The walkers problem

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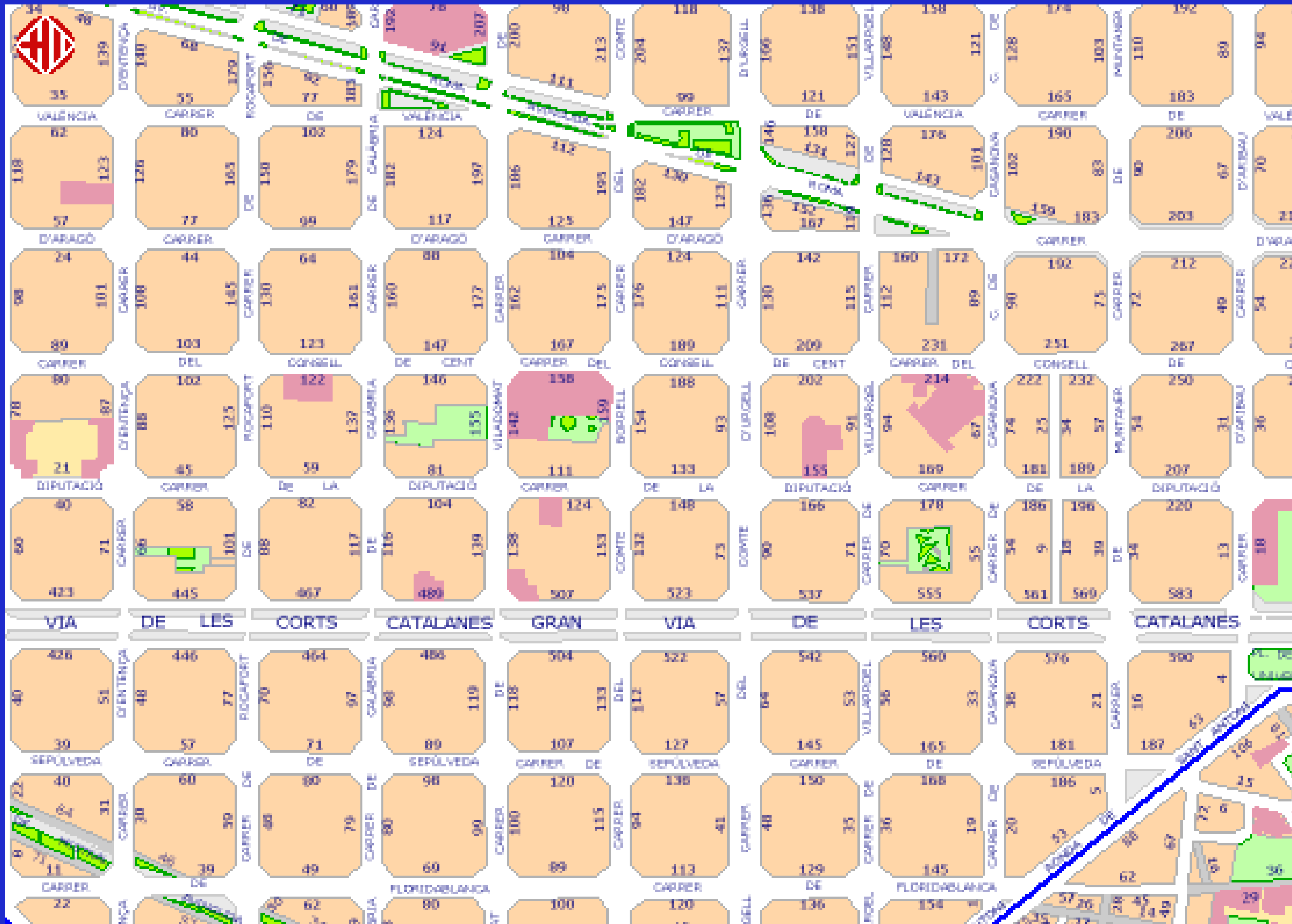

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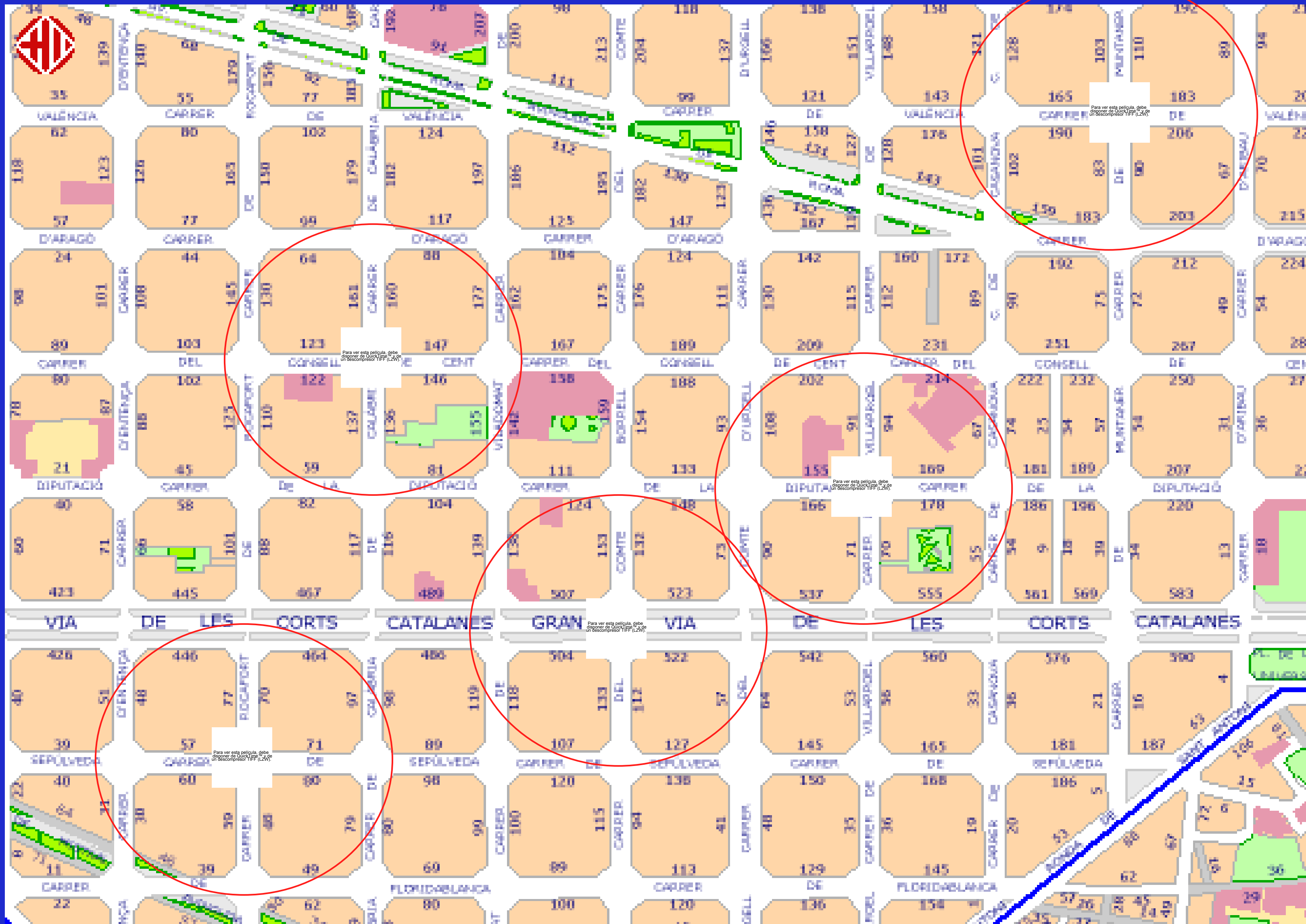

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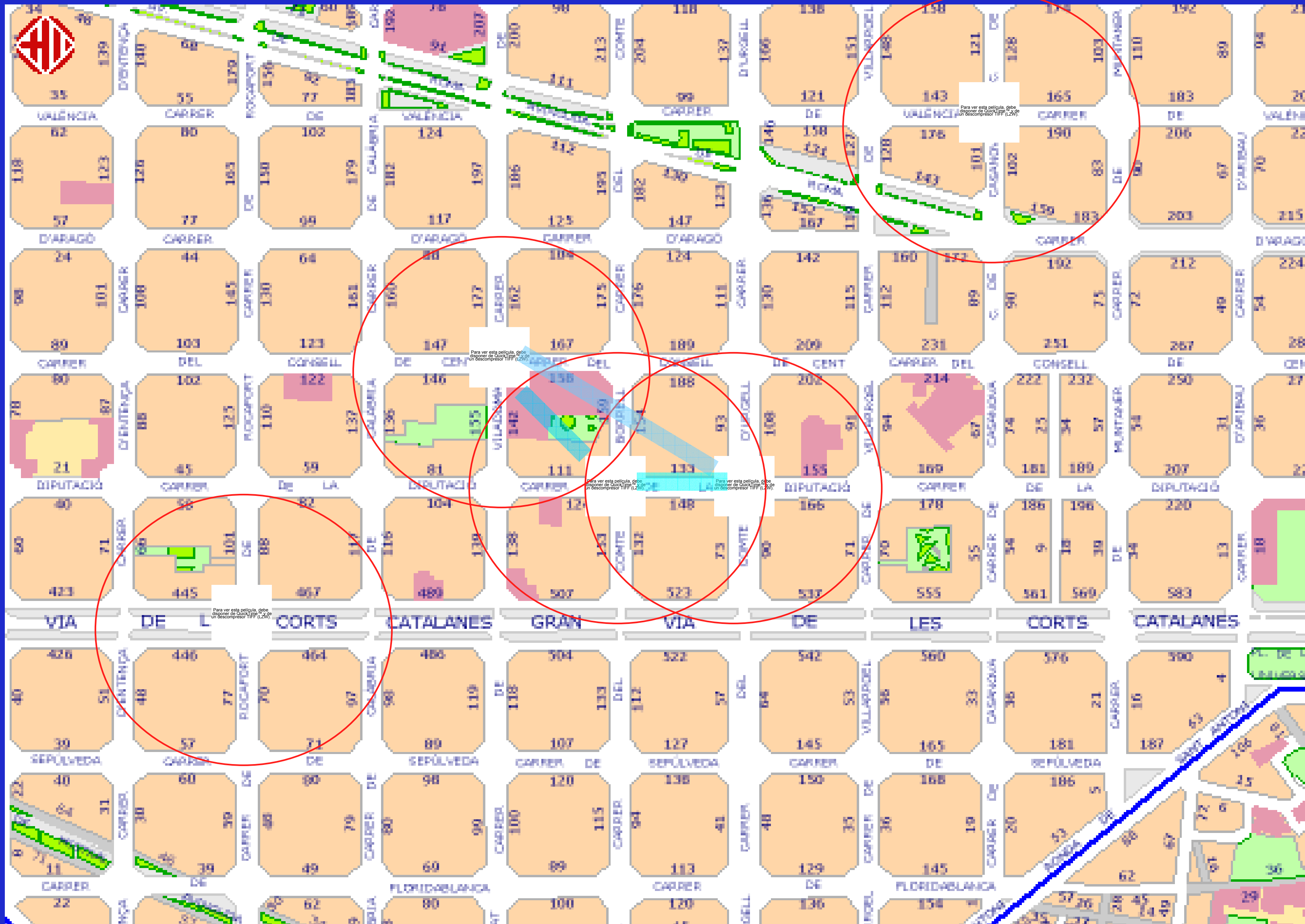

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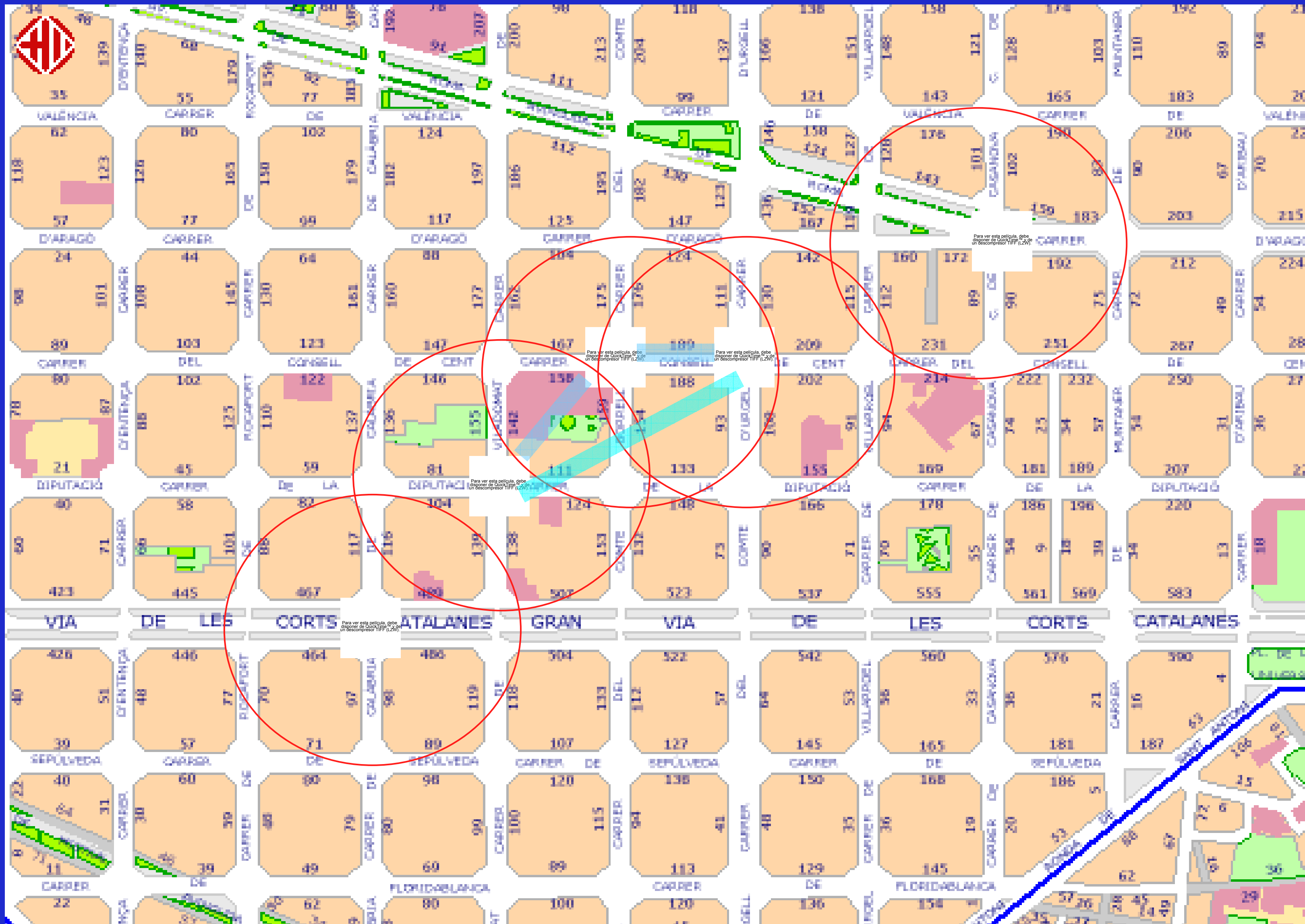

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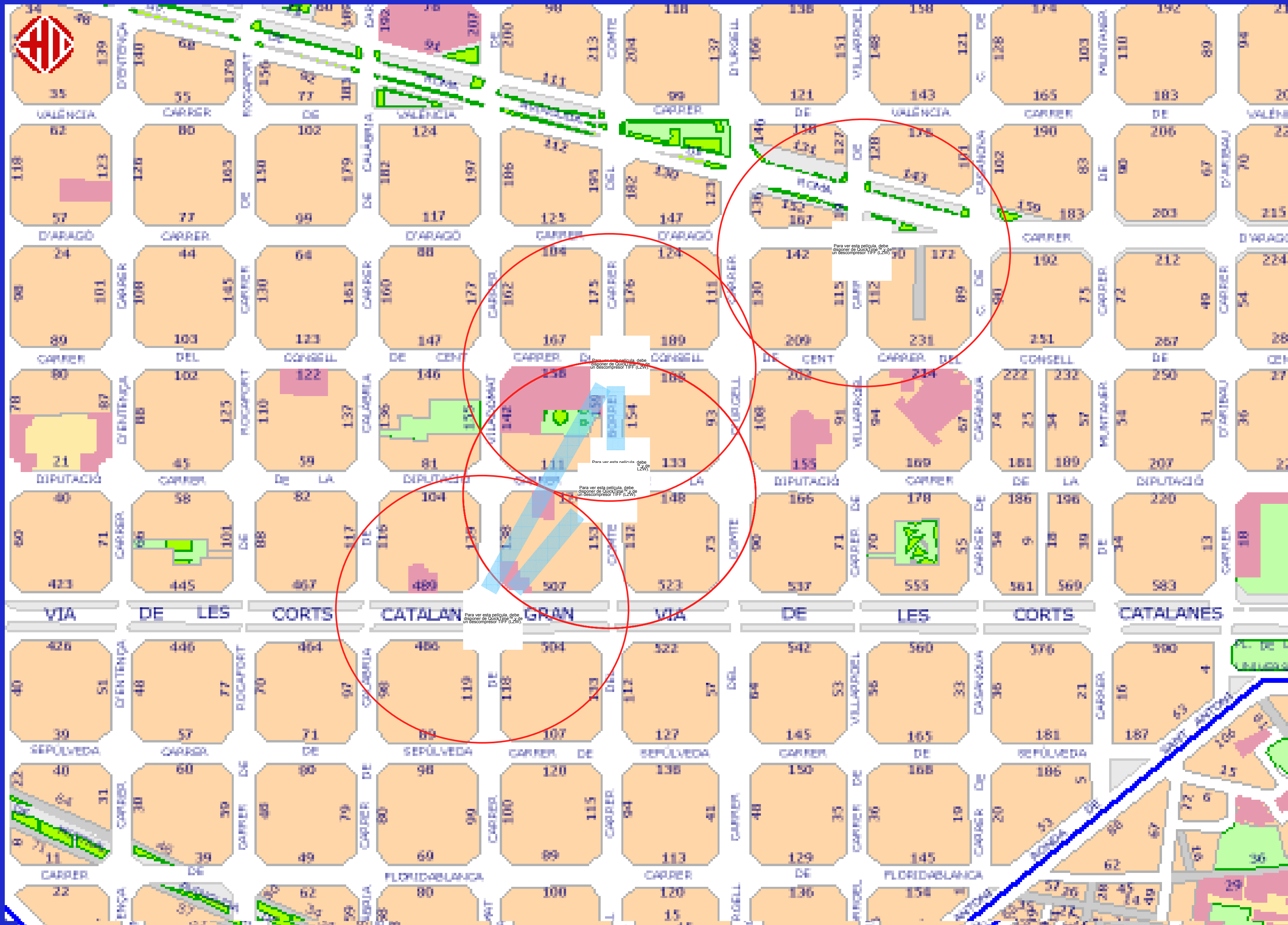
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GOAL

STUDY THE CONNECTIVITY OF THE AD-HOC NETWORK STABLISHED BETWEEN THE AGENTS, AS THESE MOVE FOLLOWING THE EDGES OF A GRAPH:

- Cycle
- Grid
- Hypercube
- Random Geometric graph

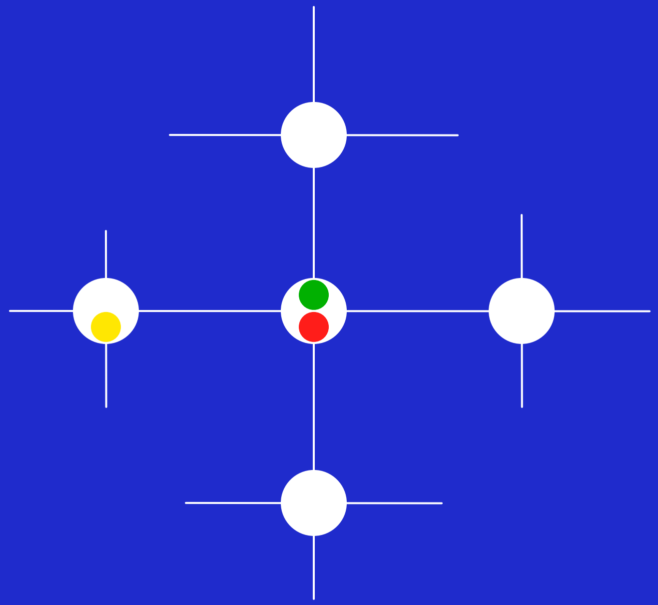
GOAL

STUDY THE ITERACTIONS OF SIMULTANEOUS RANDOM WALKS ON DIFFERENTS TOPOLOGIES.

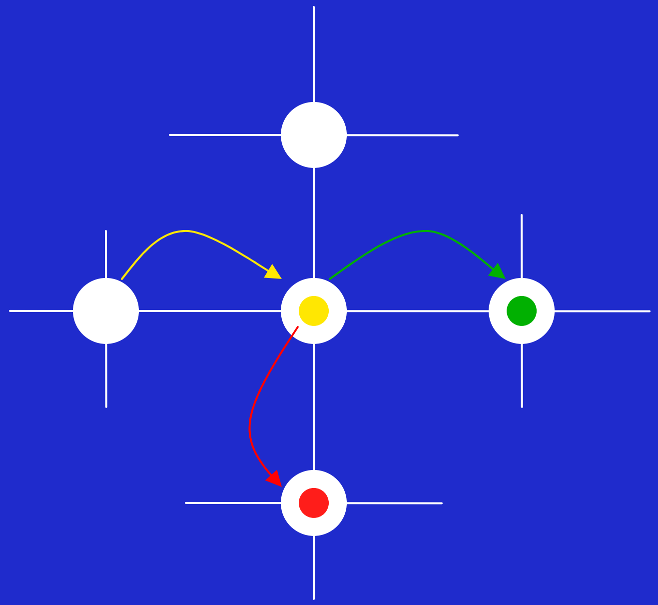
Walkers in the toroidal grid

- Given a set W ($|W|=w$) of walkers (agents, robots,..) which at each step, they can move N/S/E/W on the edges of a toroidal grid T_N, with $N=n^2$, the walkers have RF communication within a distance d (Manhatan, euclidian, etc.), we wish to study the evolution of the connectivity graph $G_t[W]$, as the walkers move.

- At step $t=0$, the w walkers are sprinkled u.a.r. on T_N. $f{:}W \rightarrow V(T_N)$ (static case)
- At each step t, every walker is forced to move

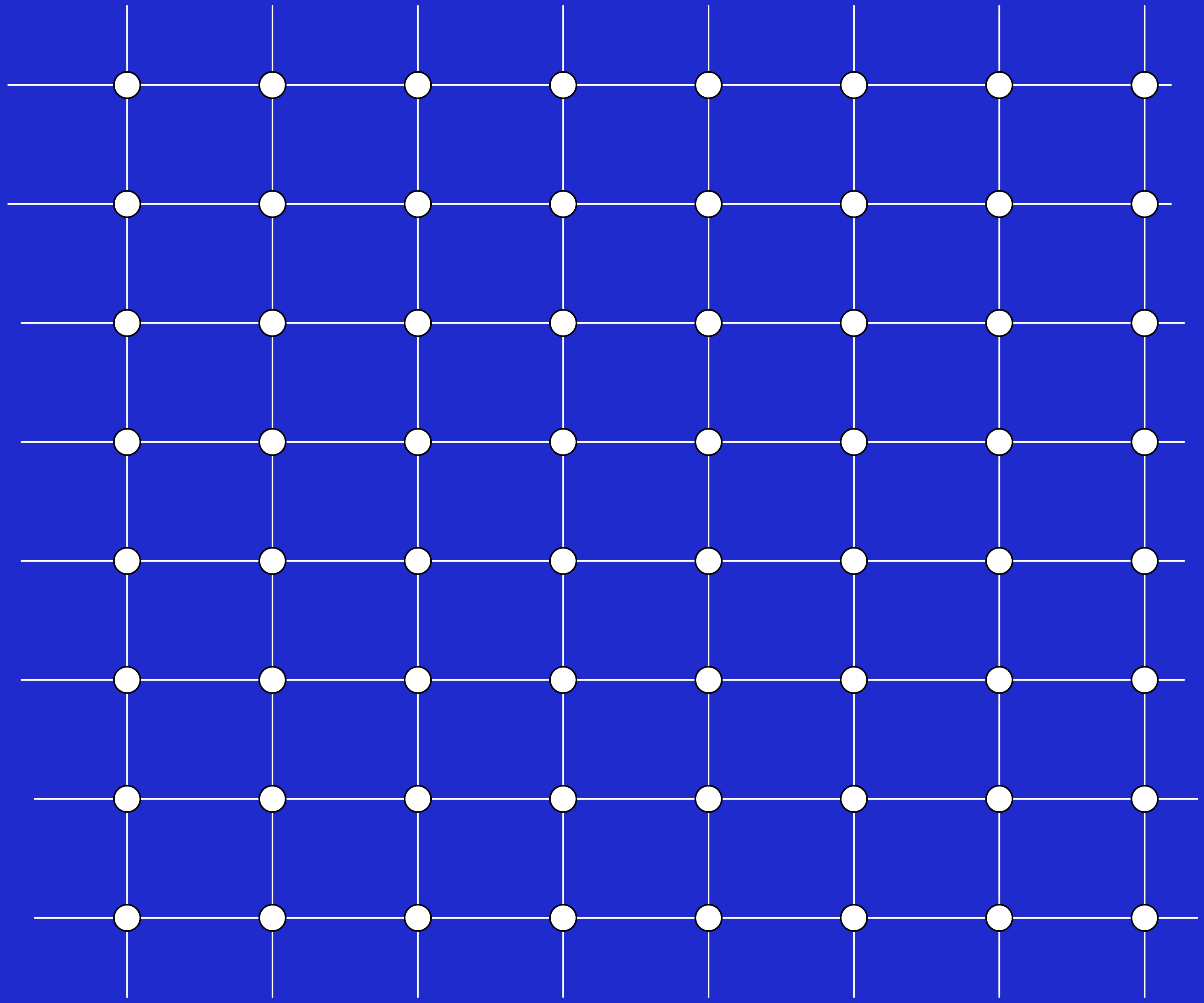


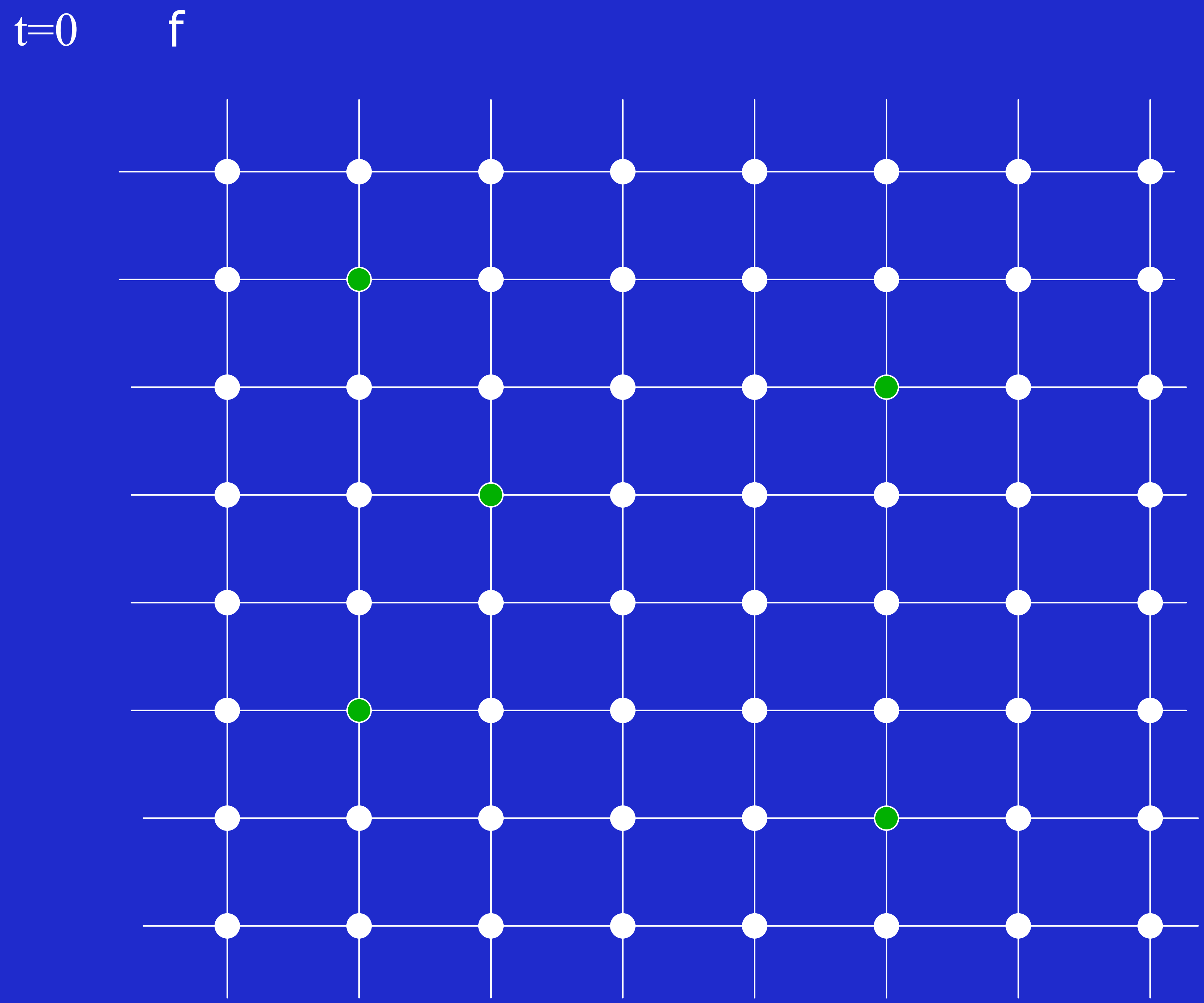
- At step $t+1$, every walker is forced to move



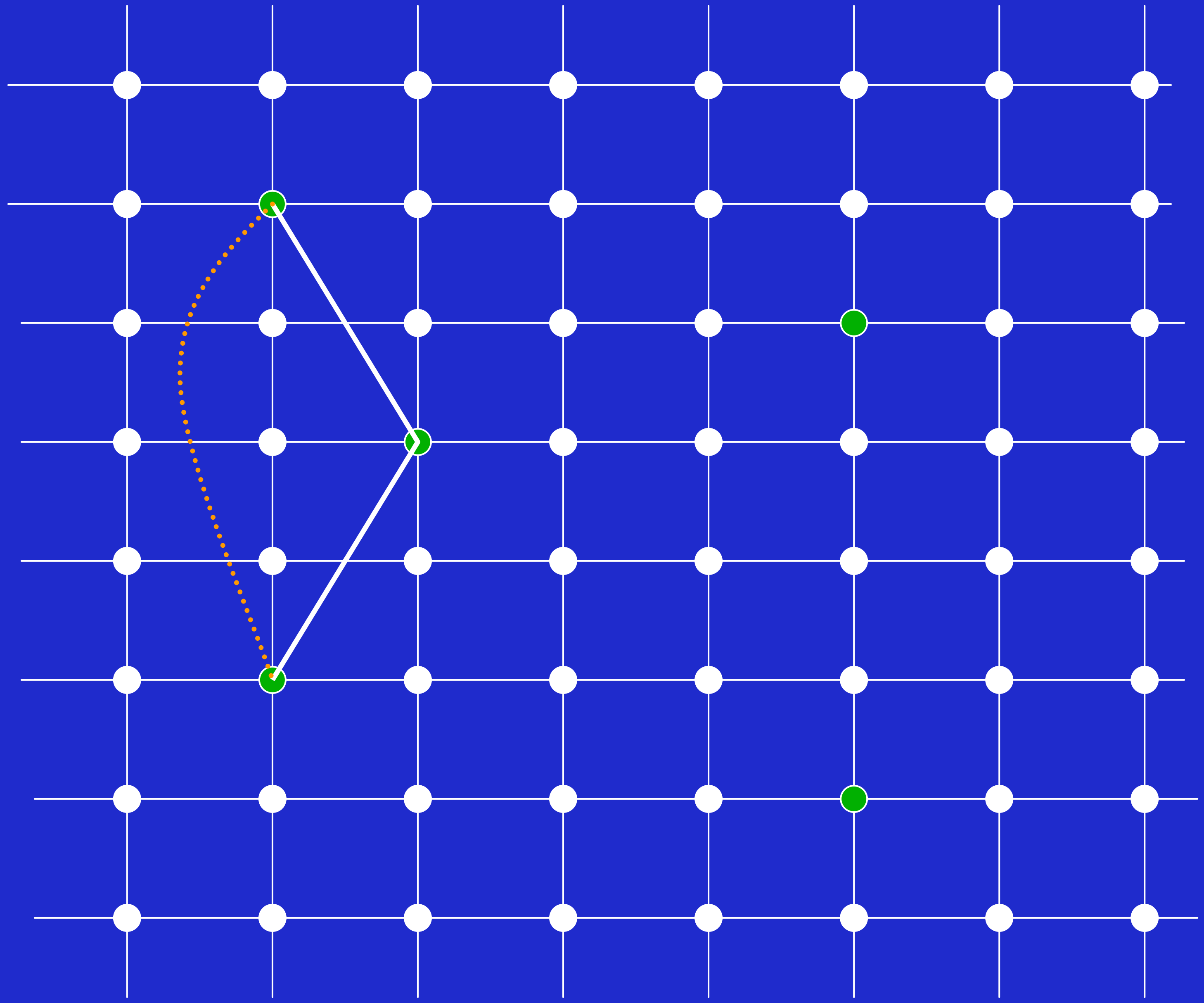
Toy example

- At $t=0$, sprinkle 5 walkers in a nxn grid, with a max communication distance $d=3$ (in the l^2 norm)
- Look evolution of $G_t[W]$ up to $t=4$.




t=0
f

t=0



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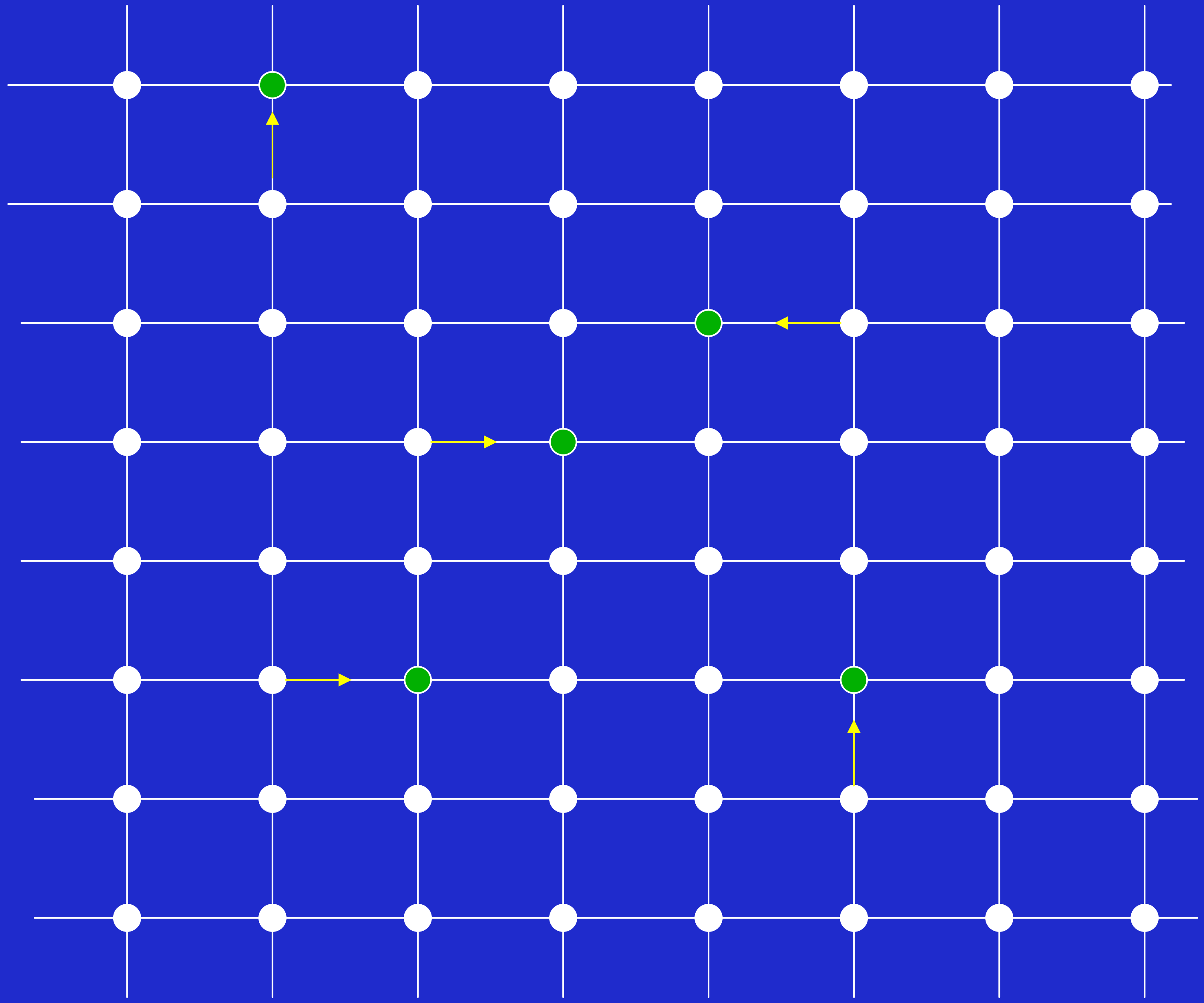
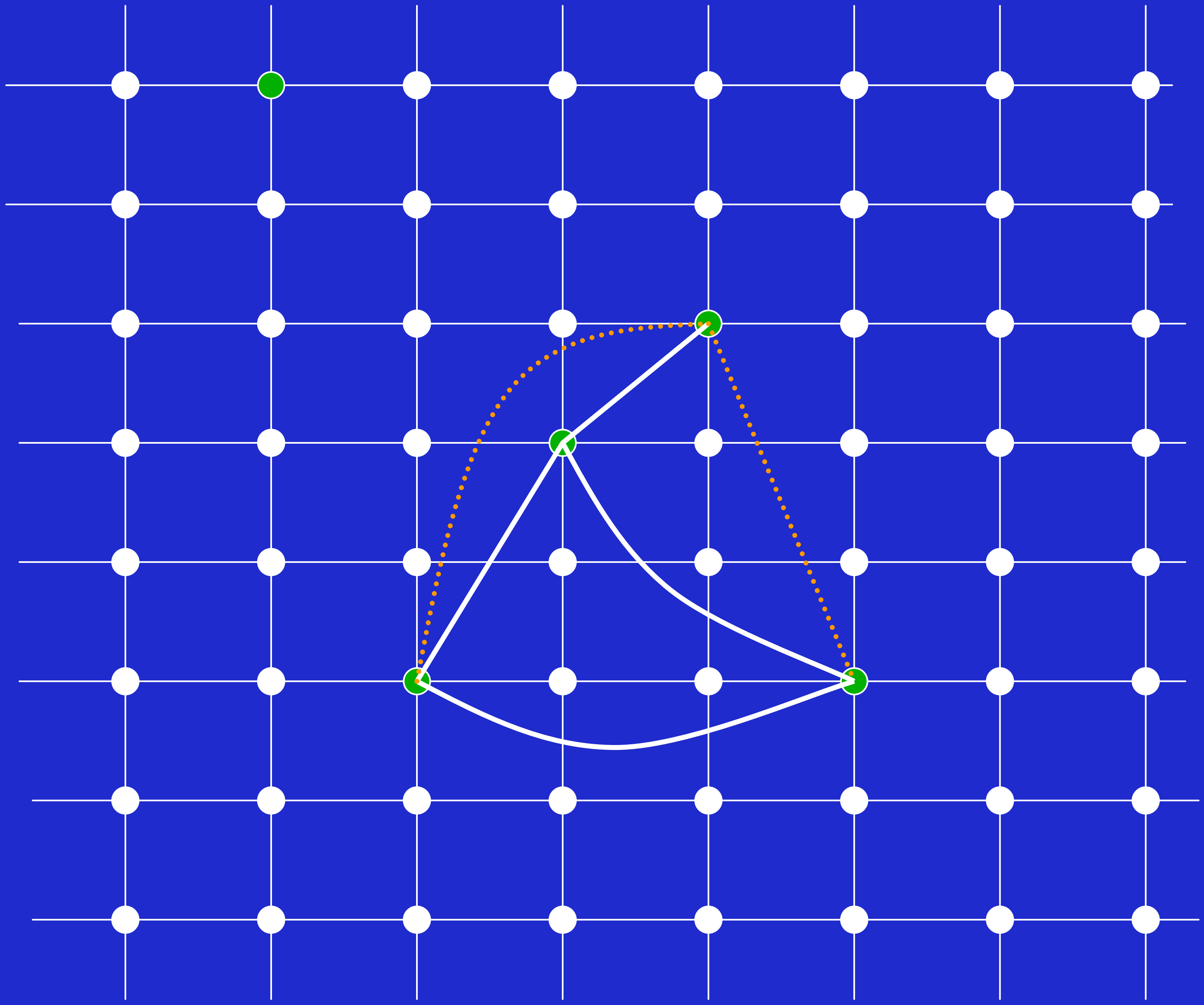
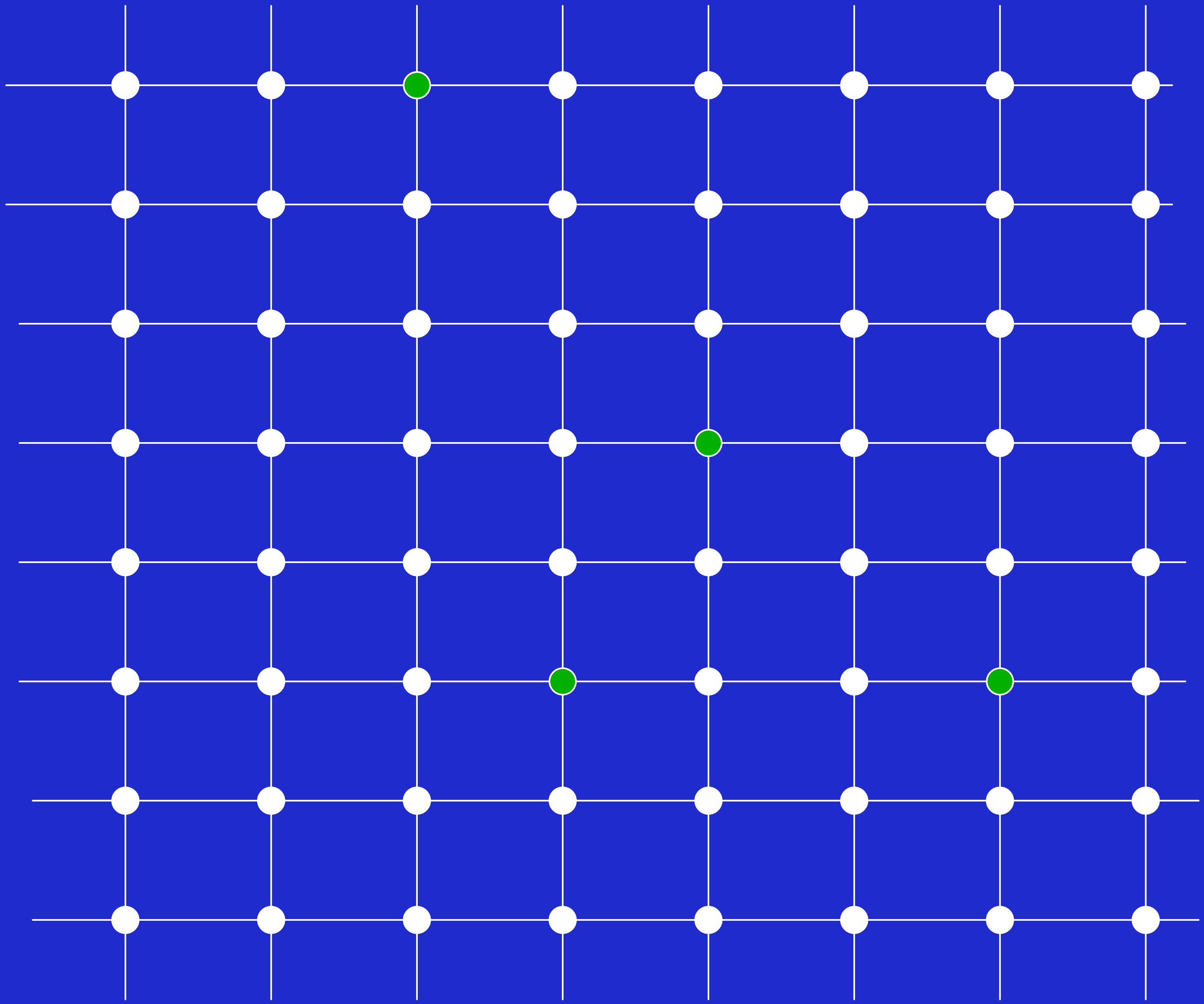


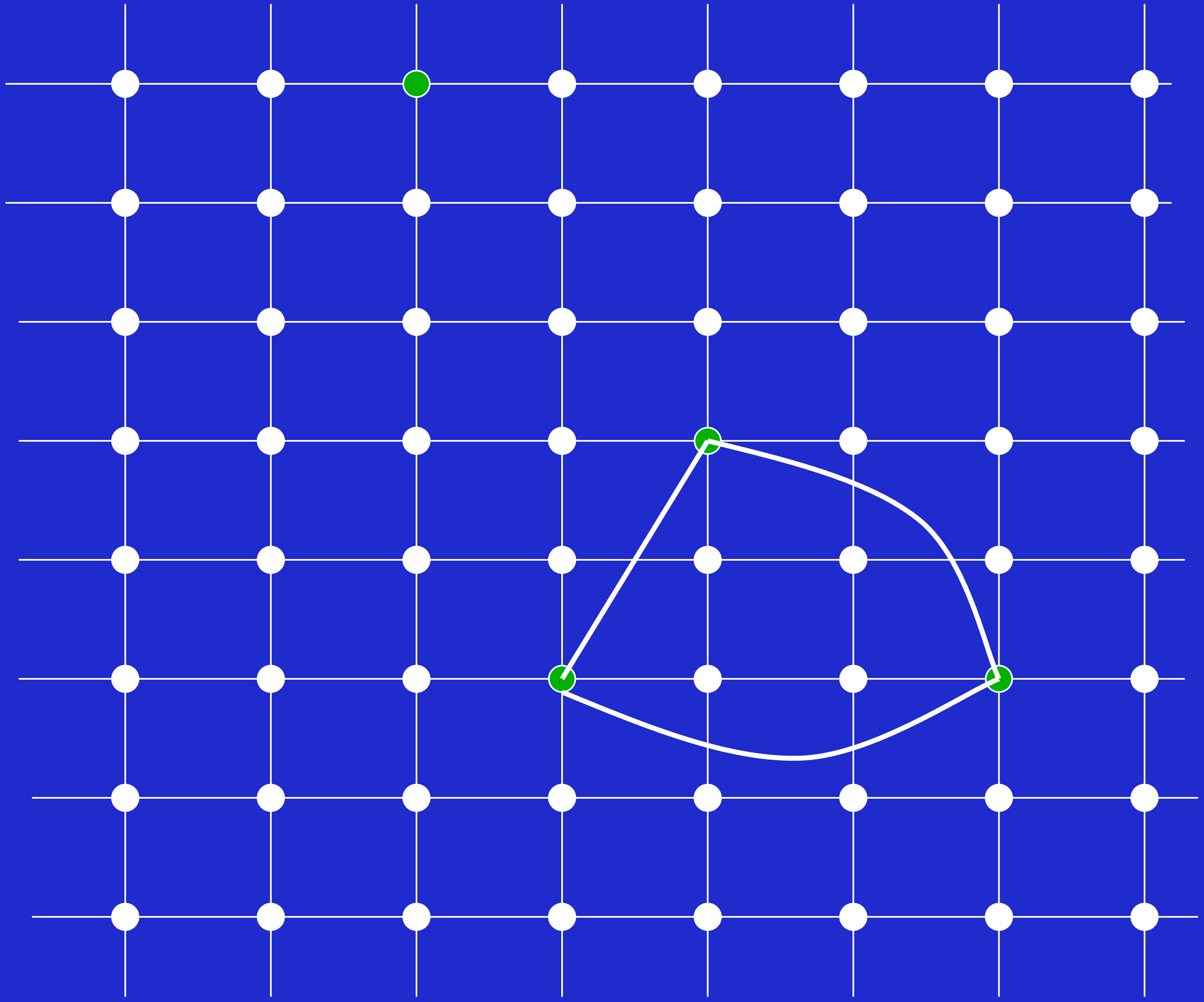
t=1



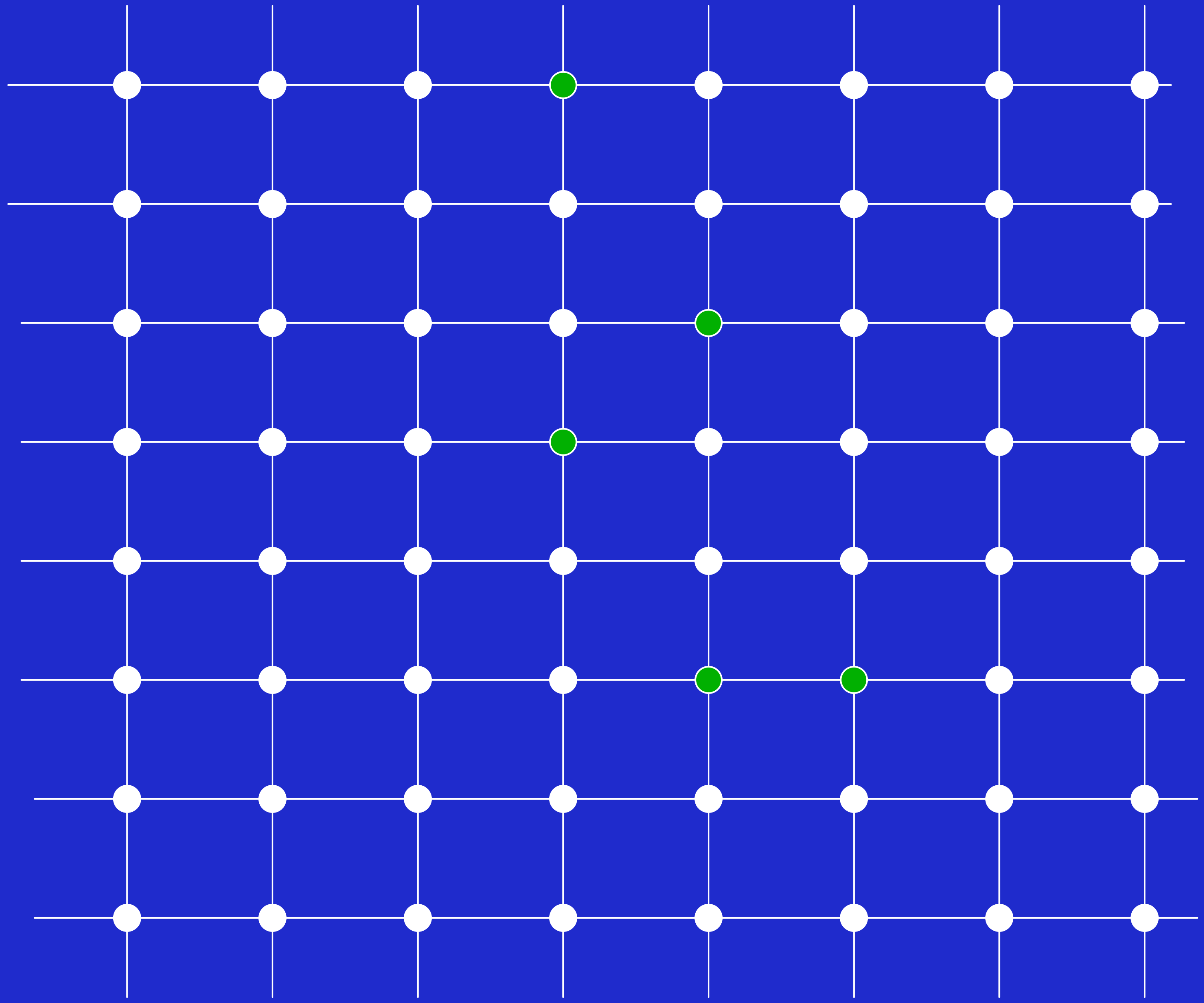
t=2



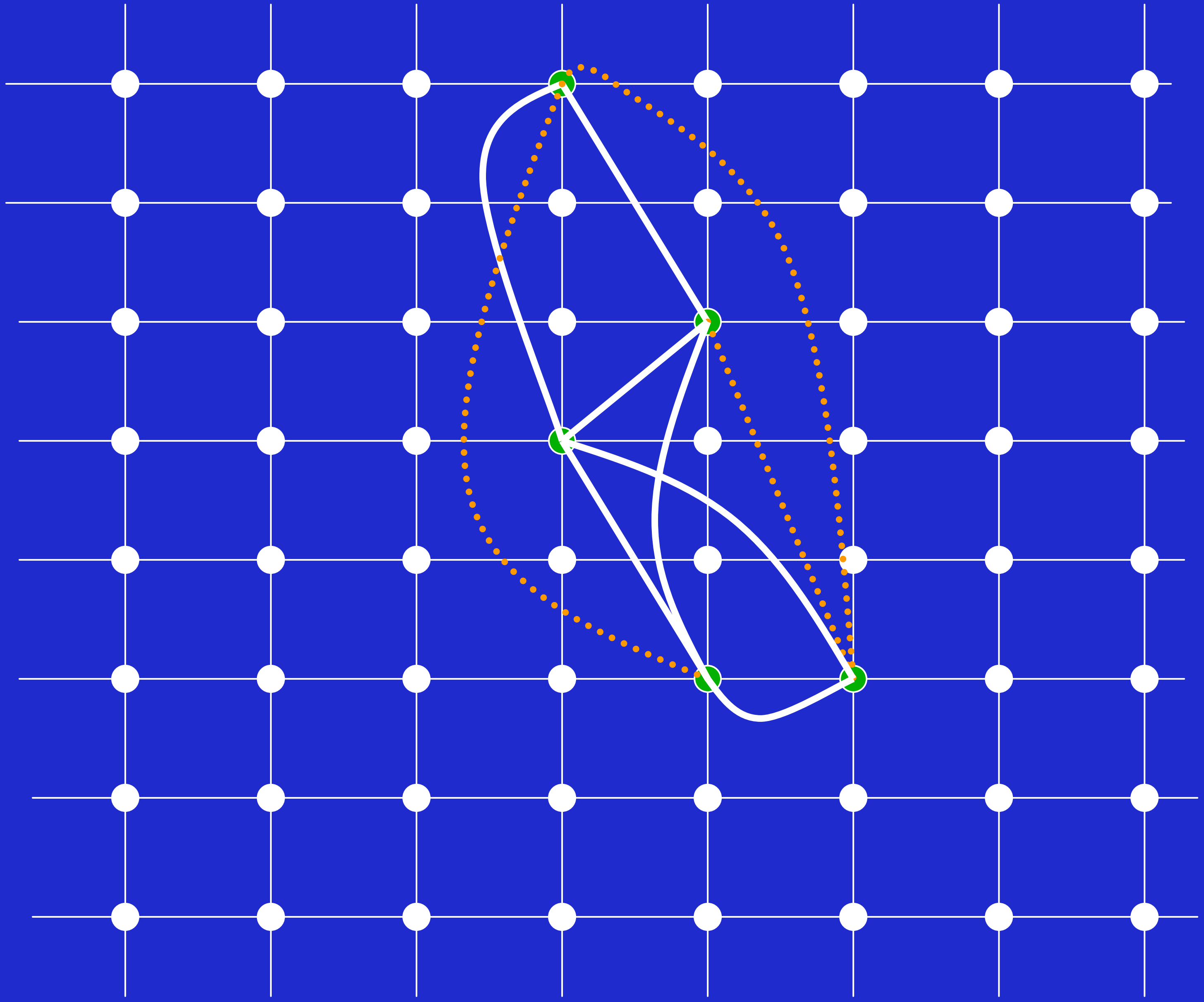
t=2



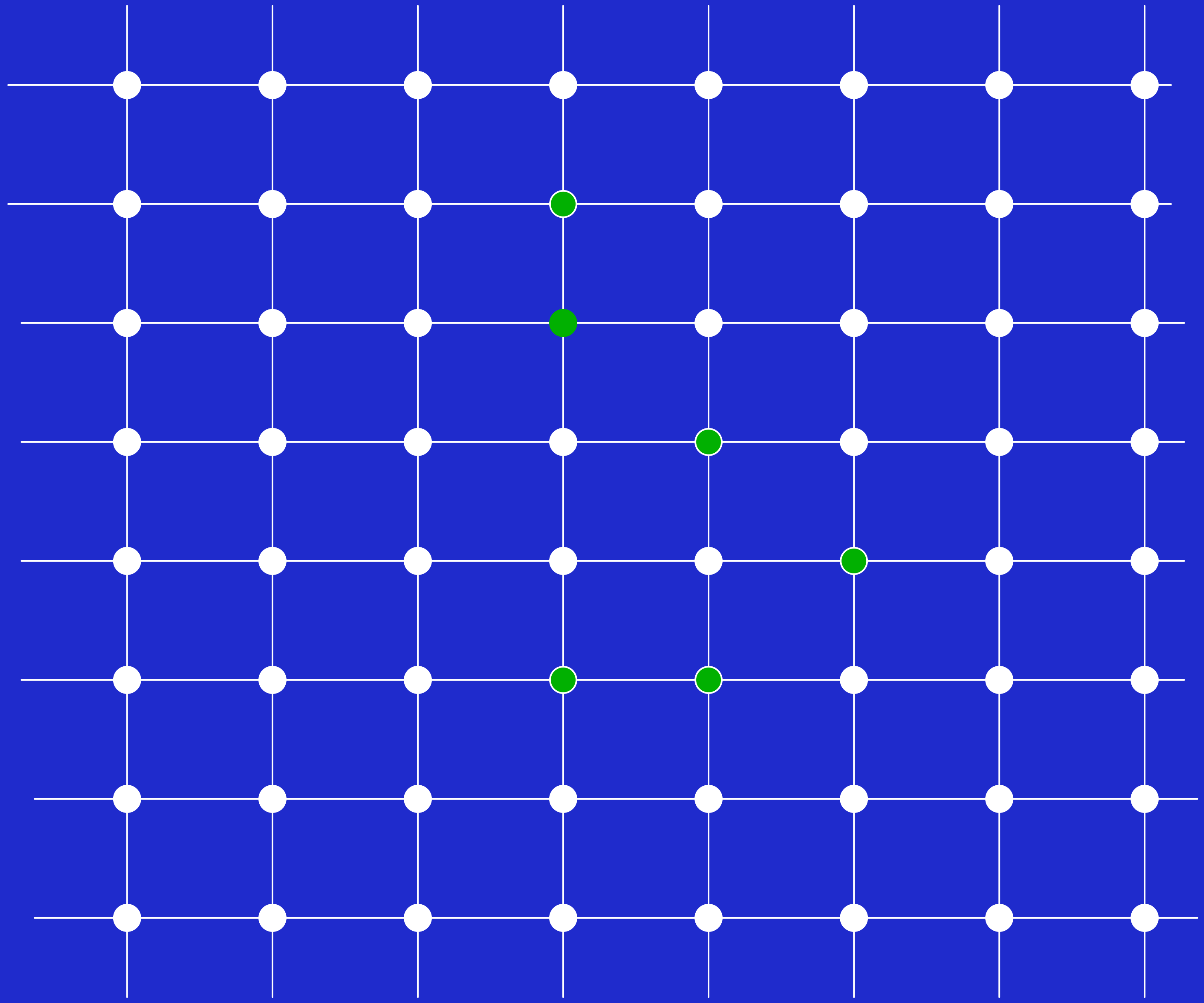
t=3



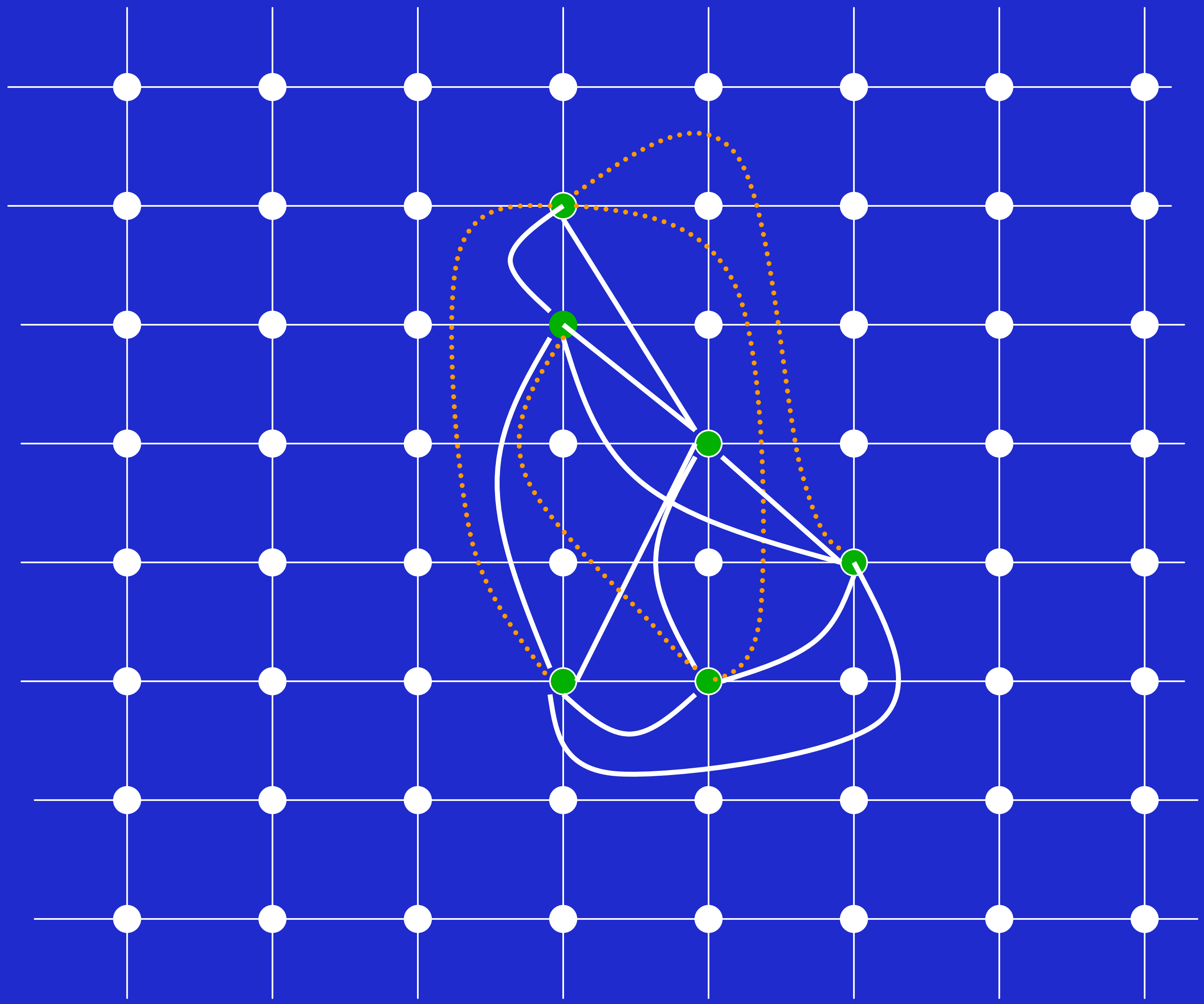
t=3



t=4



t=4

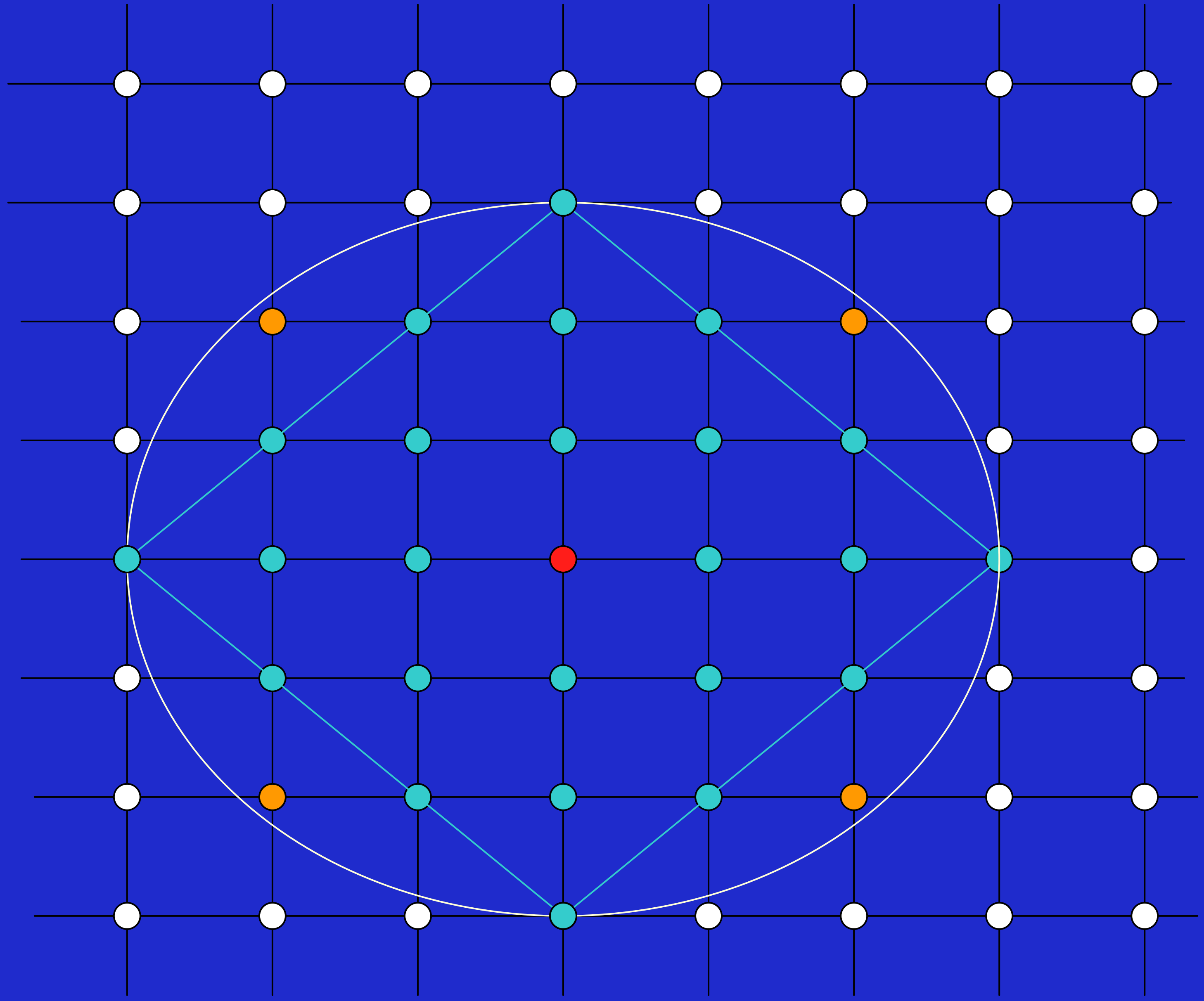


Static case

Some parameters

- K = # connected components in $G_f[W]$
  $\rho = w/N$ (expected walkers per node)
- h = minimum number vertices around a simple component.
- Simple component: isolated vertex in $G_f[W]$

h for l^1 and l^2 distances (for $d=3$)



Some values for h

l^1 $h = 2d(d+1)$

l^2 $h \sim \pi d^2$ for large d

l^∞ $h = 4d(d+1)$

Observation

- If $d \geq 2n$ → $G_f[W]$ is connected
- If $d^2 = \Omega(N/\sqrt{w})$ → $G_f[W]$ connected a.a.s. →
- Interesting case of study: $d^2 = o(N/\sqrt{w})$
  $d = o(n)$

Random variables

- X = number simple components
- K = number connected components

Let $\mu = N(1-e^{-\rho})\, e^{-h\rho}$. Then

$$\mu \sim w\, e^{-h\rho} \quad \text{if } \rho = w/N \to 0,$$

$$\mu \sim N(1-e^{-\rho})\, e^{-h\rho} \quad \text{if } \rho = w/N \to c,$$

$$\mu \sim N e^{-h\rho} \quad \text{if } \rho = w/N \to \infty.$$

Shape distribution of X

Theorem The expected number of simple Components satisfy $E[X] = N(1-e^{-\rho})(1-h/N)^{w}$

Moreover

- If $\mu \to 0$ *then* $E[X] \to 0$, there are no simple components a.a.s.
- If $\mu \to \infty$ there are simple components a.a.s.
- If $\mu = \Theta(1)$ then X is Poisson with mean μ

Corollary. The probability of not having simple Components is

$$Pr[X=0]=e^{-\mu}+o(1).$$

Sketch of proof

Compute the *K-th* moment:

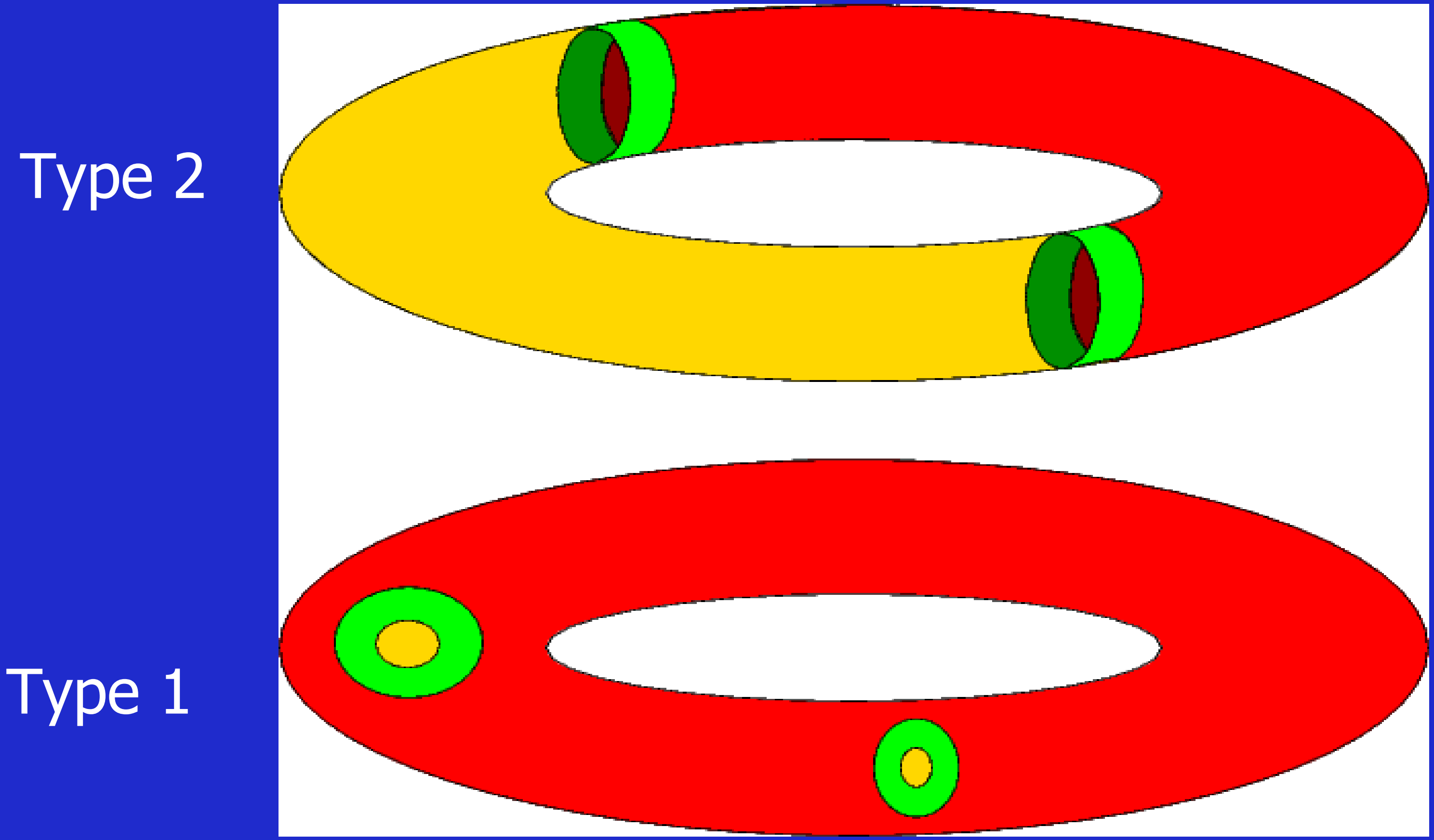
$\mu_k = E[[X]_k] = \sum Pr[S_{v1}=1 \wedge \ldots \wedge S_{vk}=1]$

where $S_{vi}=1$ if v_i is the center of a simple component, otherwise $S_{vi}=0$, and the sum is over all k-tuples of vertices which occupy different walkers.

Use inclusion-exclusion

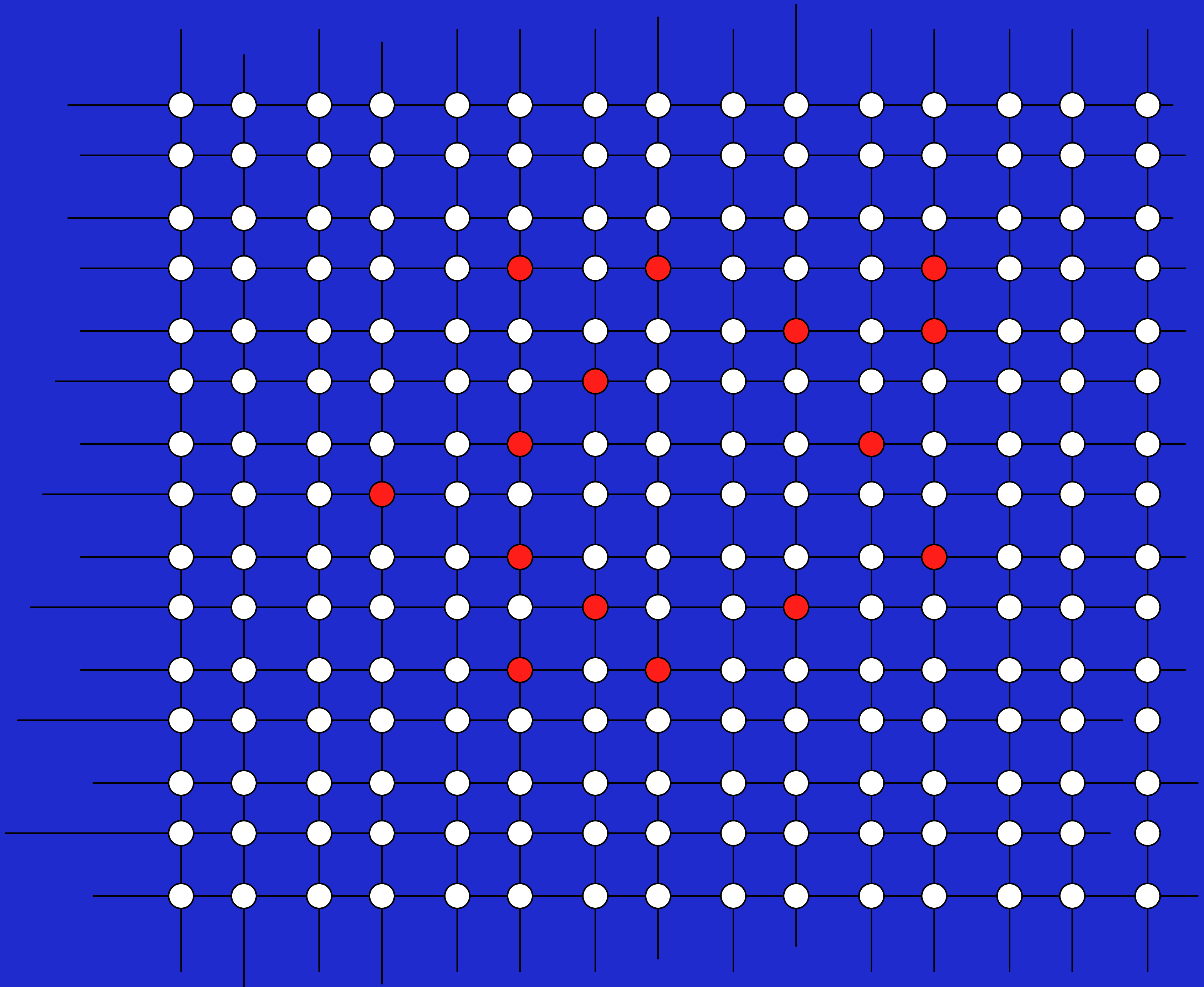
- A r-component a non-simple component which can be embedded in a i ´ j grid ($i,j<n$)
- A nr-component a non-simple component which is not r-component

nr-components

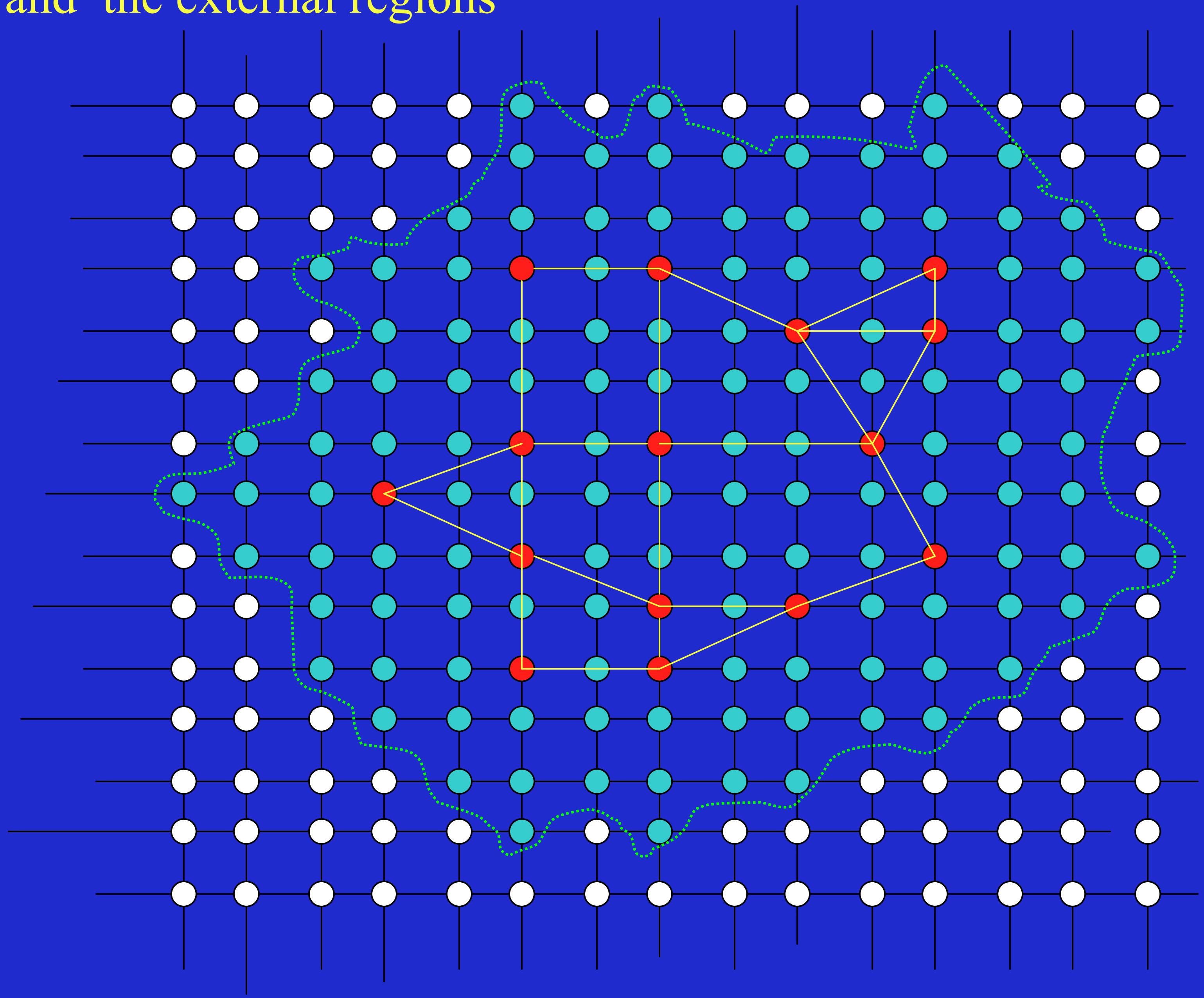


- X = # simple components
- Y = # r-components
- Z_1 = # nr-components which can not coexists with other nr-components
- Z_2 = # non type 1 nr-components

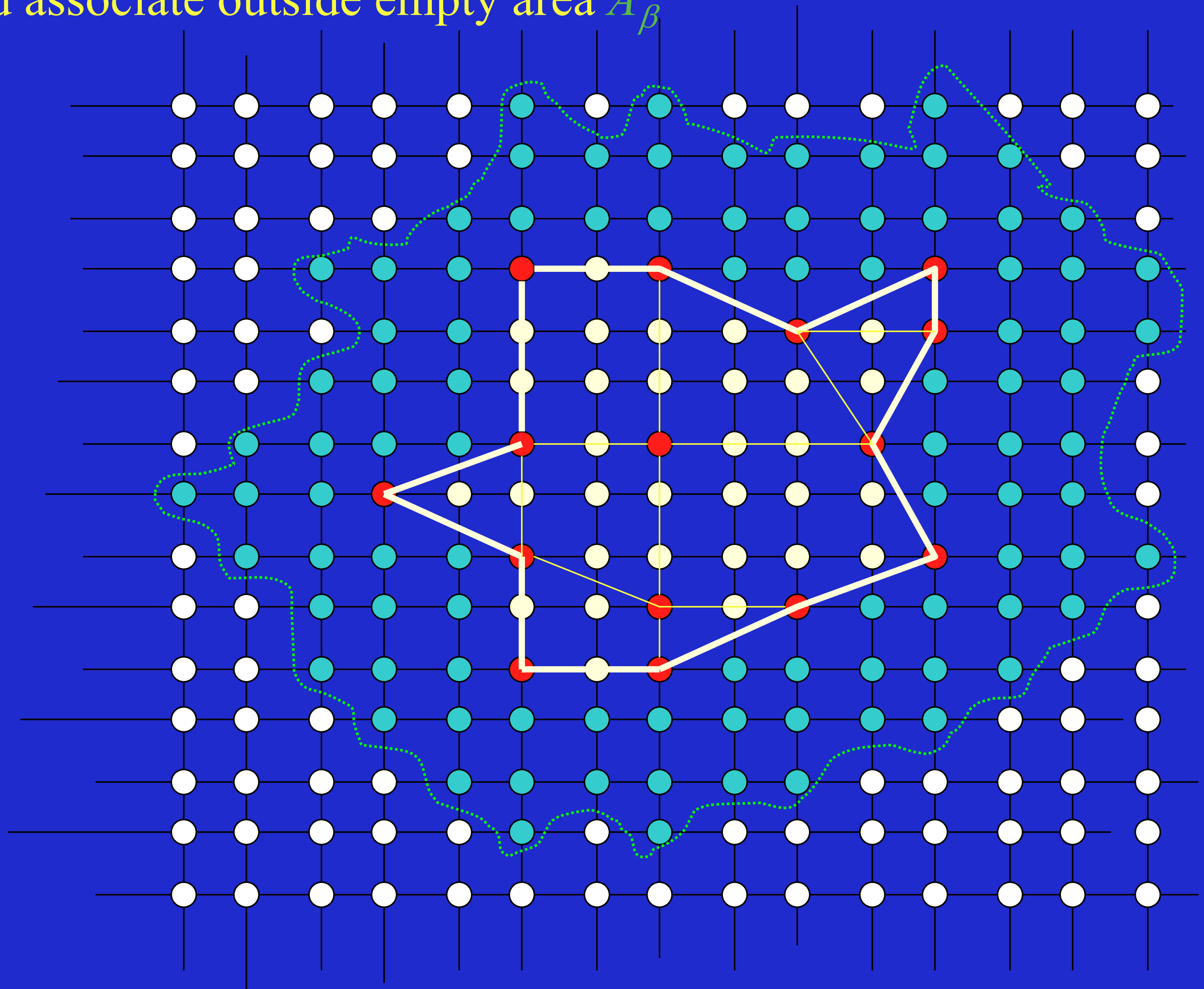
$$K = X + Y + Z_1 + Z_2$$



Connected component C with the edges of C, the associated empty area A_C and the external regions



Connected component C, maximal boundary walk β and associate outside empty area A_β



Geometric Lemma

Let C be a component in T_N with β a max. boundary walk of length l. Then

$|A_\beta| > dl/10^{10}$

If C is rectangular, then $|A_\beta| > h + dl/10^{10}$

Simple components are predominant a.a.s. in T_N

Lemma

If $h\rho = hw/N \rightarrow \infty$, then

- $E[Y] = o(E[X])$
- $E[Z_2] = o(E[X])$

Connectivity of $G_f[W]$

Theorem

For $\mu \to \Theta(1)$, aas $G_f[W]$ consists of simple components and a giant connected component

Corollary. If w walkers are placed uar on T_N, the probability that $G_f[W]$ is connected is

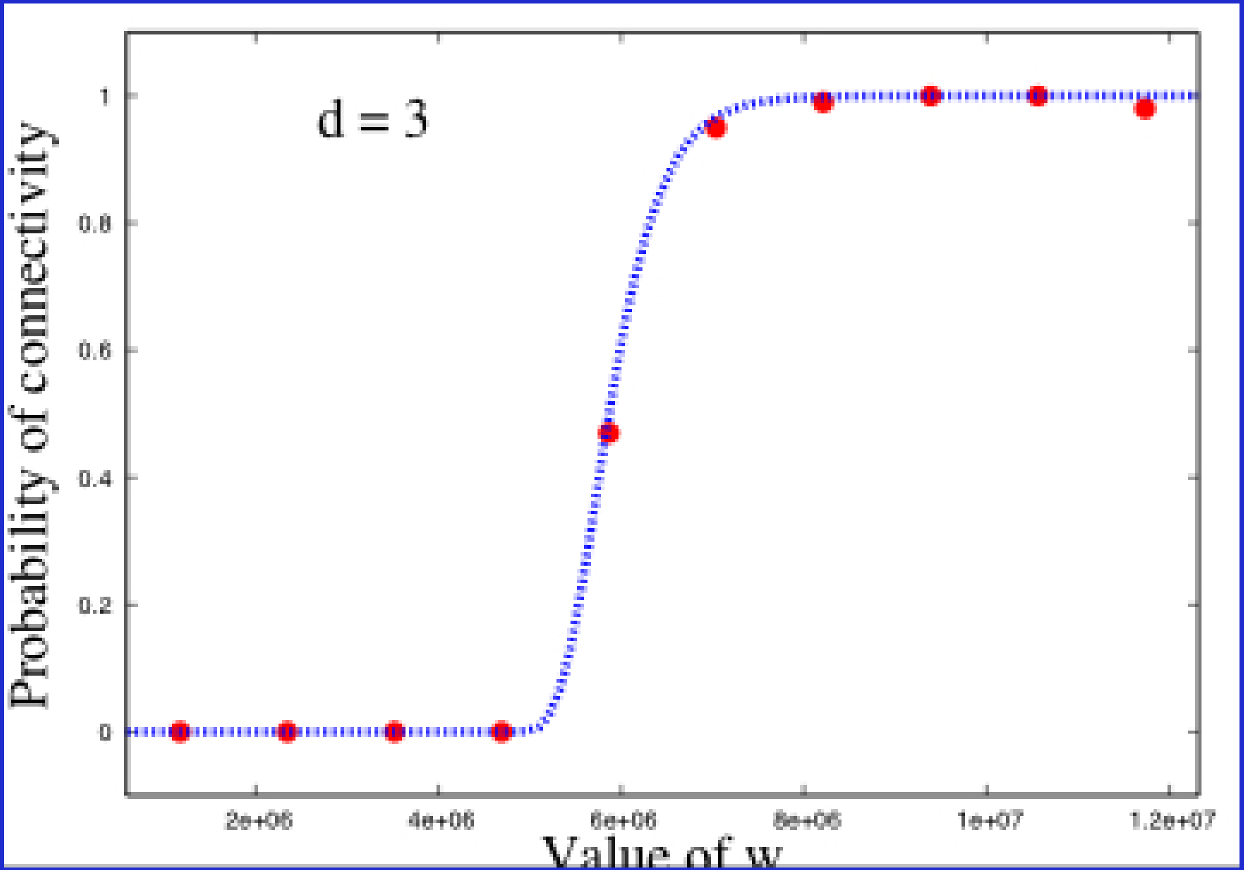
$$e^{-\mu} + o(1).$$

Threshold for connectivity d vs w

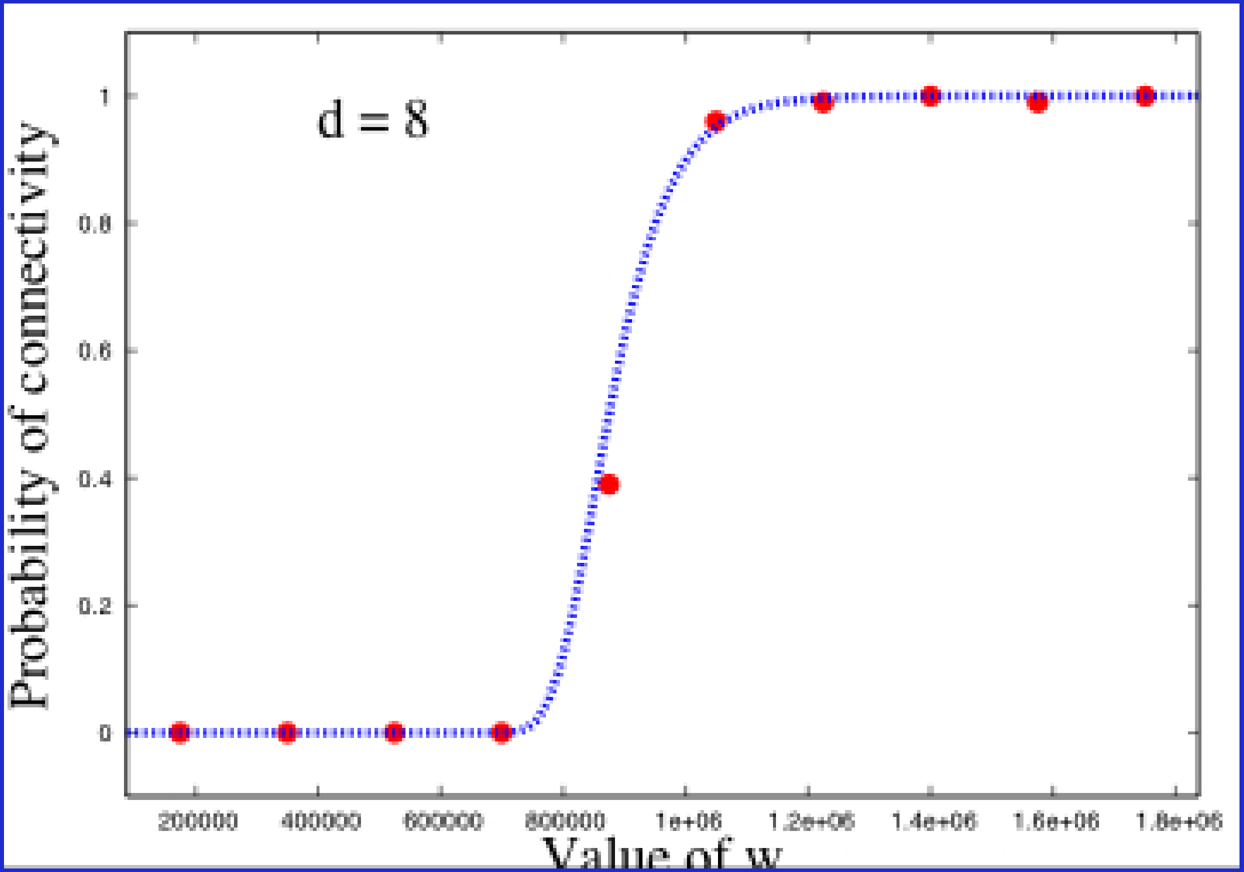
Corollary. If $\mu = \Theta(1)$:

- If $h = \Theta(1)$ iff $w = \Theta(N \log N)$
- If $h = \Theta(\log N)$ iff $w = \Theta(N)$
- If $h = \Theta(N^c \log N)$ iff $w = \Theta(N^{1-c})$
- If $h = \Theta(N/\log N)$ iff $w = \Theta(\log N \log\log N)$

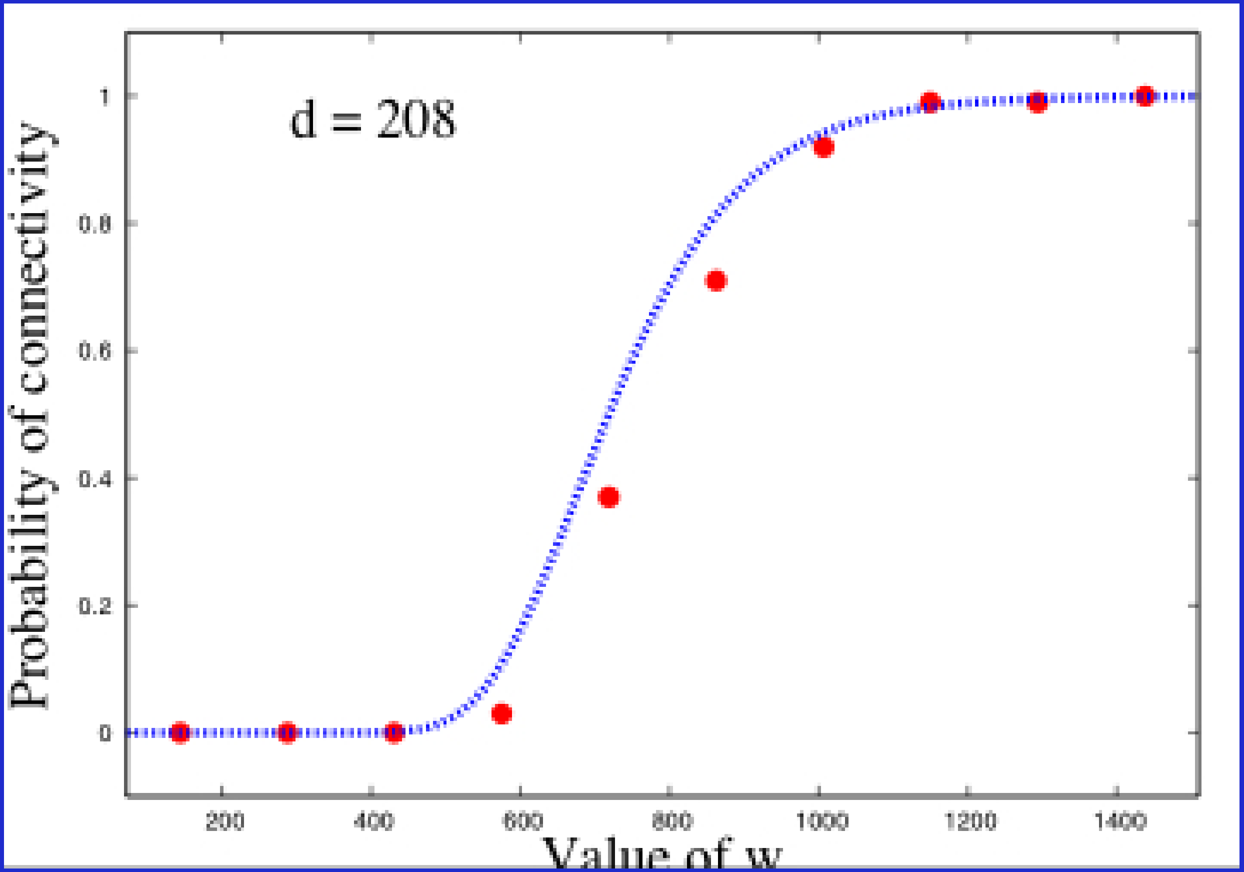
3000x3000; d=cte



3000x3000; d=log n w=875018



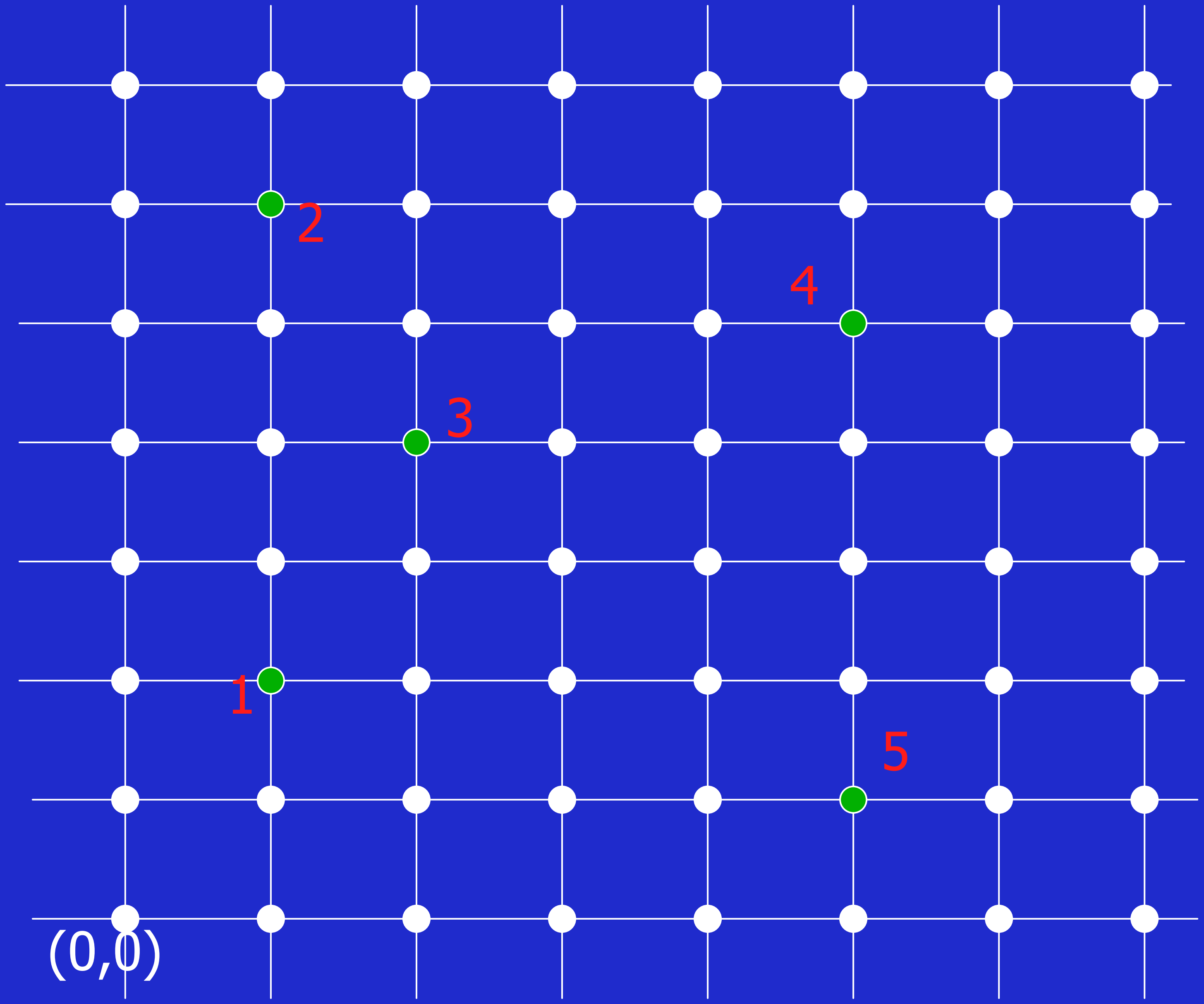
3000x3000; $n^{2/3}$ w=719



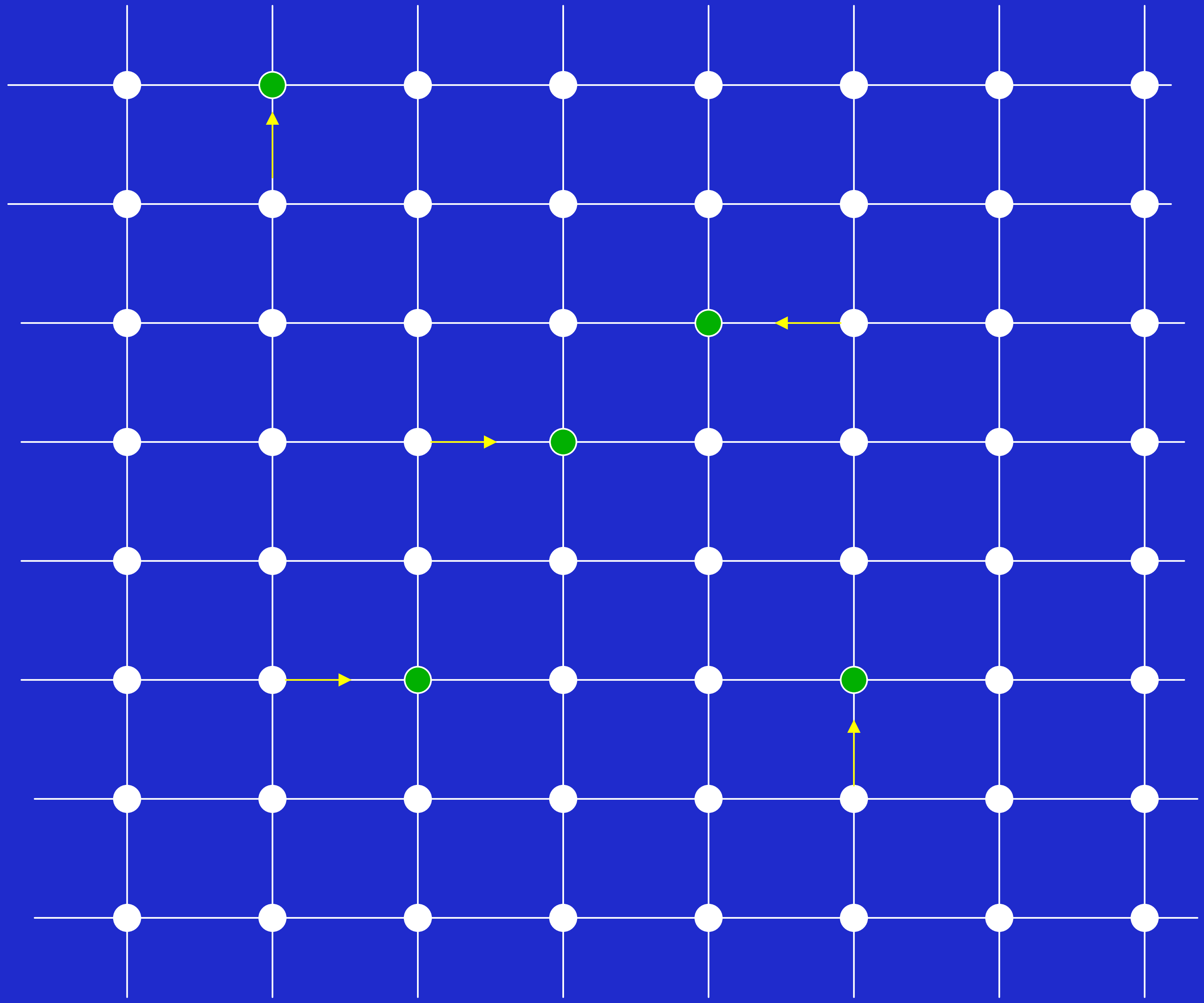
Dynamic properties

Consider labelled (x,y) all vertices in T_N
Given f of $\{1,\ldots,w\}$ on T_N, a configuration (as t evolves) is a vector $\boldsymbol{a}=(a_1,a_2,..,a_w)$, where $a_i=(a_{ix},a_{iy})$ is the label of vertex in which walker i is.

a= ((1,2),(1,6),(3,4),(5,1),(5,5))



$\mathbf{b}=((1,7),(2,2),(3,4),(4,5),(5,2))$



Let the graph M:

V(M)={configurations} and

$(\boldsymbol{a}, \boldsymbol{b})$ in E(M) if $\exists i$ dist$(a_i,b_i)=1$

Notice: $\delta(\boldsymbol{a}) = 4^w$.

The dynamic process is a random walk on M.

Hitting time h_{ab} in M

If N is even, ***a*** and ***b*** have the same parity iff □ i,j $(a_{ix}-a_{jx})+(a_{iy}-a_{jy})=(b_{ix}-b_{jx})+(b_{iy}-b_{jy})$ mod 2.

Lemma Given ***a, b*** in M,
If N is odd, M is ergodic and h_{ab} is finite.
If N is even, M no es ergodic but if ***a*** and ***b*** have the same parity, h_{ab} is finite.

Therefore, the system always reaches a state representing a single connected component, within finite expected time

Notice The initial uniform distribution stays invariant as *t* evolves.

So we need to consider only the case $\mu=\Theta(1)$.

(if $\mu \to 0$ then Gt[W] aas connected
if $\mu \to \infty$ then Gt[W] aas disconnected)

Dynamic random variables

- $X(t)$ = number simple components at time t
- $S(t)$ = number simple components surviving between t and $t+1$
- $B(t)$ = number simple components born between t and $t+1$
- $D(t)$ = number simple components dying between t and $t+1$

Theorem. *S(t), B(t), D(t)* are asymptotically jointly independent Poisson and

$E[S(t)] \sim \mu$ if $dw/N \to 0$,

$E[S(t)] \sim \mu - \lambda$ if $dw/N \to c$,

$E[S(t)] \sim 4\mu(1-e^{-\rho/4})/(1-e^{-\rho})$ if $dw/N \to \infty$,

---

$E[B(t)] = E[D(t)] \sim d\mu\rho$ if $dw/N \to 0$,

$E[B(t)] = E[D(t)] \sim \lambda$ if $dw/N \to c$,

$E[B(t)] = E[D(t)] \sim \mu$ if $dw/N \to \infty$,

with $\lambda = (1- e^{-d\rho})\mu$.

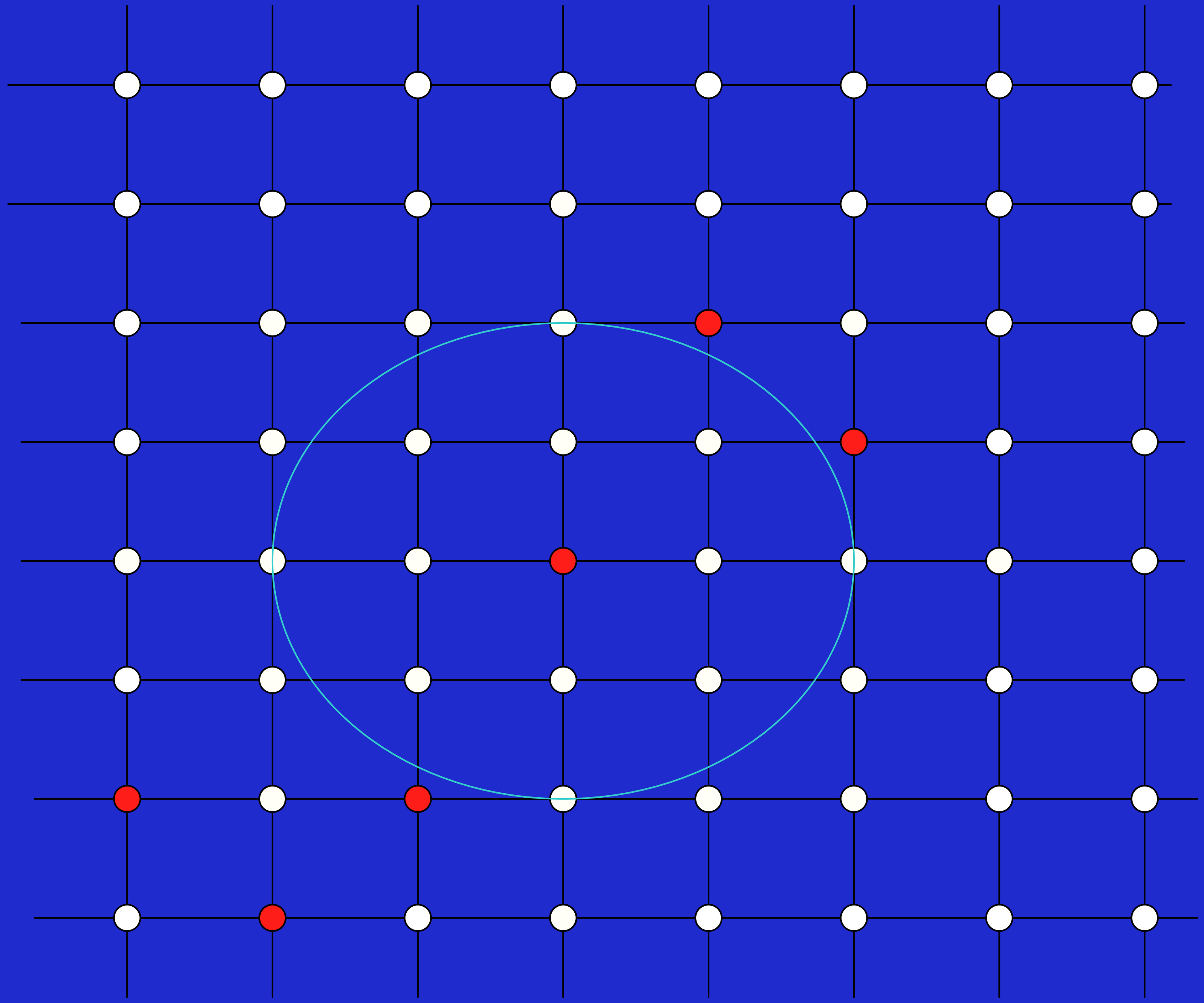
Sketch of proof: consider all cases of S, B, D

Show that S, B, B are jointly asymptotic Poisson

$$E[[S]_q.[B]_r.[D]_p] = \mu_1^{\,q}.\mu_1^{\,r}.\mu_1^{\,p}$$

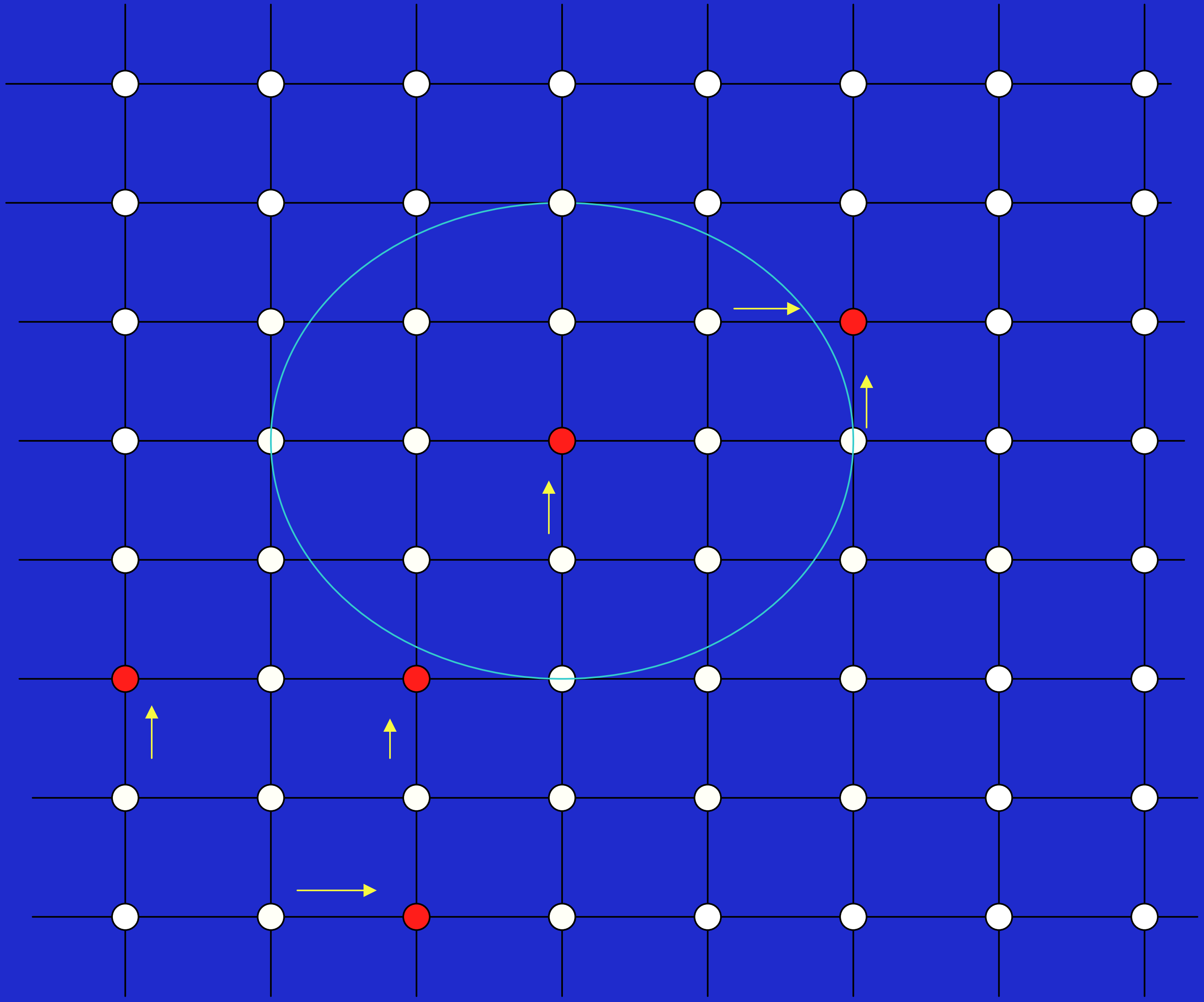
t

Survival sc in l^2 (for $d=2$)



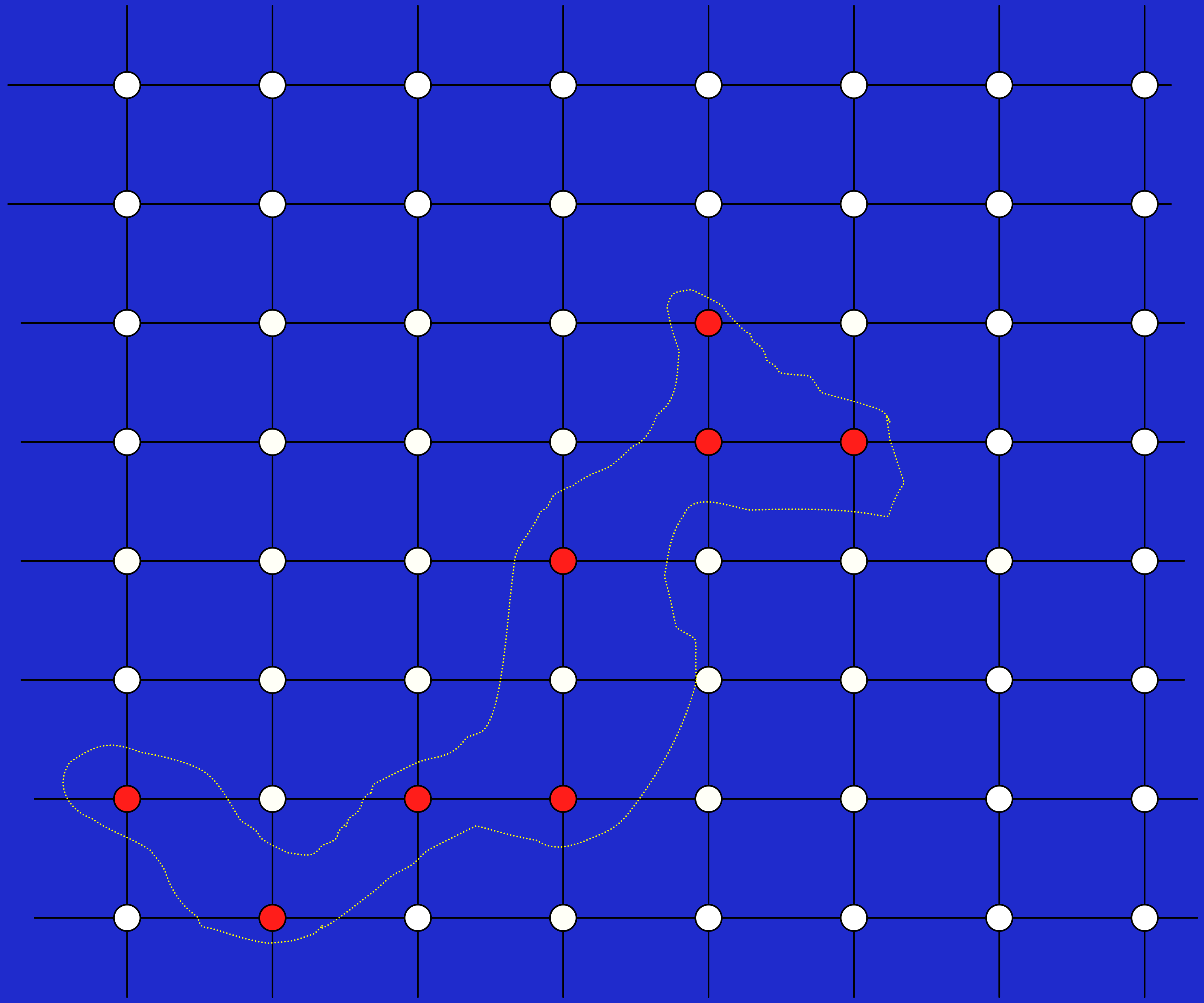
t+1

Survival sc in l^2 (for $d=2$)



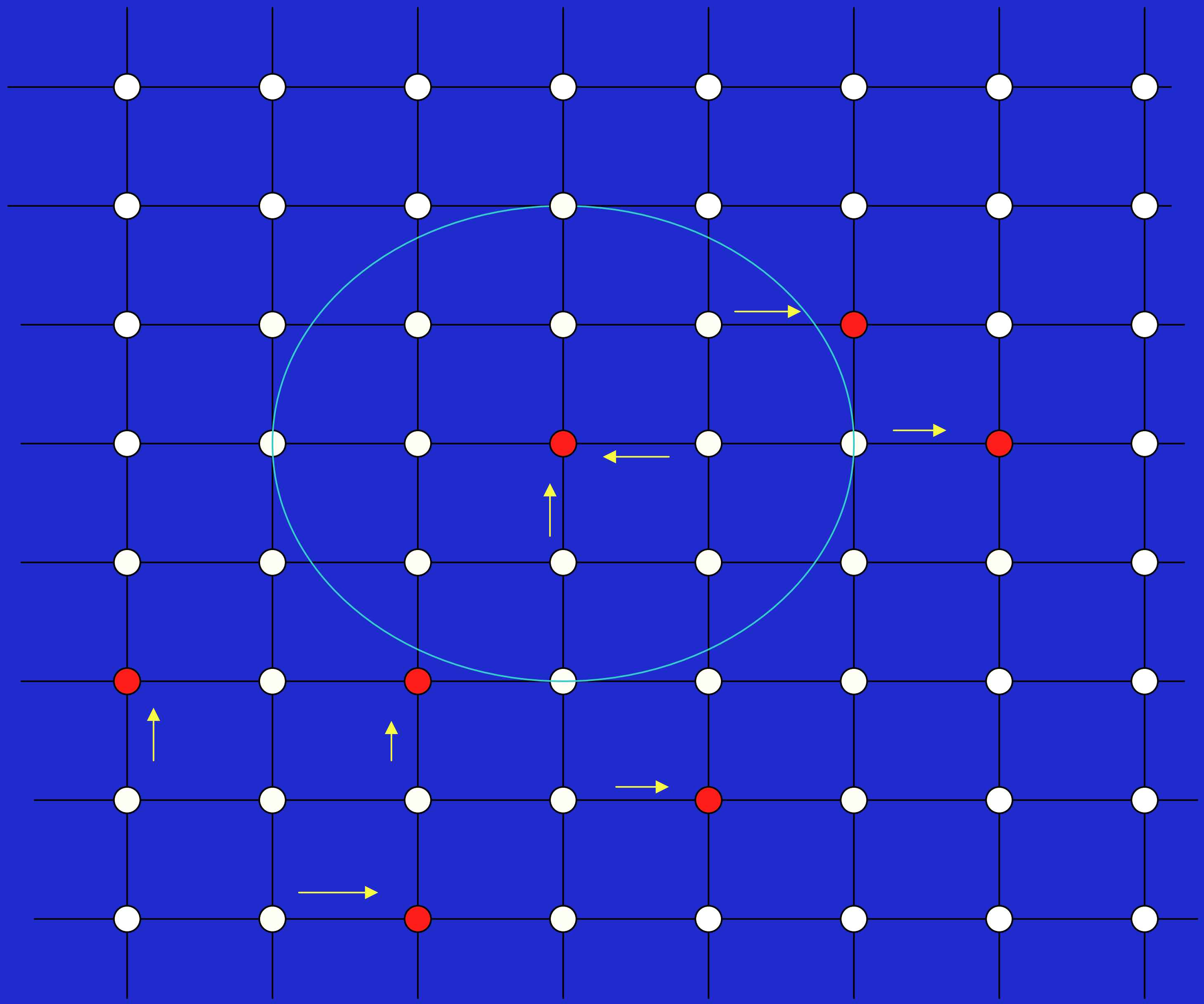
t

Creation sc in l^2 (for $d=2$)



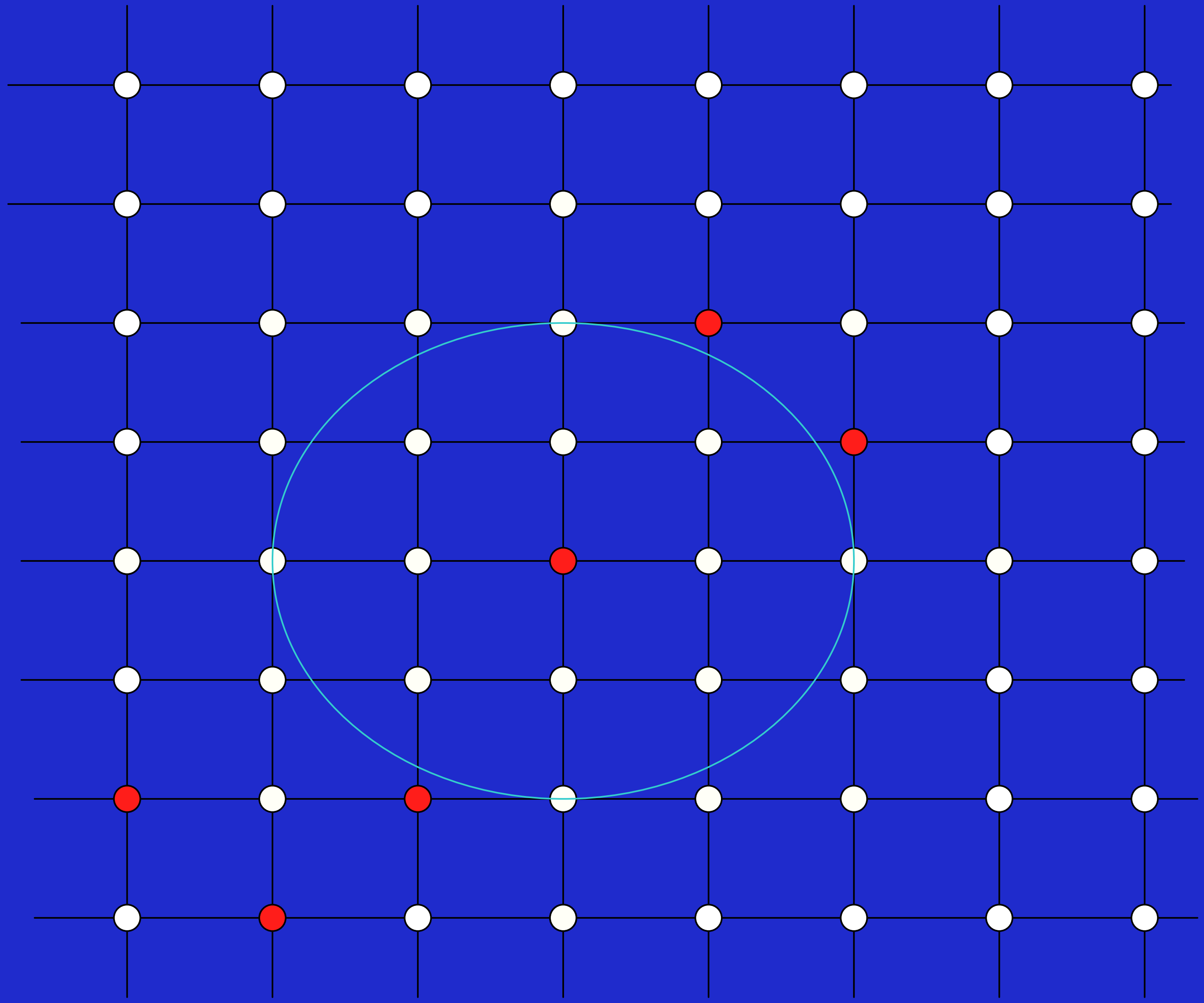
t+1

Survival sc in l^2 (for $d=2$)



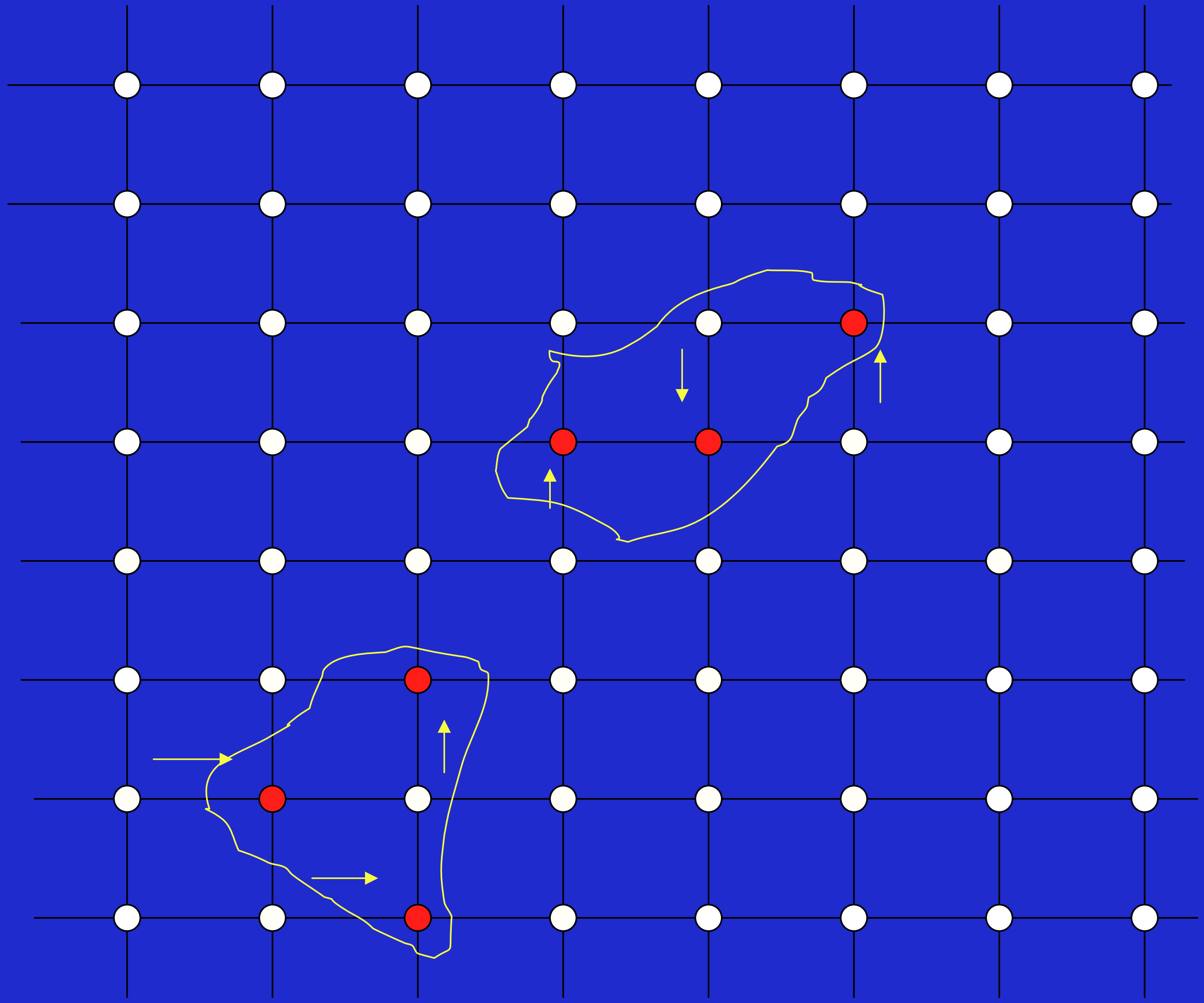
t

Destruction sc in l^2 (for $d=2$)



t+1

Destruction sc in l^2 (for $d=2$)



Prob $G_t[W]$ connected and $G_{t+1}[W]$ disconnected

Theorem.

$Pr[X(t+1) \geq 1 \text{ and } X(t)=0] \sim$

$$\begin{cases} \mu e^{-\mu} b\rho & \text{if } d\rho \to 0, \\ e^{-\mu}(1-e^{-\lambda}) & \text{if } d\rho \to c, \\ e^{-\mu}(1-e^{-\mu}) & \text{if } d\rho \to \infty. \end{cases}$$

Average lifespan of simple component

Lifespan of simple component: number of steps from creation to destruction.

L_{vt}: lifespan simple component at *v*, between t and t+1.

Average lifespan L_T of simple components born in [0,T-1]

$$L_T = (\sum_t \sum_v L_{vt}) / |\{(v,t) : L_{vt} > 0\}|$$

Average lifespan of simple component

Theorem

$L \sim 1/d\rho$ if $d\rho \rightarrow 0$

$L \sim \mu/\lambda$ if $d\rho \rightarrow c$

$L \sim 1$ if $d\rho \rightarrow \infty$

Average connectiveness

Let C be the average connectivity of $G_f[W]$

The random variable counting the expected length of any connected period

Theorem

$C \sim 1/d\rho\mu$ if $d\rho \to 0$

$C \sim 1/(1-e^{-\lambda})$ if $d\rho \to c$

$C \sim 1/(1-e^{-\mu})$ if $d\rho \to \infty$

Average disconnectiveness

Let D be the average disconnectivity of $G_f[W]$

Theorem

$D \sim (e^{-\mu} - 1)/d\rho\mu$ if $d\rho \to 0$

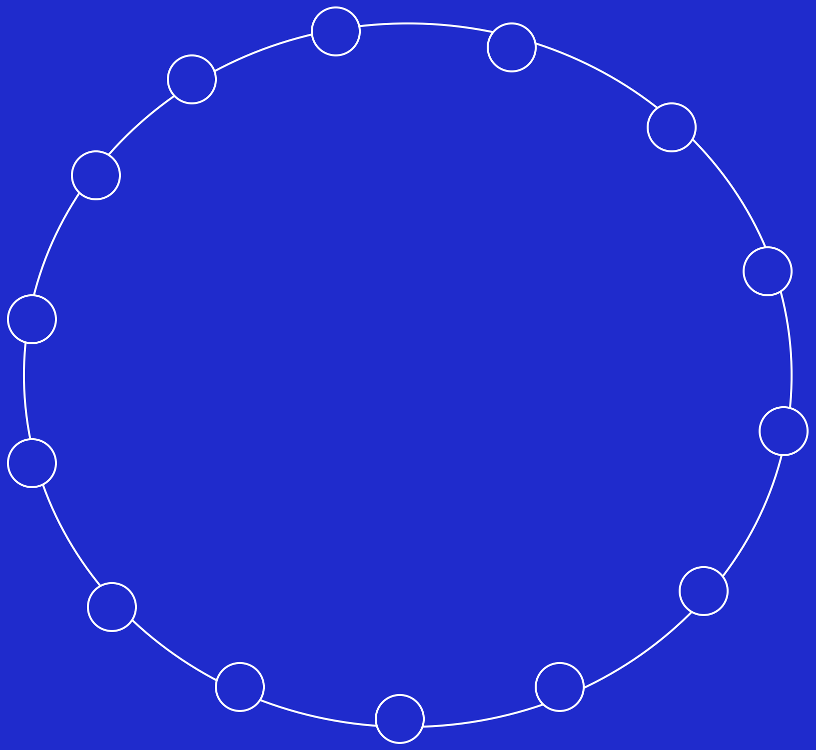
$D \sim (e^{-\mu} - 1)/(1 - e^{-\lambda})$ if $d\rho \to c$

$D \sim e^{-\mu}$ if $d\rho \to \infty$

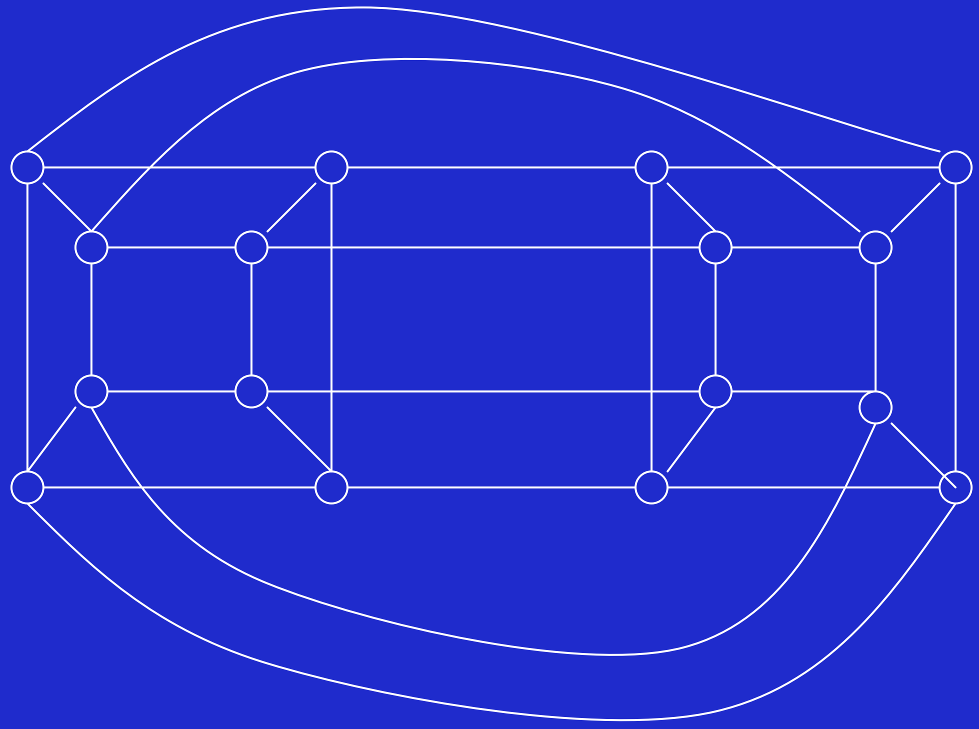
$N = 1000 \times 1000$		Experimental average	Modified prediction
$d = 3$	Time $G_{f_t}[W]$ stays connected	1.93	2.08
$w = 555377$	Time $G_{f_t}[W]$ stays disconnected	2.14	2.02
$d = 7$	Time $G_{f_t}[W]$ stays connected	2.05	2.01
$w = 106128$	Time $G_{f_t}[W]$ stays disconnected	2.70	2.88
$d = 10$	Time $G_{f_t}[W]$ stays connected	2.28	2.20
$w = 50804$	Time $G_{f_t}[W]$ stays disconnected	3.17	2.49
$d = 32$	Time $G_{f_t}[W]$ stays connected	4.89	4.97
$w = 4113$	Time $G_{f_t}[W]$ stays disconnected	7.56	8.15
$d = 100$	Time $G_{f_t}[W]$ stays connected	14.14	15.26
$w = 301$	Time $G_{f_t}[W]$ stays disconnected	27.86	33.42
$d = 145$	Time $G_{f_t}[W]$ stays connected	18.97	21.35
$w = 122$	Time $G_{f_t}[W]$ stays disconnected	55.20	63.51

Similar results obtained for:

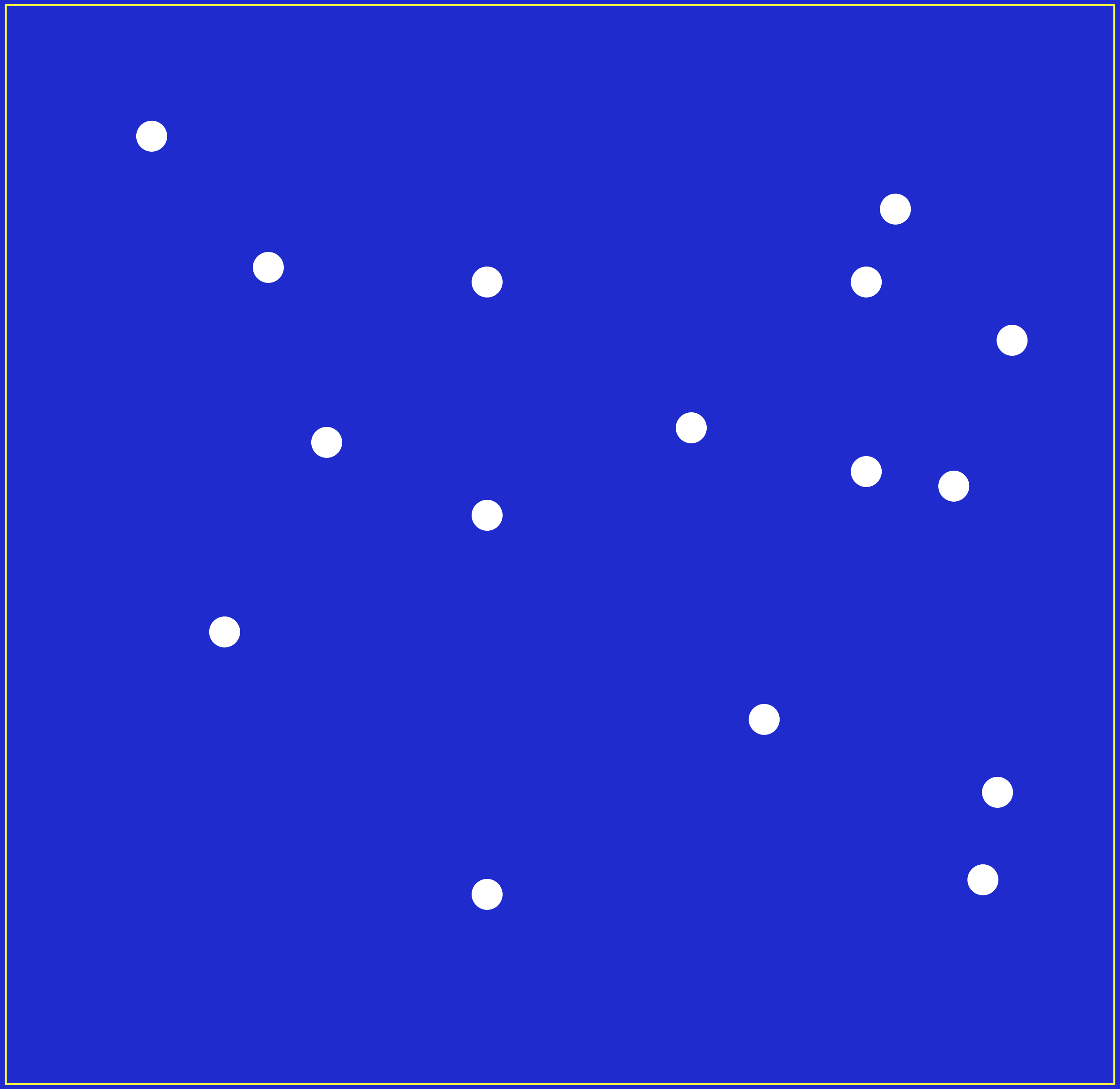
Cycle C_N

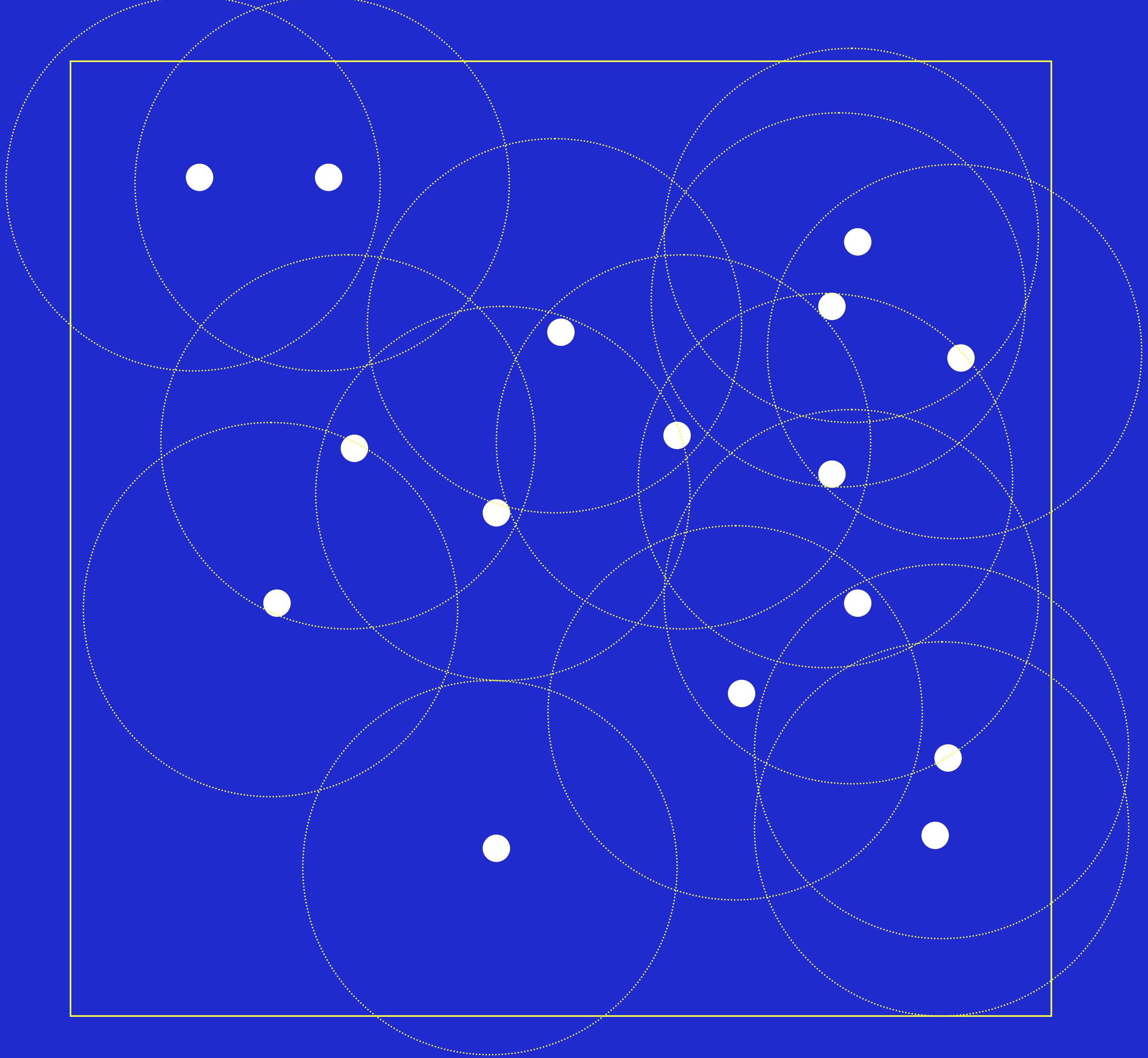


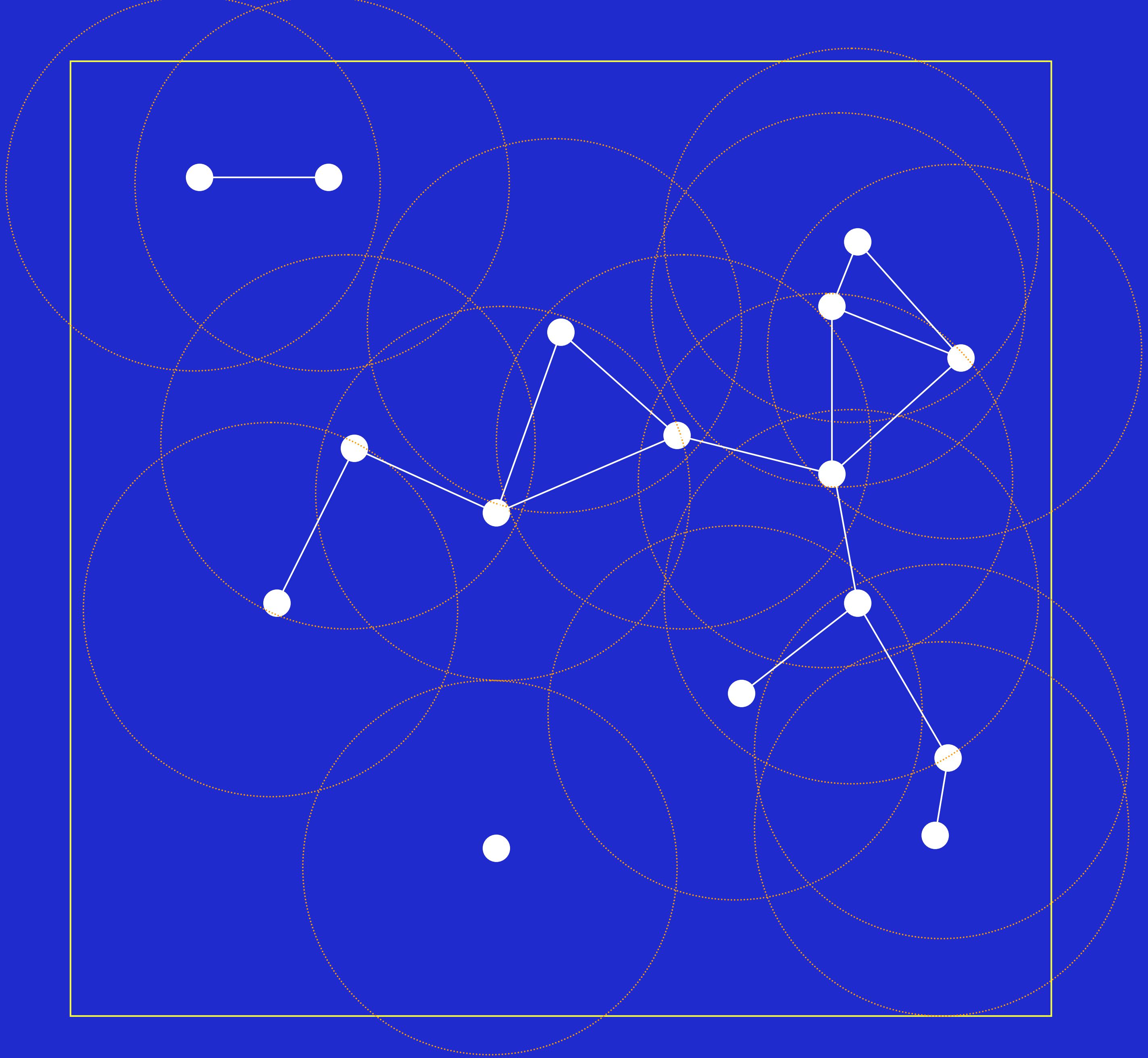
n-dimensional hypercube: H_N

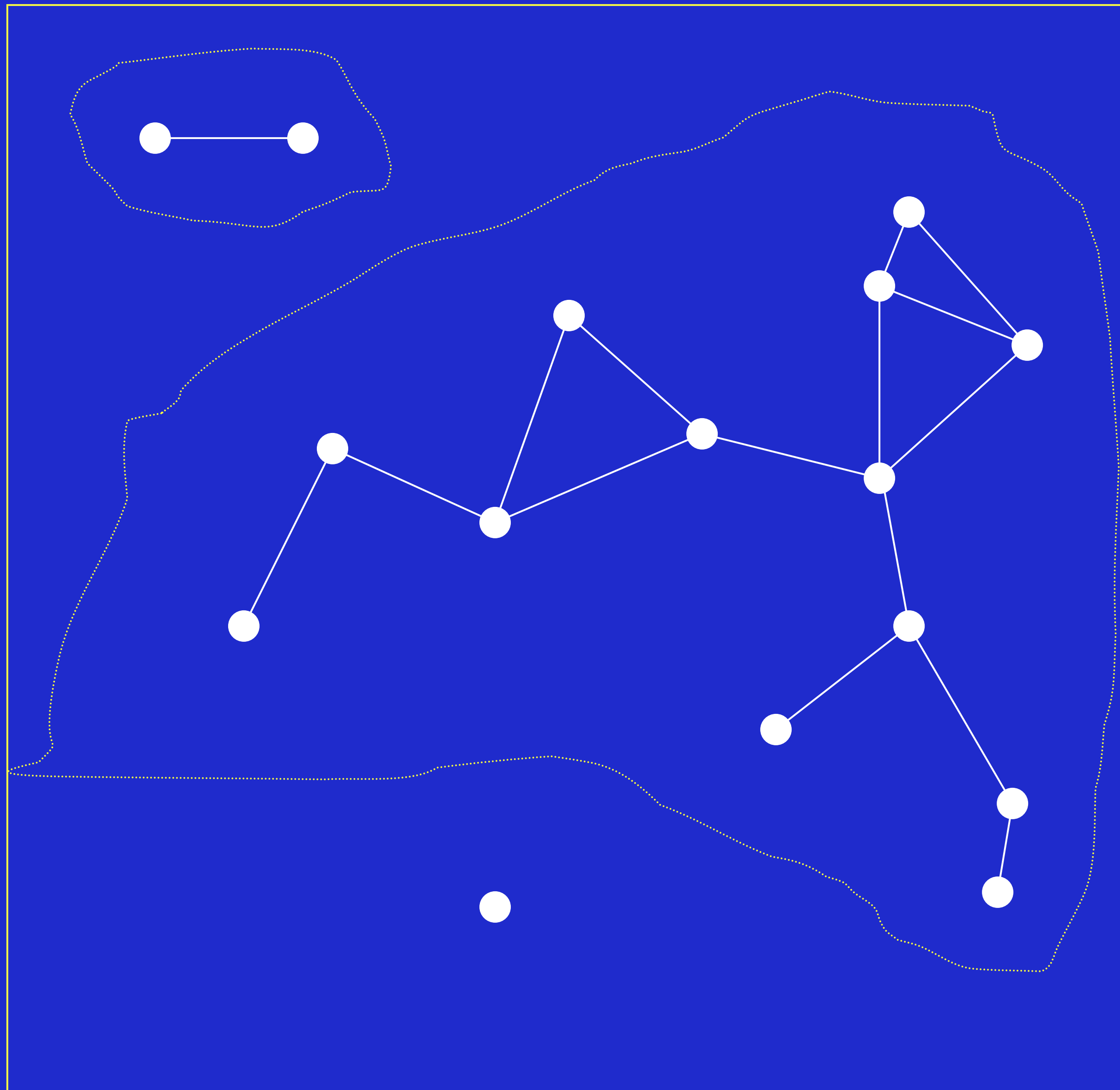


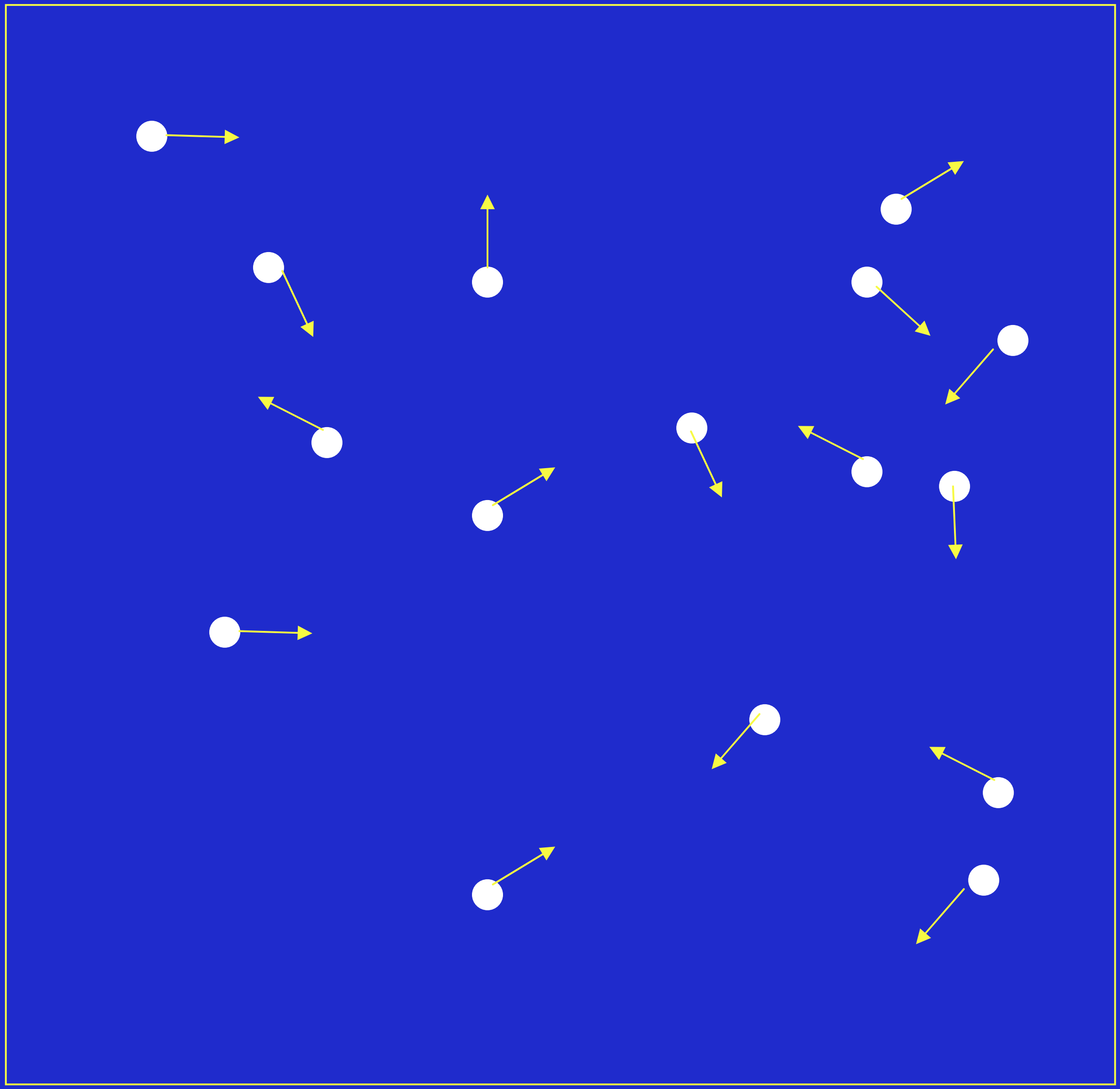
Future work on Random Geometric Graphs

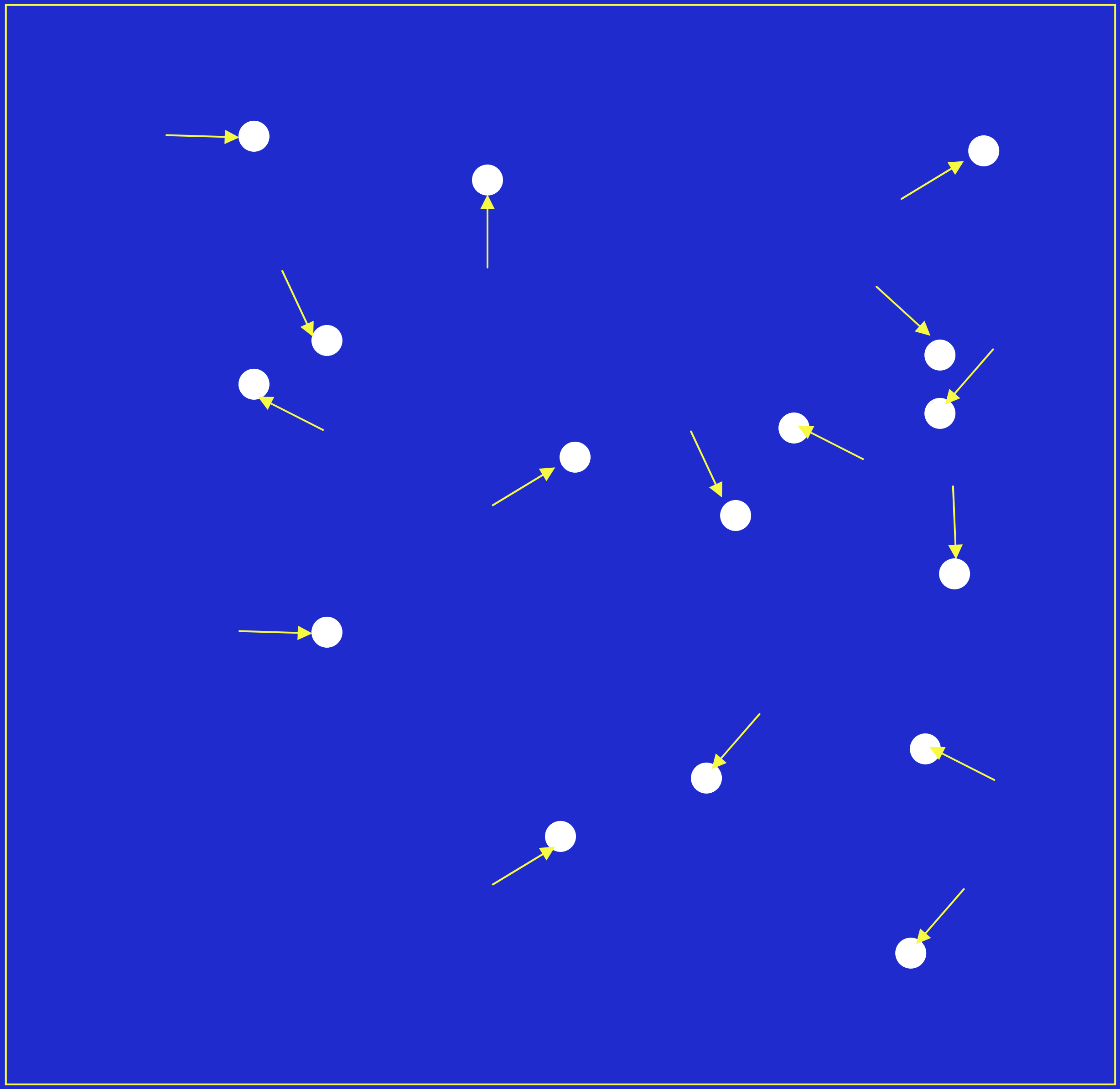


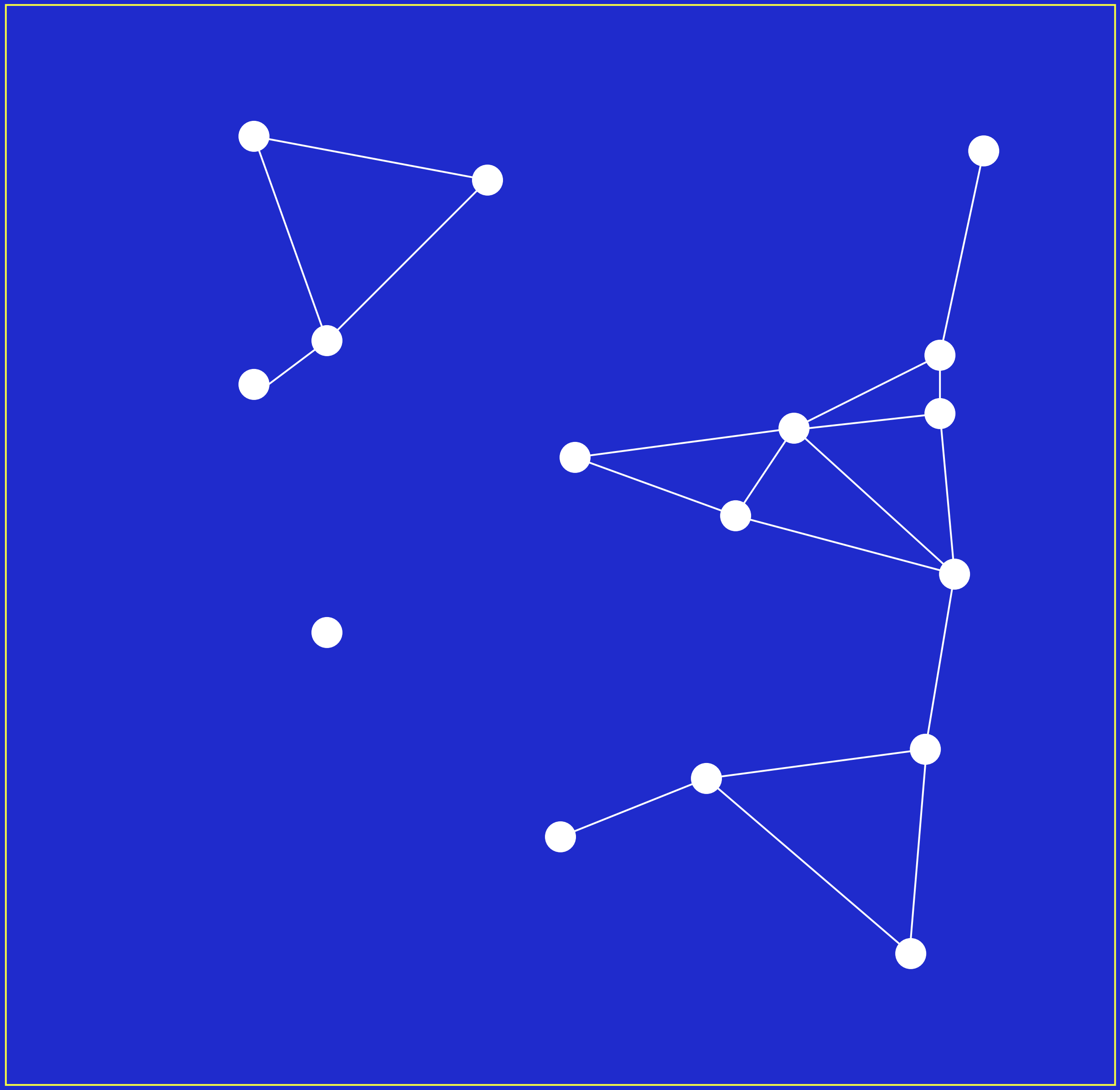












ありがとう