is a cycle

Two cycle share at most one vertex which is a separator

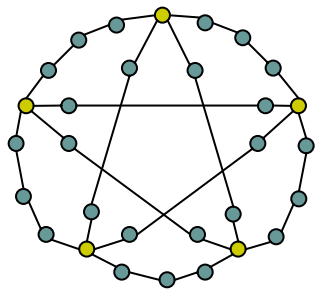
cactus



G



G'



The longest path between u and v on G = *The shortest path between u and v on G'*

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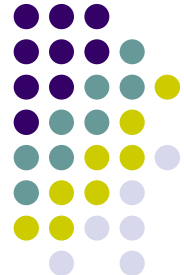
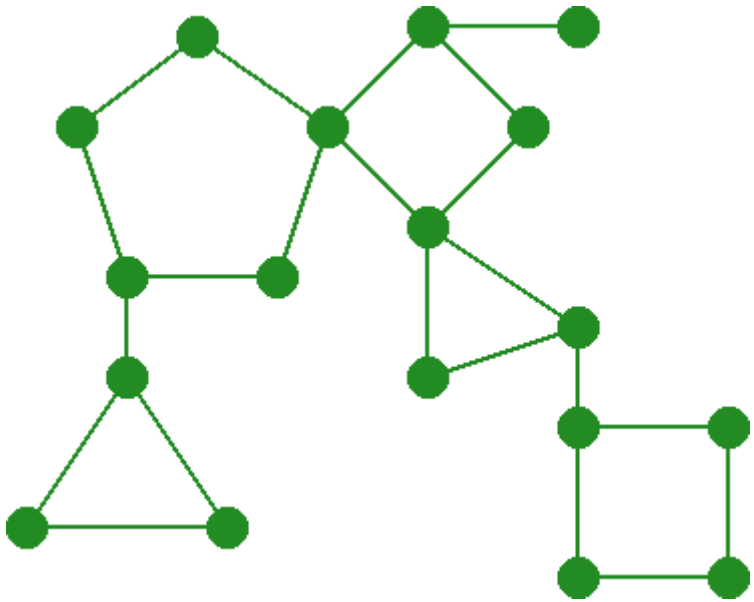
1. Ex of Dijkstra's Alg (Cacti)



Sample



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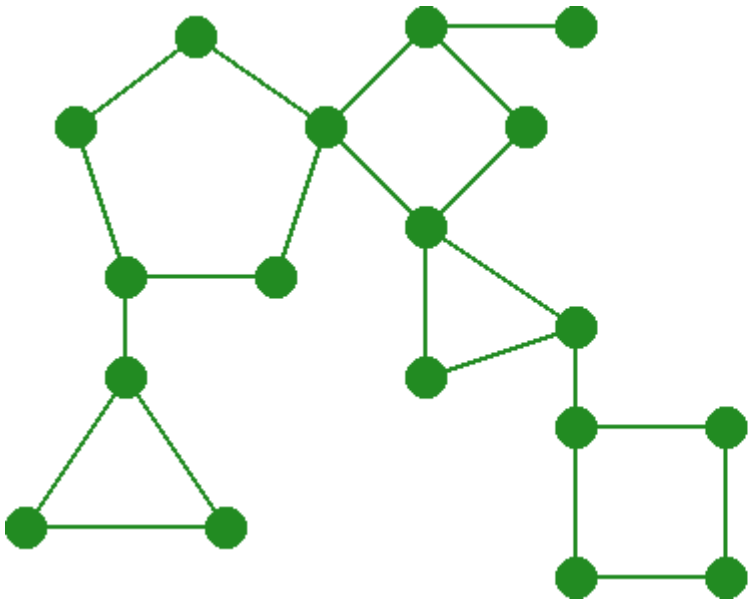

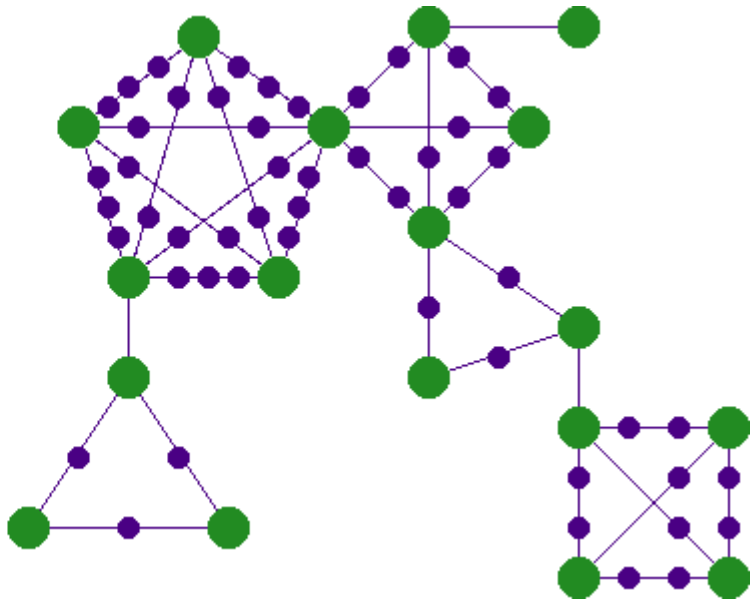

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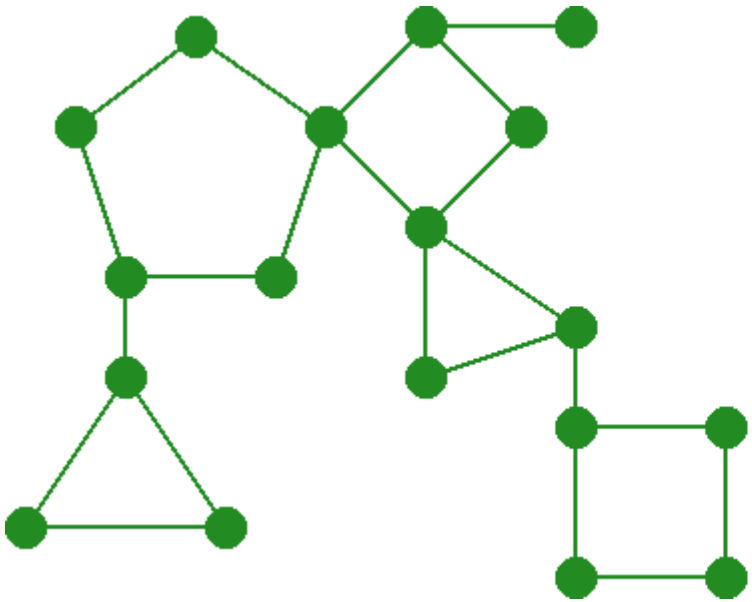

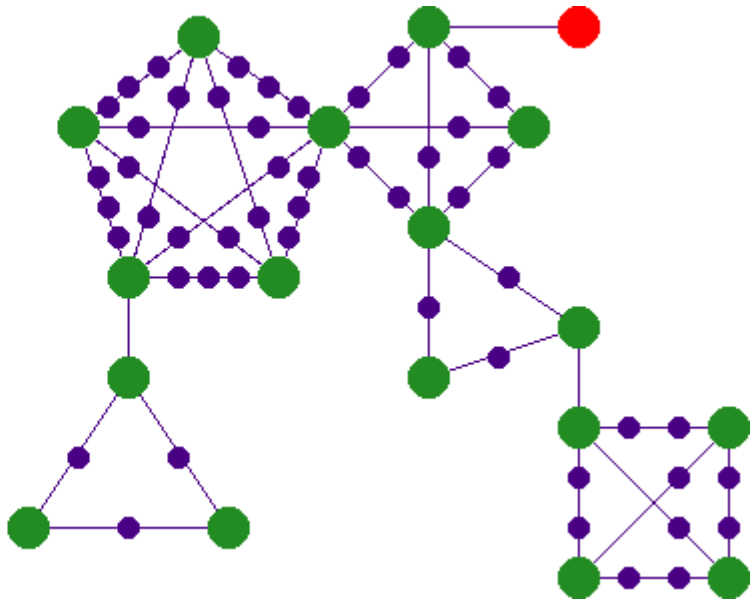

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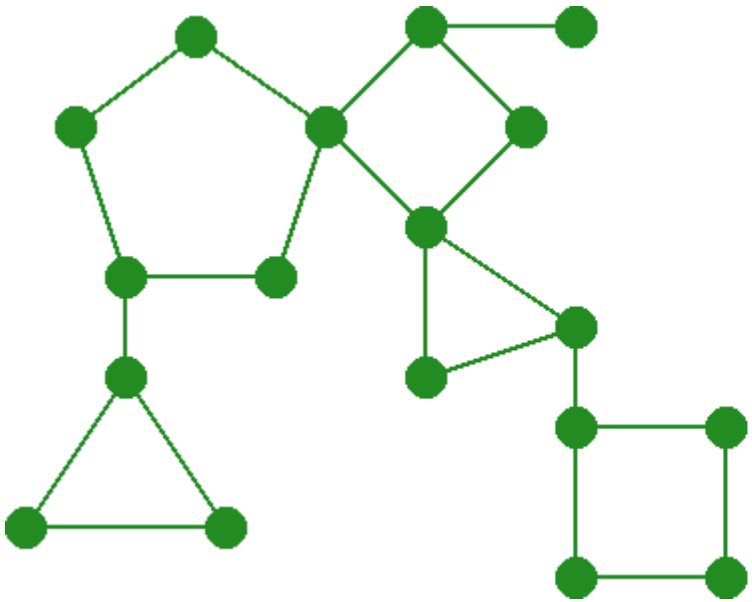

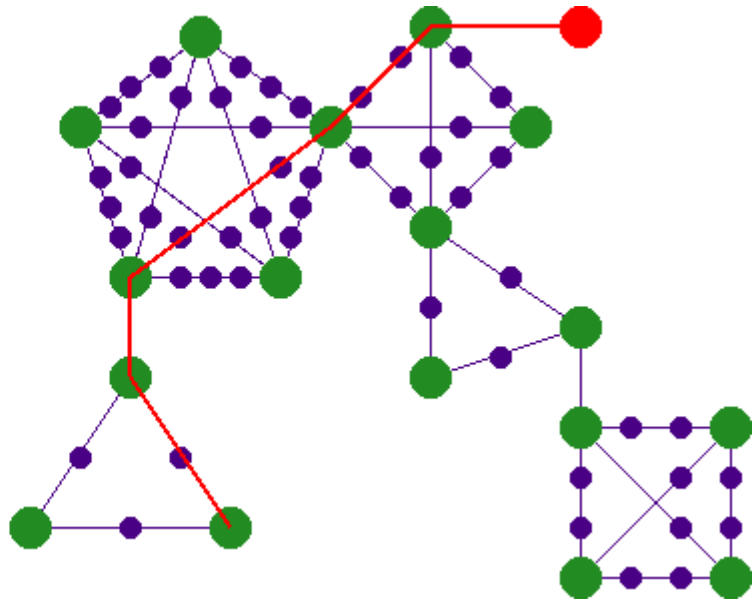

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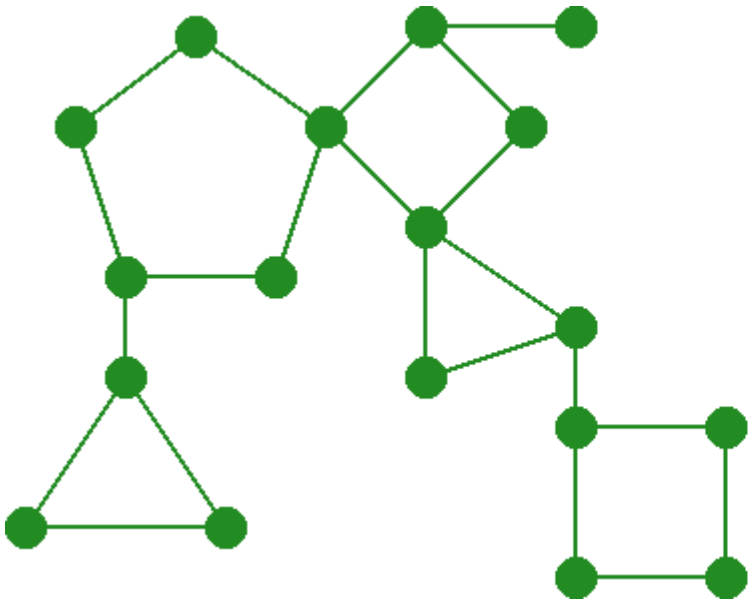

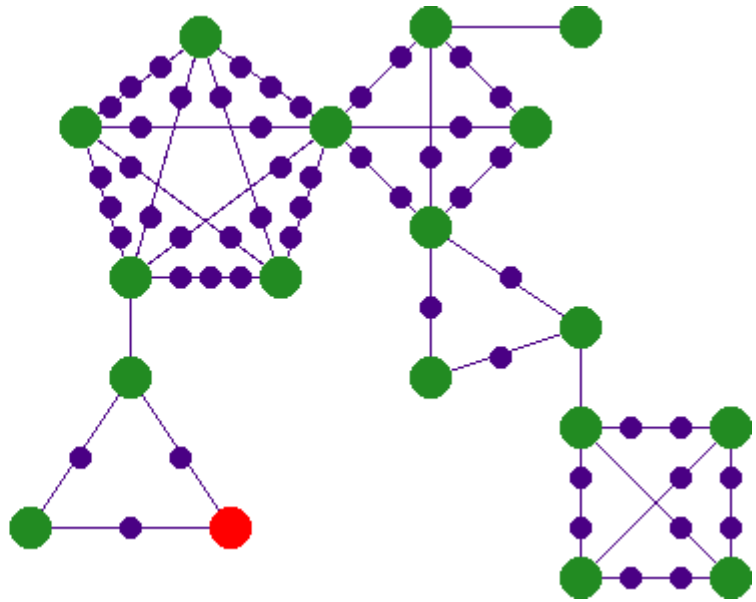

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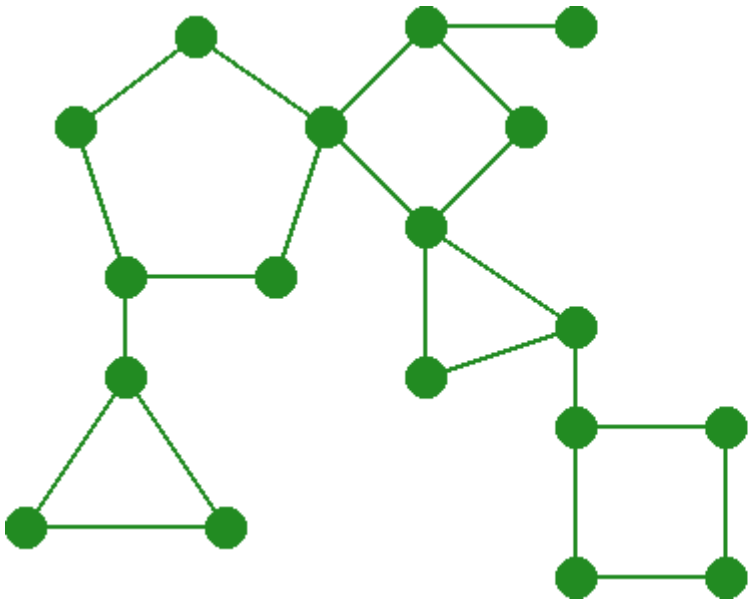

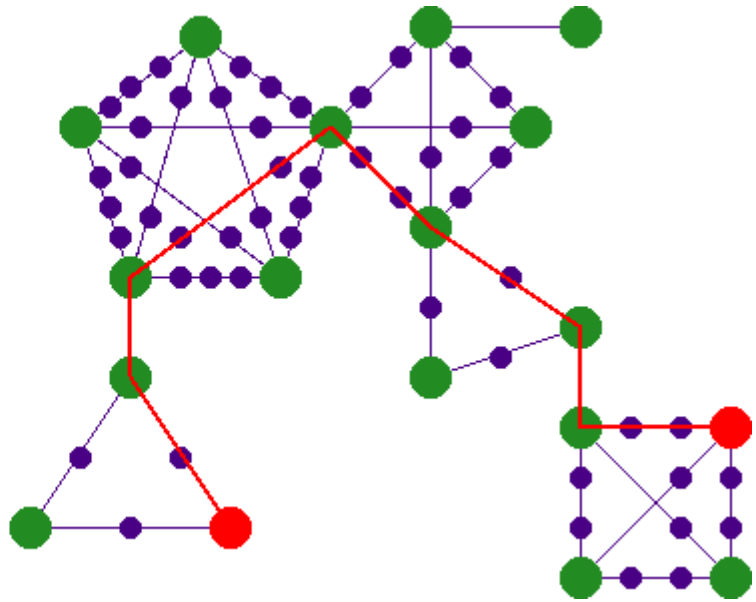

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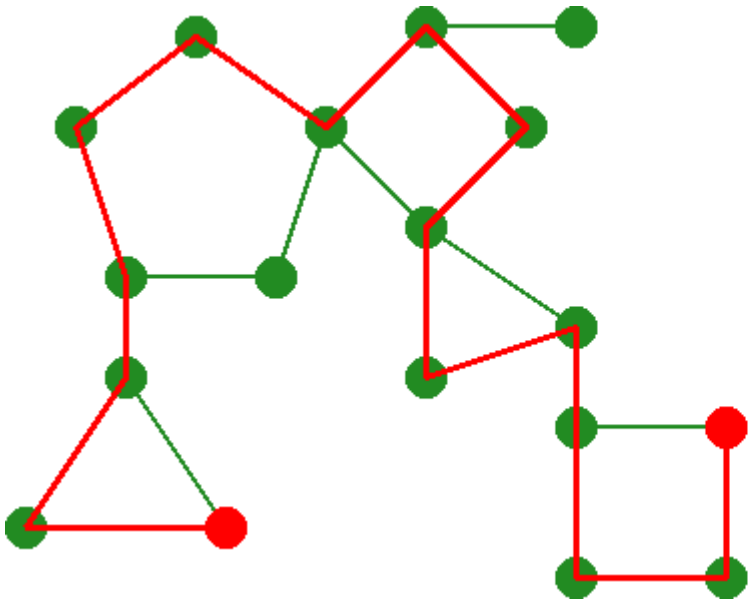
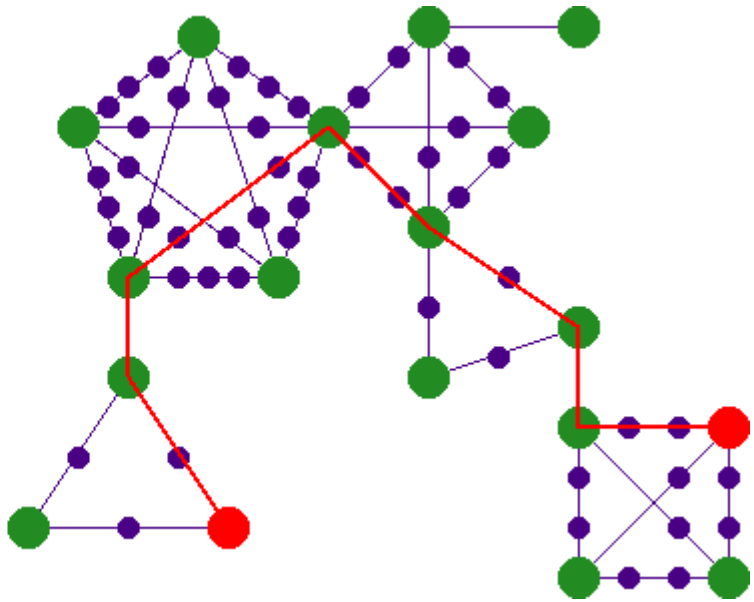




Graph classes s.t. Hamiltonian Path can be found in poly time

Fact 1:

Hamiltonian Path is NP-hard on a chordal graph.
(In fact, strongly chordal split graph[Müller,1997].)

Fact 2:

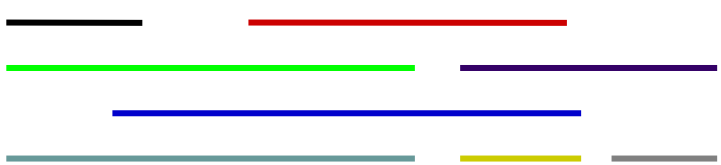
Hamiltonian Path is solvable on an interval graph in linear time. [Damaschke, 1993].

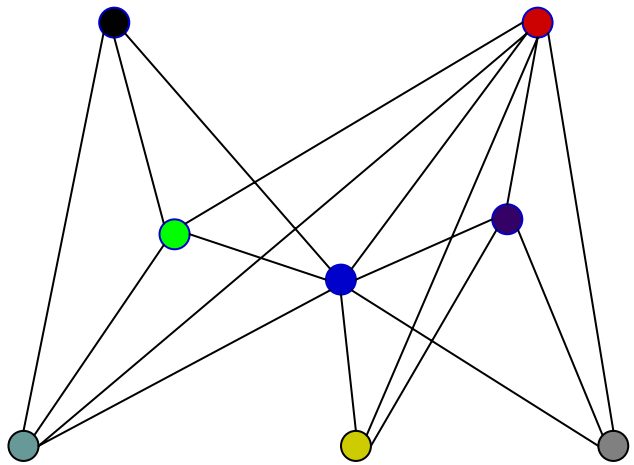
Our goal:

Poly-time algorithm for Longest Path on an interval graph.

Interval Graphs

- An interval graph $G=(V,E)$ has an interval representation *s.t.* $\{u,v\} \in E$ iff $I_u \cap I_v \neq \emptyset$





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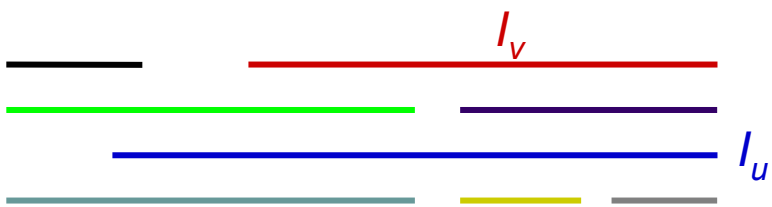
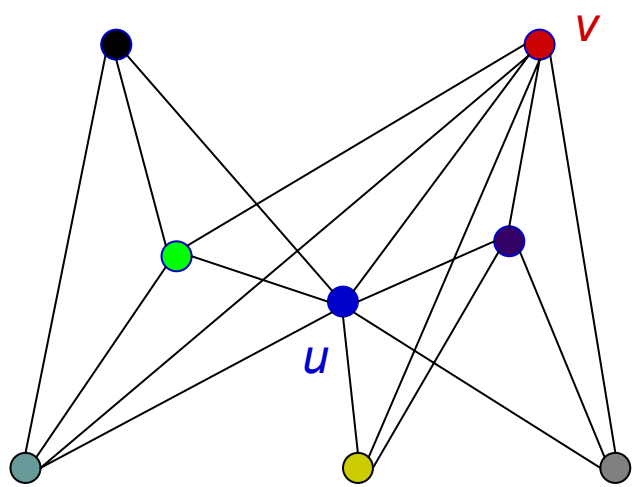




Hamiltonian Path: linear time solvable.

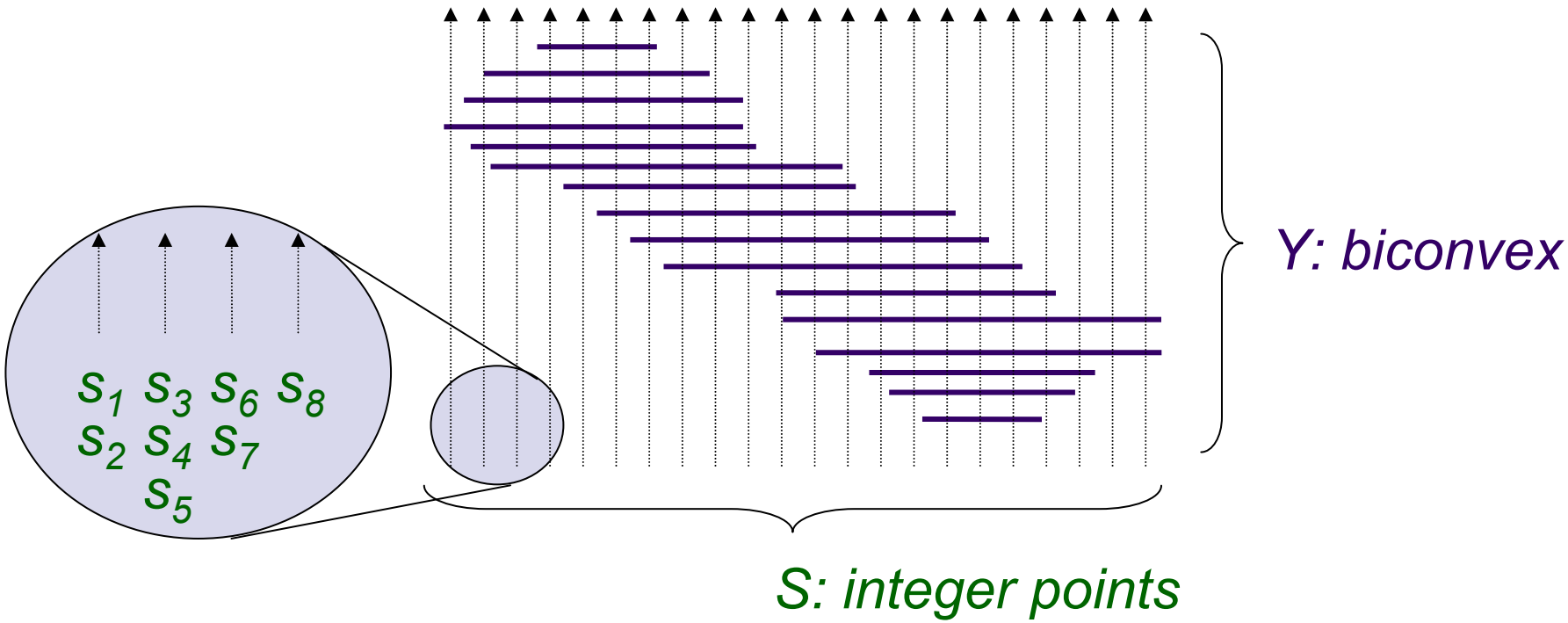
Longest Path: ????

⇒ Restricted interval graphs…

http://www.jaist.ac.jp/~uehara/ps/longest.pdf

Restricted Interval Graphs

- An interval biconvex graph $G=(S \cup Y, E)$ has an interval representation *s.t.…*



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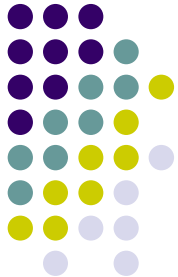
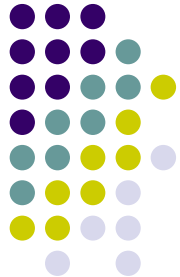


- Interval biconvex graph $G=(S \cup Y, E)$ is introduced [Uehara, Uno; 2004] from graph theoretical viewpoints;

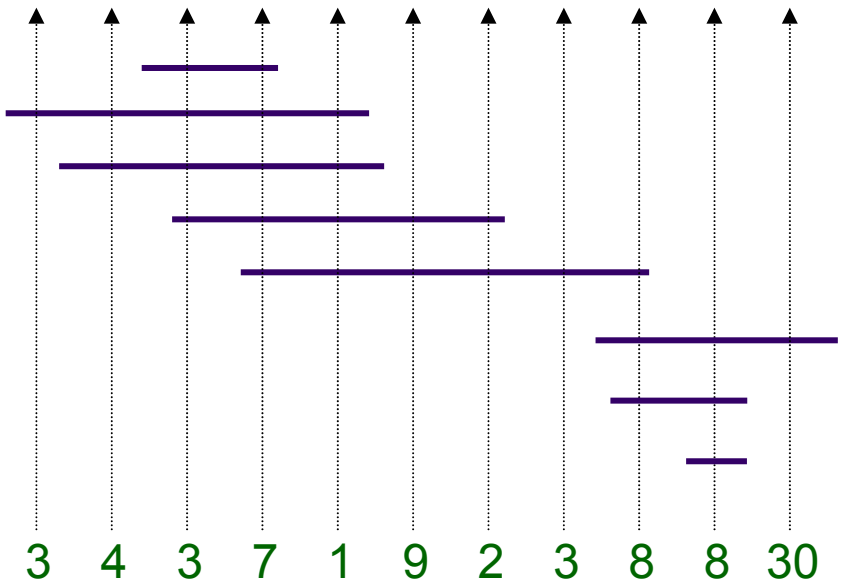
  - ✓ *Natural analogy of biconvex graphs (bipartite graph class)*
  - ✓ *Generalization of proper interval graphs*
  - ✓ *Generalization of threshold graphs*

  - ✓ *Best possible class longest path can be found in poly time…*

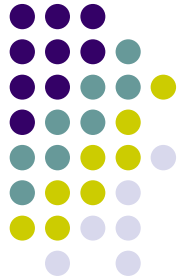
Poly-time alg for longest path on an interval biconvex graph (idea)



- Find the trivial longest path P on $G[Y]$;
- Embed the vertices in S into P as possible;
- Adjust endpoints if necessary.



Poly-time alg for longest path on an interval biconvex graph (idea)



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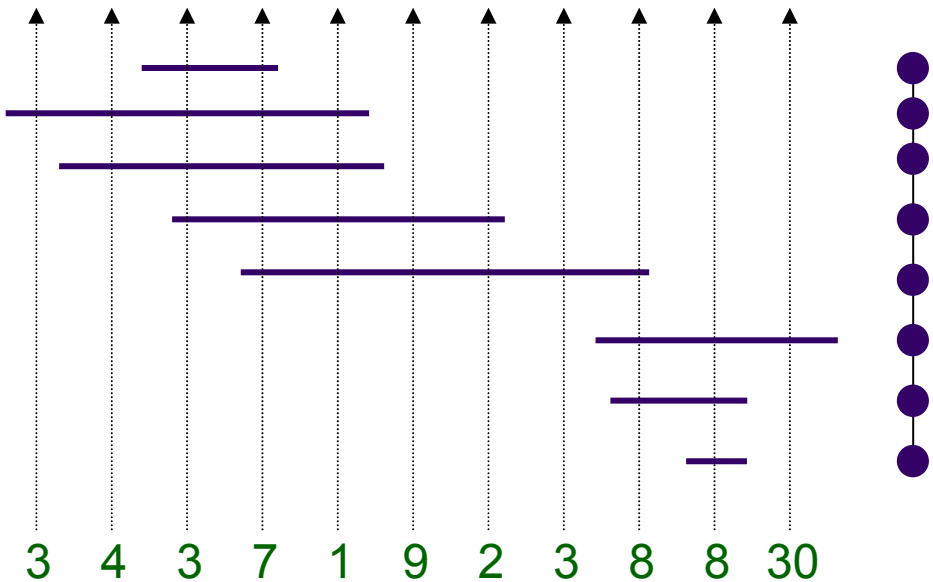

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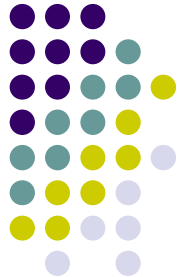

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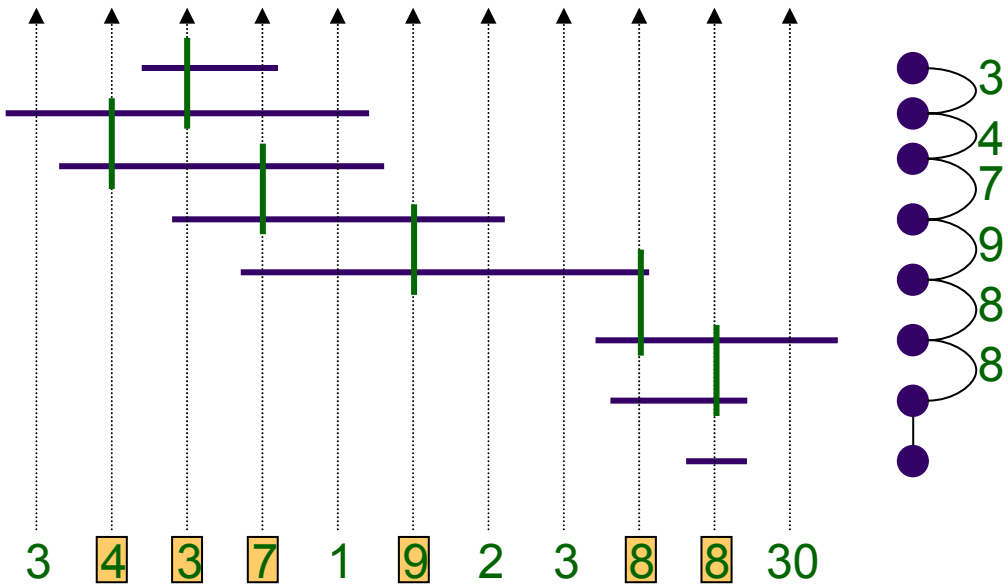

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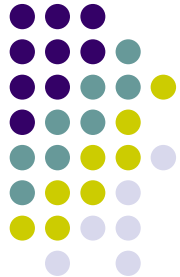

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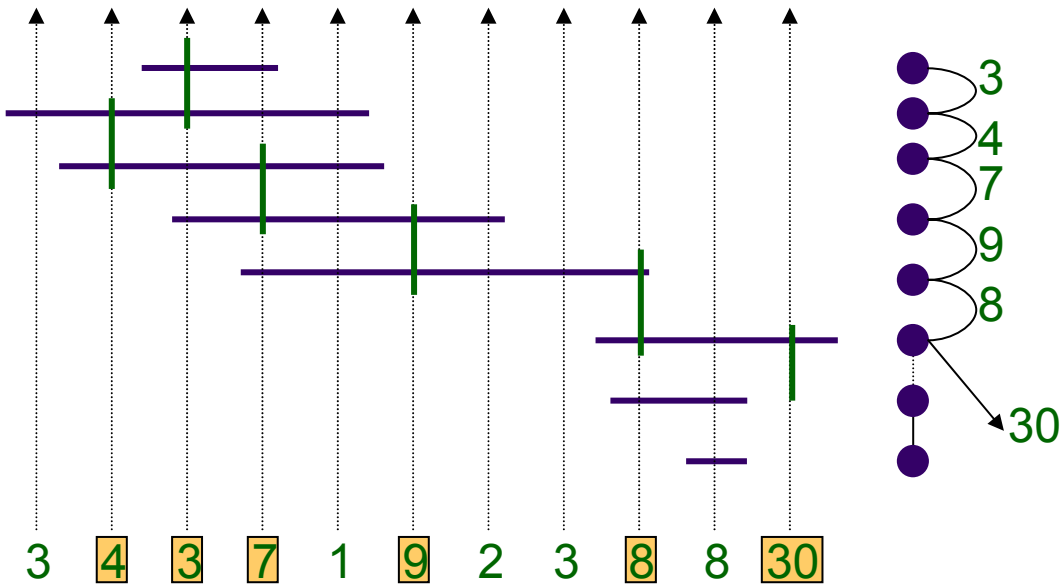

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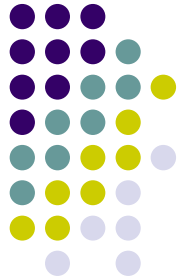

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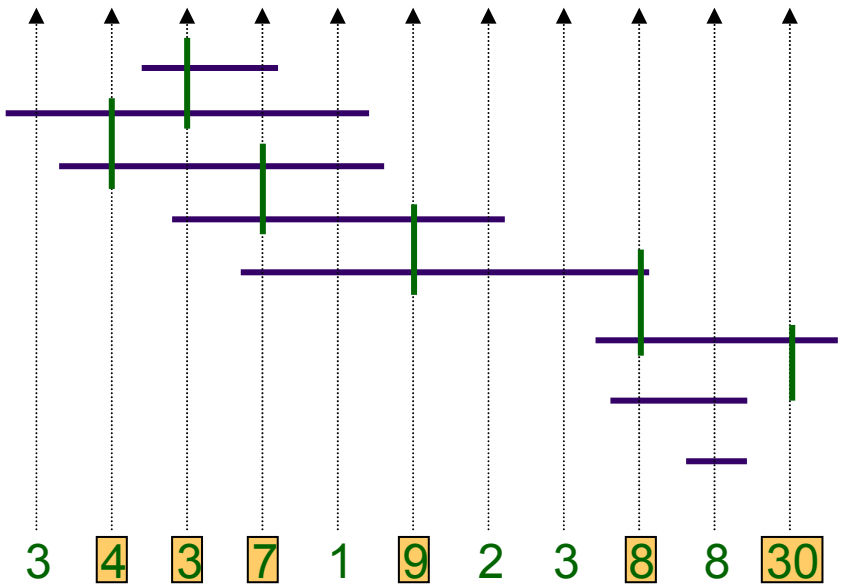
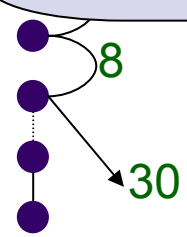


✓ How can we determine the vertices in S?
✓ Where do we embed them?



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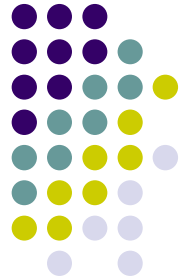
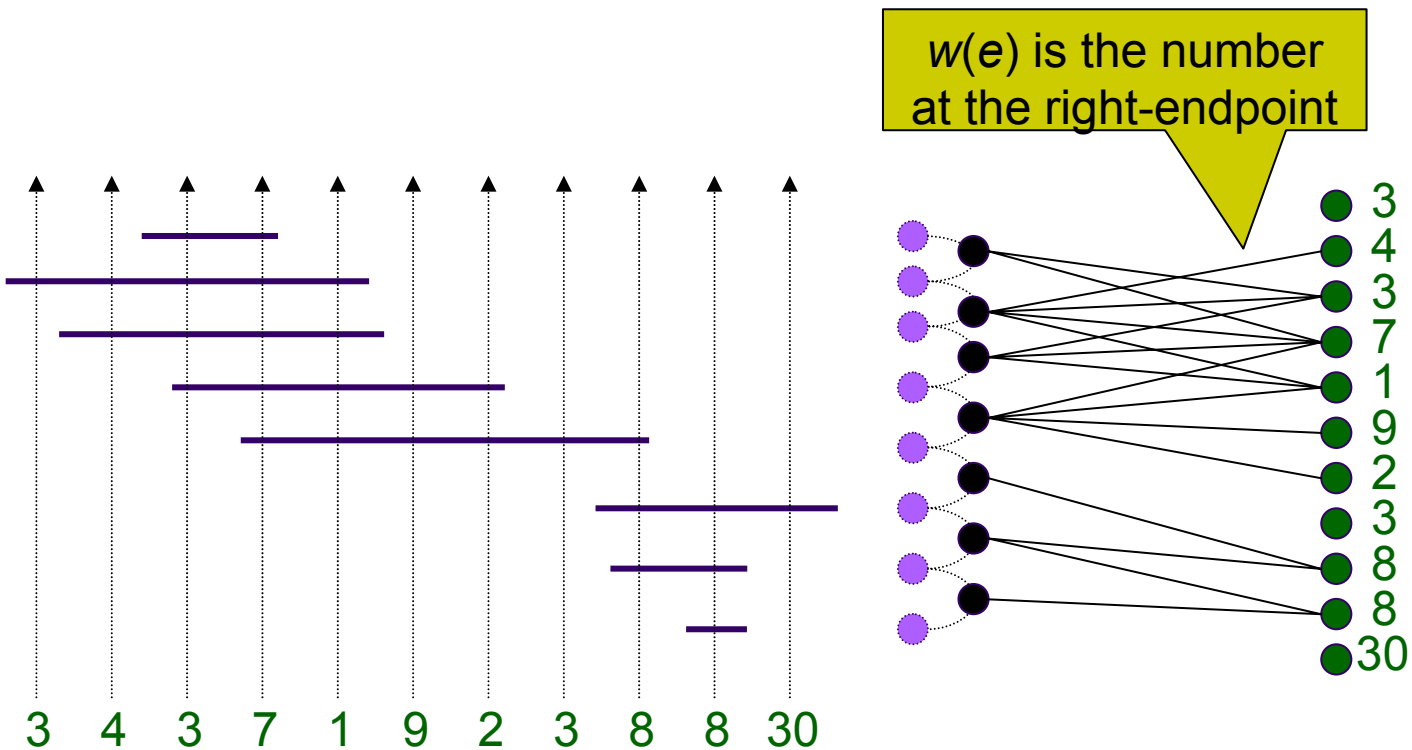
Poly-time alg for longest path on an interval biconvex graph (idea)



- Embed the vertices in *S* into *P* as possible;



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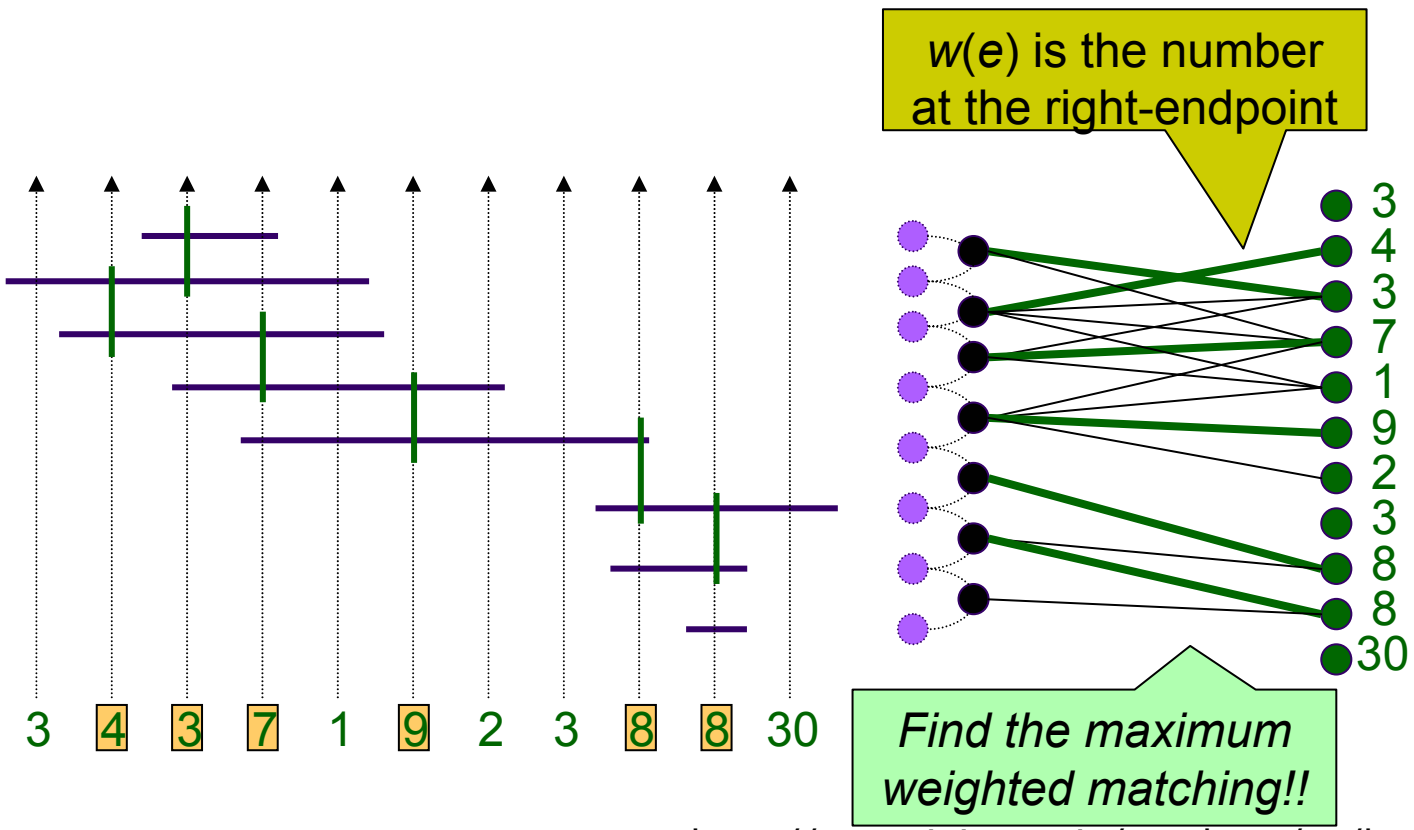
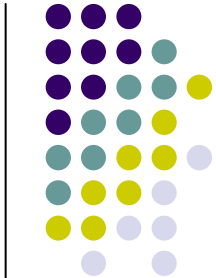


Open Problems



- Longest Path on an interval graph??
  - Combination of *DP/Dijkstra* and *weighted maximum matching* on MPQ-tree representation?
  - Related to the following open problem?

    Hamiltonian Path *with a start point* on an interval graph*?* [Damaschke, 1993].

- Extension to
  - Longest cycle on some graph classes
  - Hamiltonian cycle/path on some graph classes