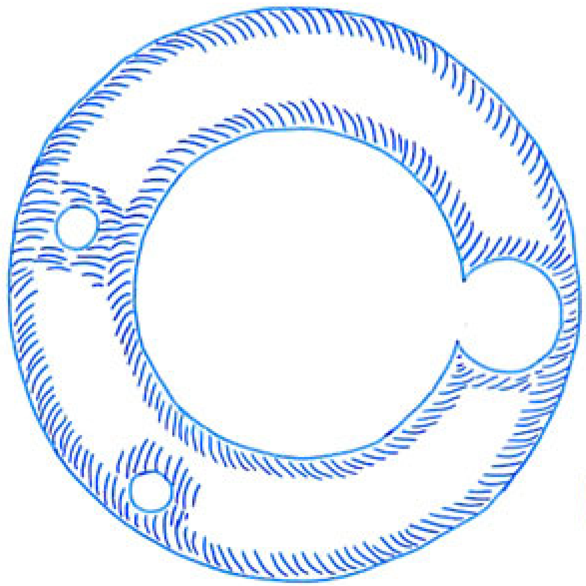
Steps into Computational Algebraic Topology

I. Topological Persistence

II. Stable Diagrams

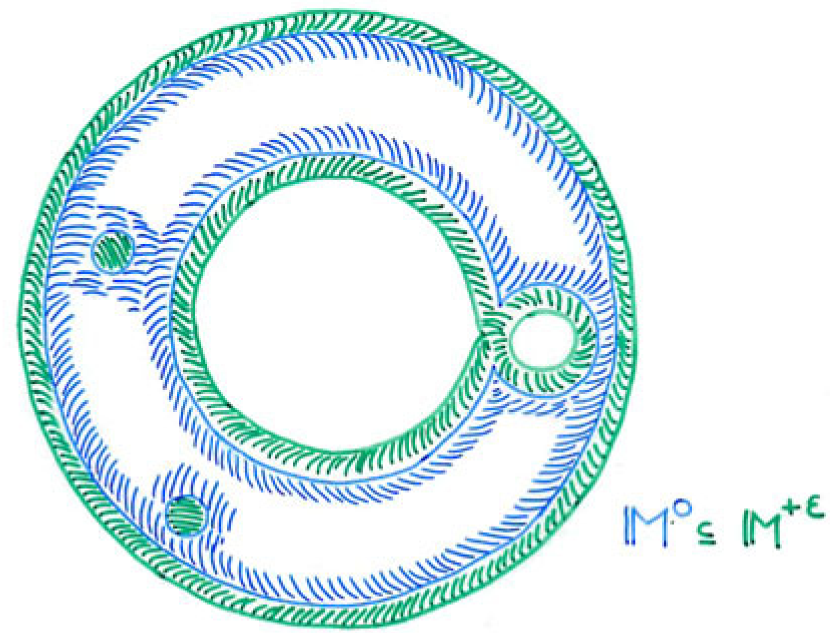
III. Applications

I.1 Small'n'Big Holes



IM°

I.1 Small'n'Big Holes



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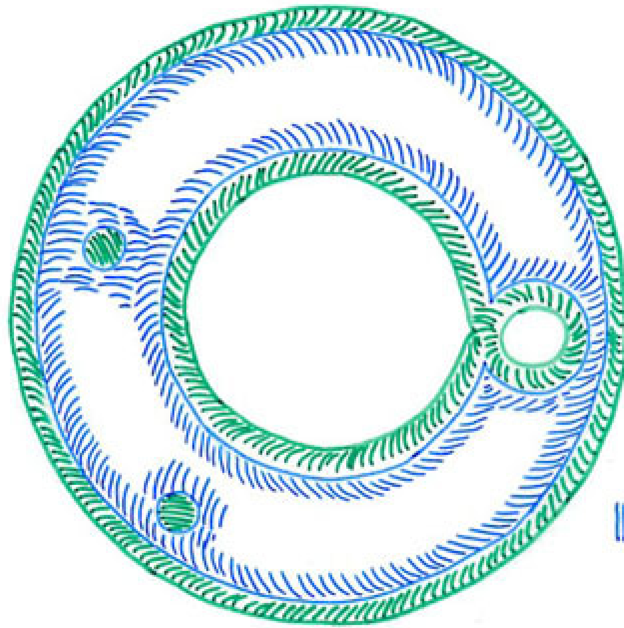


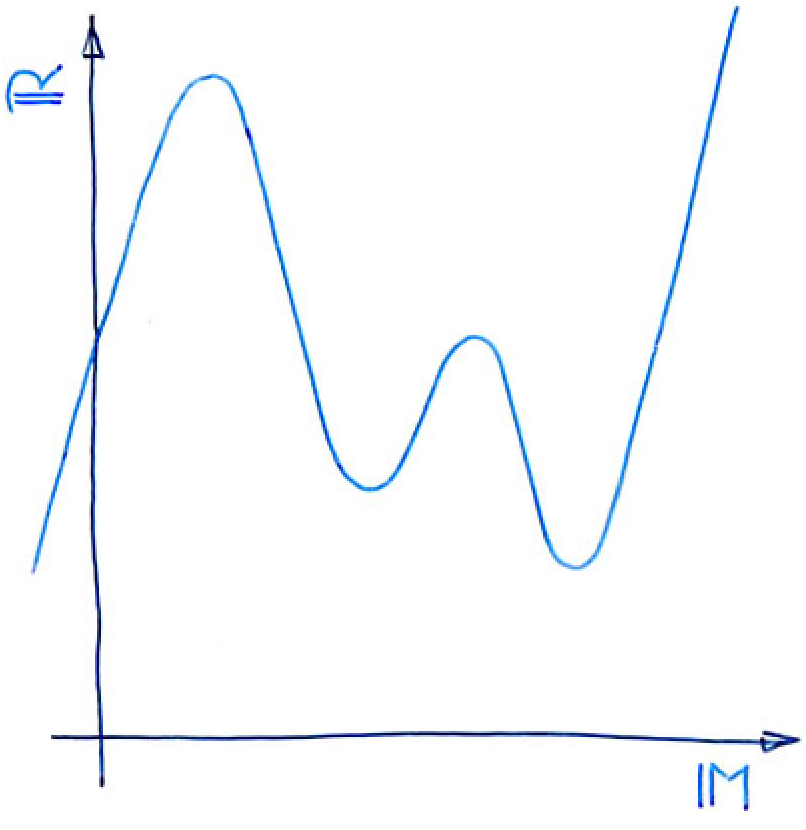
	$H_1(\mathbb{M}^0)$	$H_1(\mathbb{M}^{+\varepsilon})$	F_0^ε
dim	3	2	1

where $F_0^\varepsilon = \mathrm{im}\left(H_1(\mathbb{M}^0) \to H_1(\mathbb{M}^{+\varepsilon}) \right)$.

I.2 Function

$$f : \mathbb{M} \to \mathbb{R}$$

$$\mathbb{M}^x = f^{-1}(-\infty, x]$$



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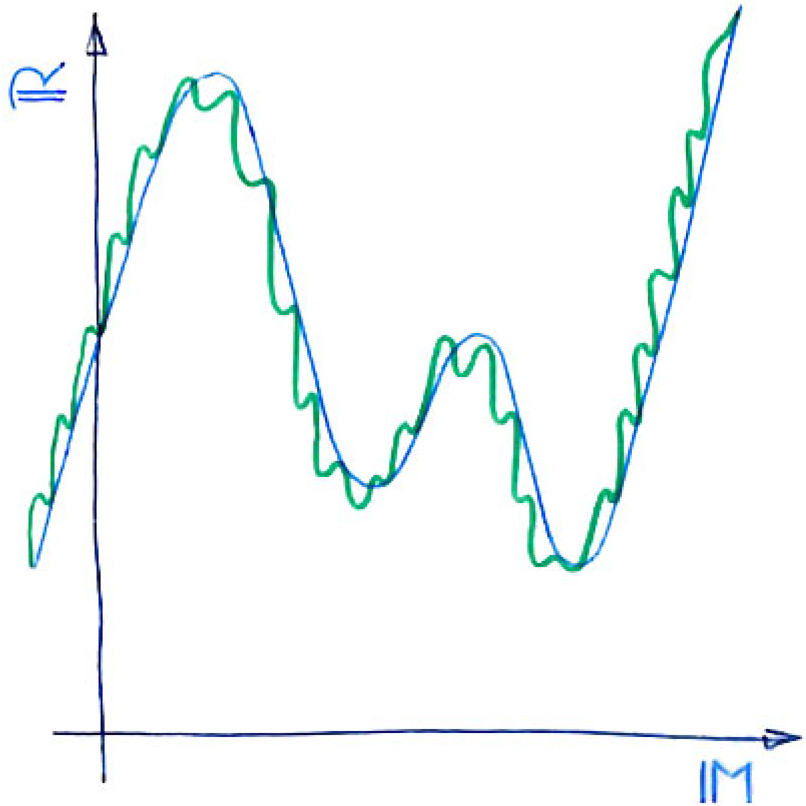


I.3 Persistent Homology

$$\dots \subset \mathbb{M}^x \subset \dots \subset \mathbb{M}^y \subset \dots$$

$$\dots \to H^x \to \dots \to H^y \to \dots$$

<u>persistent</u> <u>homology</u> <u>groups</u> are

$$F_x^y = \mathrm{im}\,(H^x \to H^y) \subseteq H^y$$

<u>persistent</u> <u>Betti</u> <u>numbers</u> are

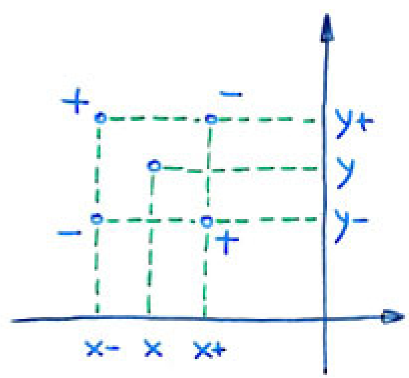
$$\beta_x^y = \dim F_x^y$$

I.4 CRIT. VALUE PAIRS

$$\dots \to H^{x-} \to H^{x+} \to \dots \to H^{y-} \to H^{y+} \to \dots$$

<u>multiplicity</u> is

$$\mu_x^y = \beta_{x-}^{y+} - \beta_{x-}^{y-} - \beta_{x+}^{y+} + \beta_{x+}^{y-}$$



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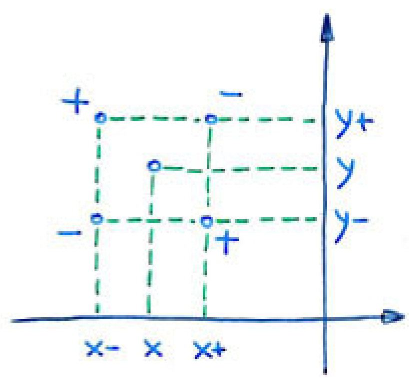
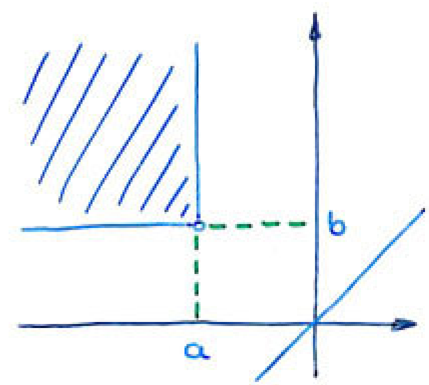


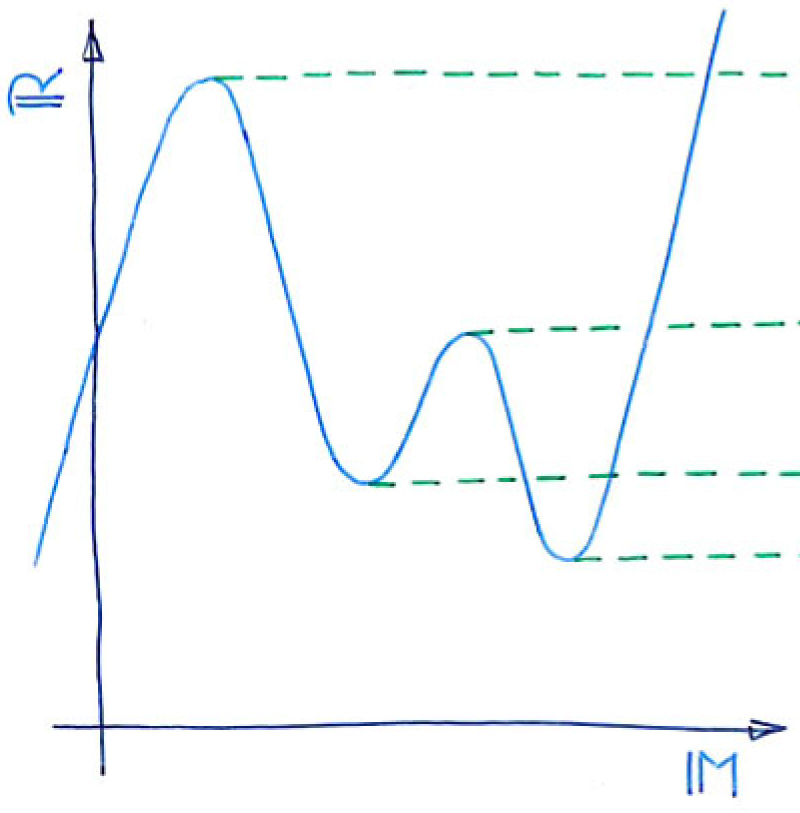


$$\beta_a^b = \sum_{x \le a \le b \le y} \mu_x^y$$

I.2 FUNCTION

$f : \mathbb{M} \to \mathbb{R}$

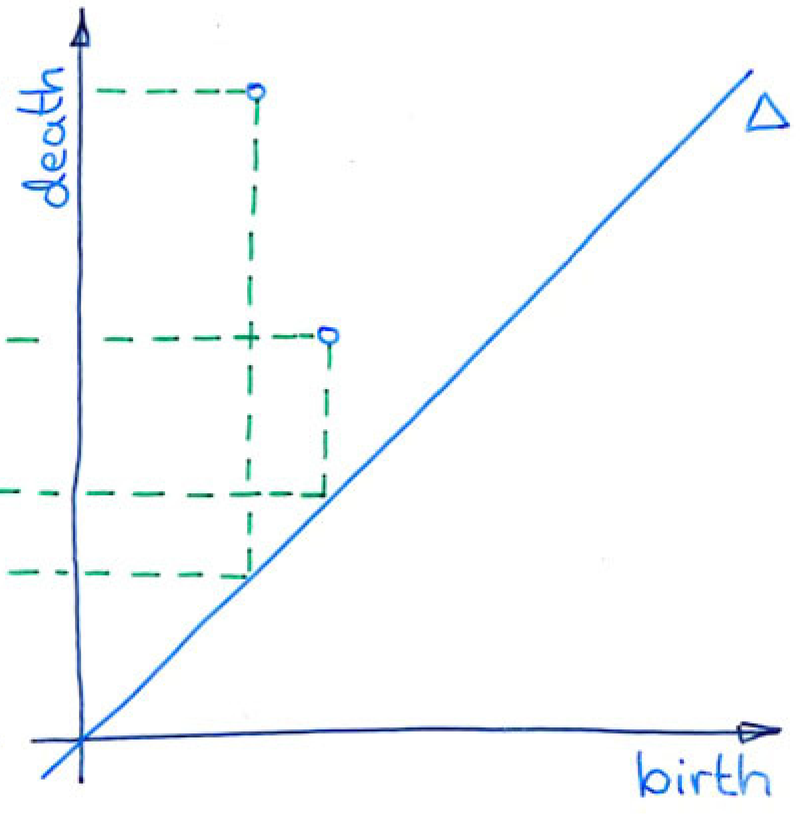
$\mathbb{M}^x = f^{-1}(-\infty, x]$



II.1 DIAGRAM

persistence diagram of f is

$D(f) = \Delta \cup \{\text{bd-pairs}\}$



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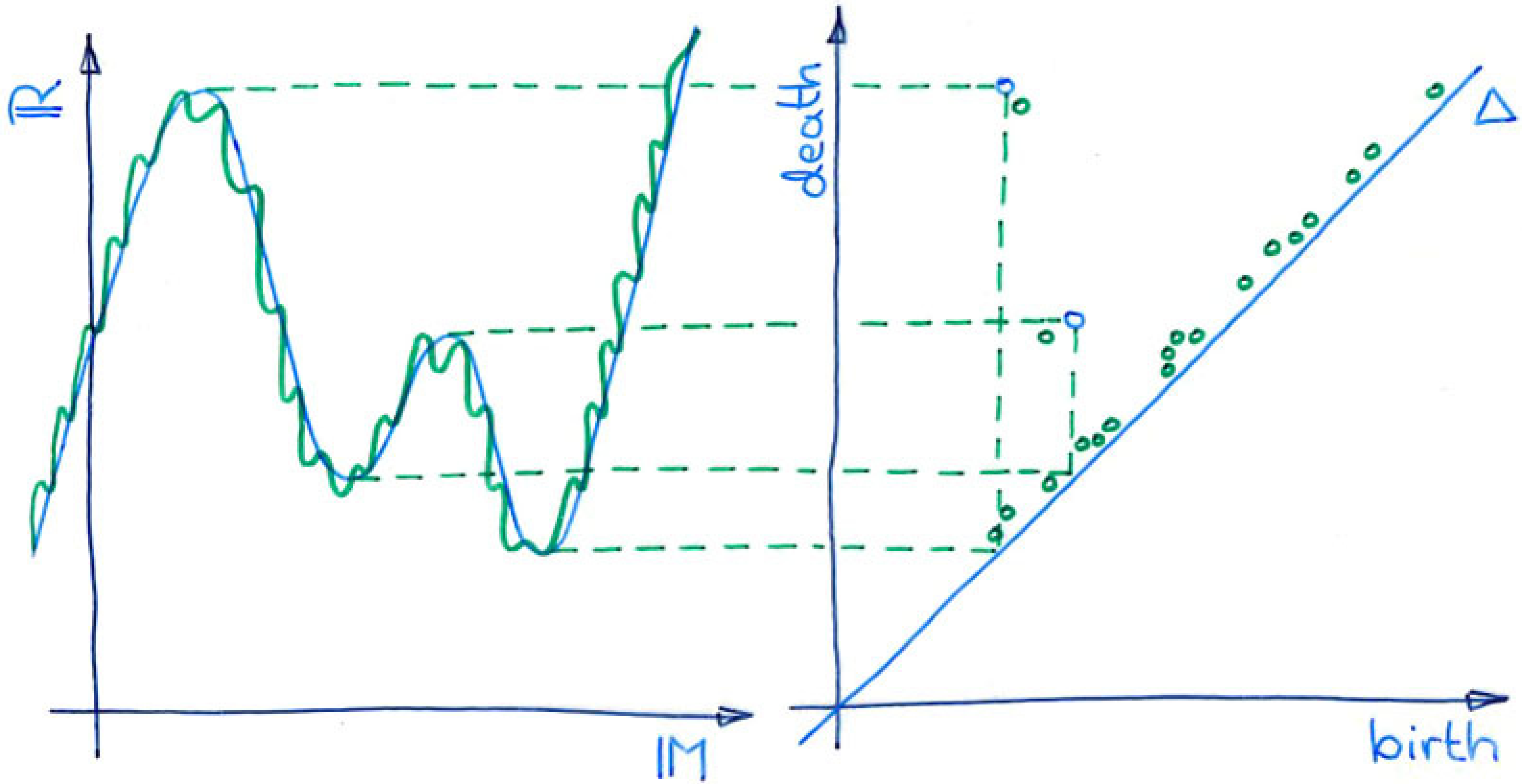

II.2 METRICS

<u>L_∞-distance</u> between functions

is

$$\|f-g\|_\infty = \sup_{x \in \mathbb{M}} |f(x)-g(x)|.$$

<u>Fréchet</u> <u>bottleneck</u> <u>distance</u>

between diagrams is

$$d_F(D(f), D(g)) = \inf_{\gamma: D(f) \to D(g)} \sup_{p \in D(f)} \|p-\gamma(p)\|_\infty.$$

II.3 Stability Theorem

Thm [Cohen-Steiner, E, Harer 2004]

$\mathbb{M}$ is triangulable space,

$f, g : \mathbb{M} \to \mathbb{R}$ are continuous and tame.

Then

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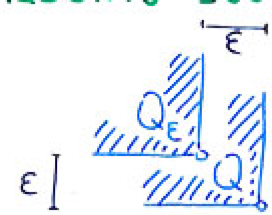
$\Longrightarrow$ **Q LEMMA** [Robins 1999]

$\varepsilon = \|f-g\|_\infty$



Then

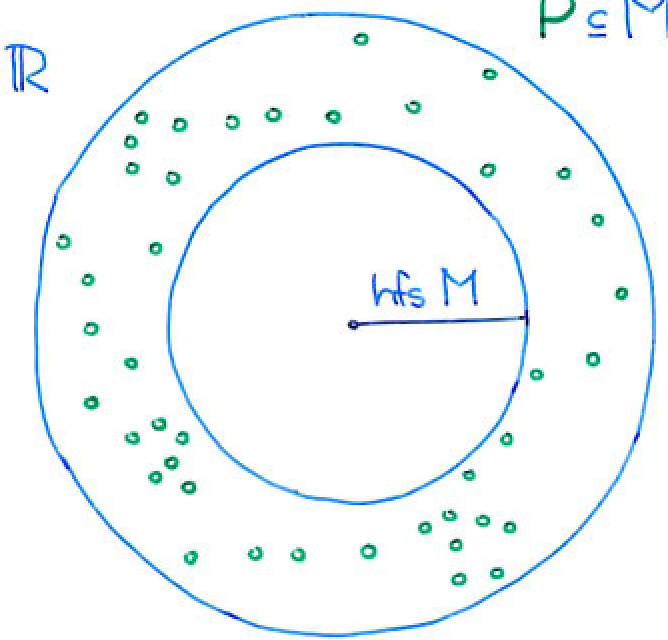
$$\#(D(f) \cap Q_\varepsilon) \leq \#(D(g) \cap Q).$$

III.1 Homology Estimation

$P \subseteq M$

$d^M, d^P: \mathbb{R}^d \to \mathbb{R}$

distance function of M, P



<u>hom.</u> <u>feature</u> <u>size</u> of d^M is

$\operatorname{hfs} M$ = min. pos. hom. crit. value

M_x^y, P_x^y ... persistent hom. groups

III.2 APPROXIMATION

THM. $d(P, M) < \varepsilon < \frac{\mathrm{hfs}\, M}{4}$.

Then

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PROOF. $\| d^P - d^M \| < \varepsilon$.

Q Lemma implies

$$\dim M_0^{4\varepsilon} \leq \dim P_{\varepsilon}^{3\varepsilon} \leq \dim M_{2\varepsilon}^{2\varepsilon}$$

$$= \qquad\qquad =$$

because $4\varepsilon < \mathrm{hfs}\, M$

$$\dim(H(M)) = \dim M_0^0 = \dim M_0^{4\varepsilon}$$

$\square$

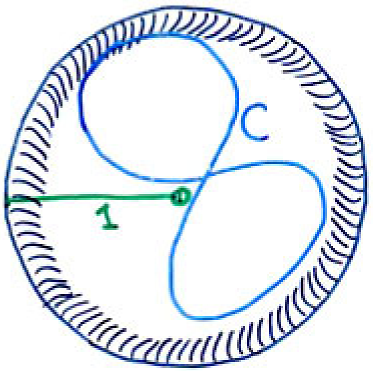
III.3 CURVES

$$K(C) = \int \kappa(s)\, ds$$

$$L(C) = \frac{1}{2} \int \chi(h \cap C)\, dh$$

Fary's THM

$L(C) \leq K(C).$



THM $|L_1 - L_2| \leq (K_1 + K_2 - 2\pi) \cdot d_F(C_1, C_2).$

III.4 Surfaces

$$G(S) = \int |\kappa_1(s) \cdot \kappa_2(s)| \, ds$$

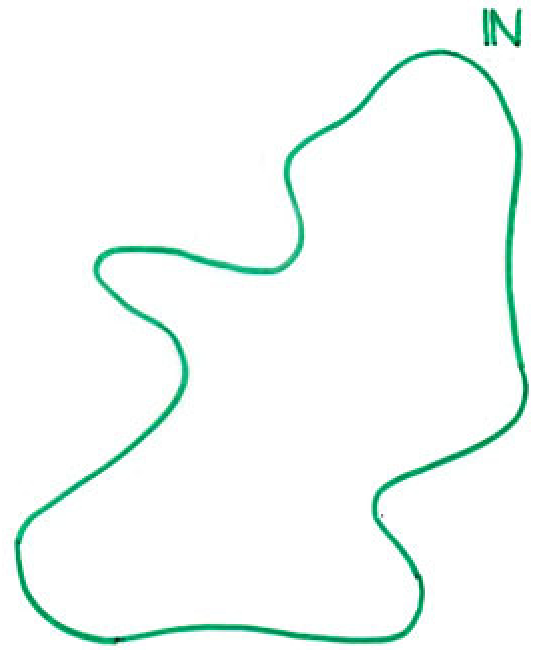
$$= 2\pi \, (\text{avg. \# crit. pts. of height f.})$$

$$H(S) = \frac{1}{2} \int (\kappa_1(s) + \kappa_2(s)) \, ds$$

$$= \int \chi(h \cap \bar{S}) \, dh$$

Thm. $|H_1 - H_2| \leq (G_1 + G_2 - 2\pi(1+g)) \cdot d_F(\bar{S}_1, \bar{S}_2).$

III.5 Protein Interaction



docking = comput. prediction of interacting configuration

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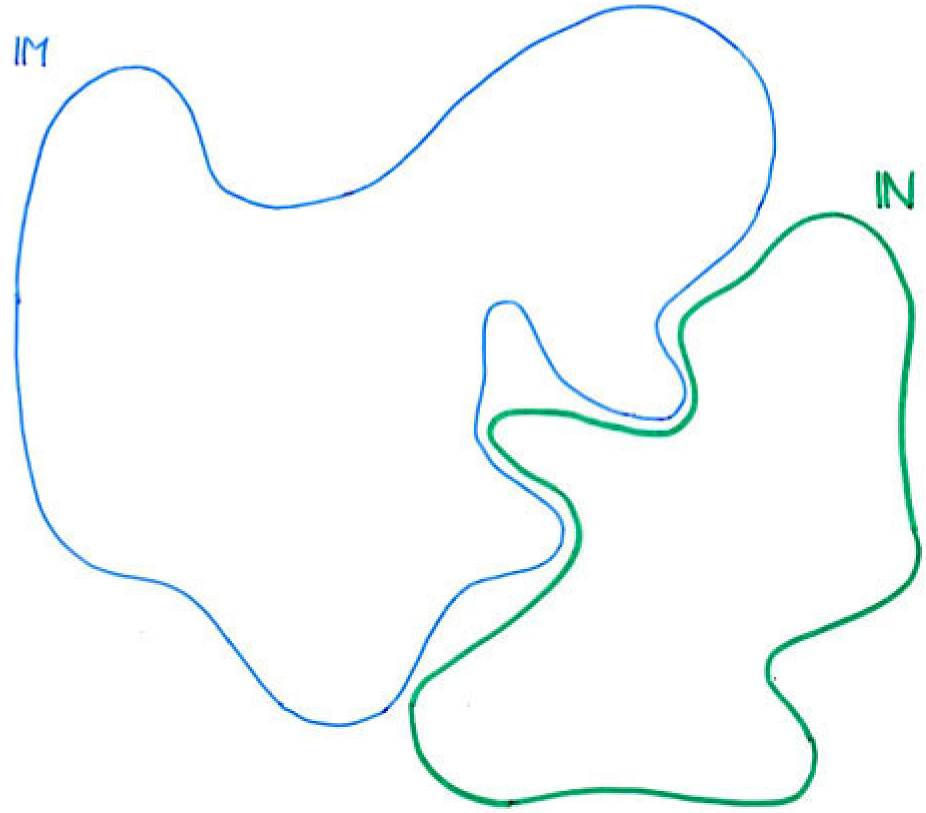

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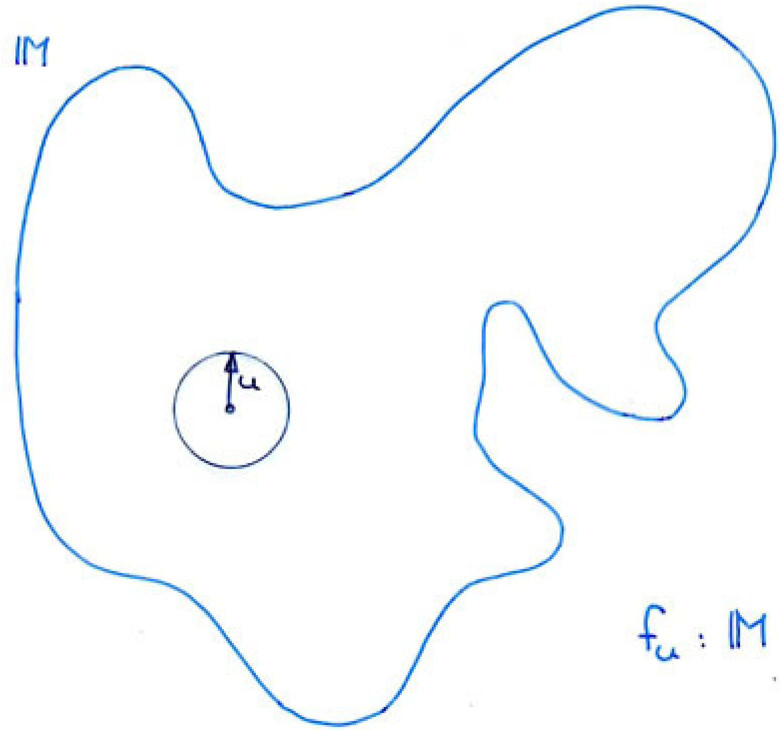
III.6 Height'n' Elevation



$f_u : \mathbb{M} \to \mathbb{R}$

<u>elevation</u> is $E : \mathbb{M} \to \mathbb{R}$ defined by

$E(x) = |f_u(x) - f_u(y)|$ with

(i) $u = \pm n_x = \pm n_y$, (ii) x, y paired.

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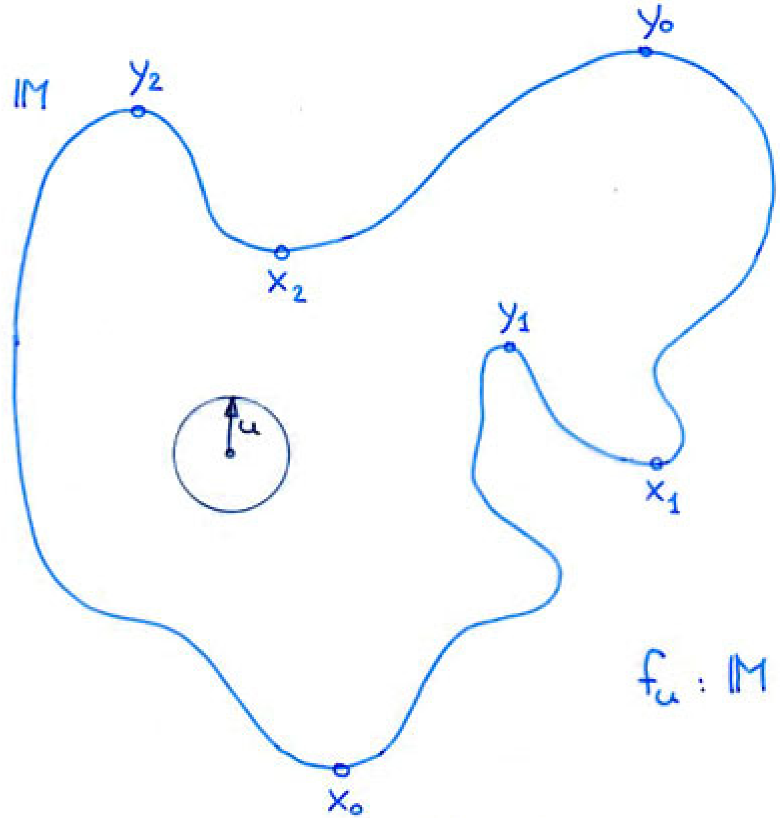

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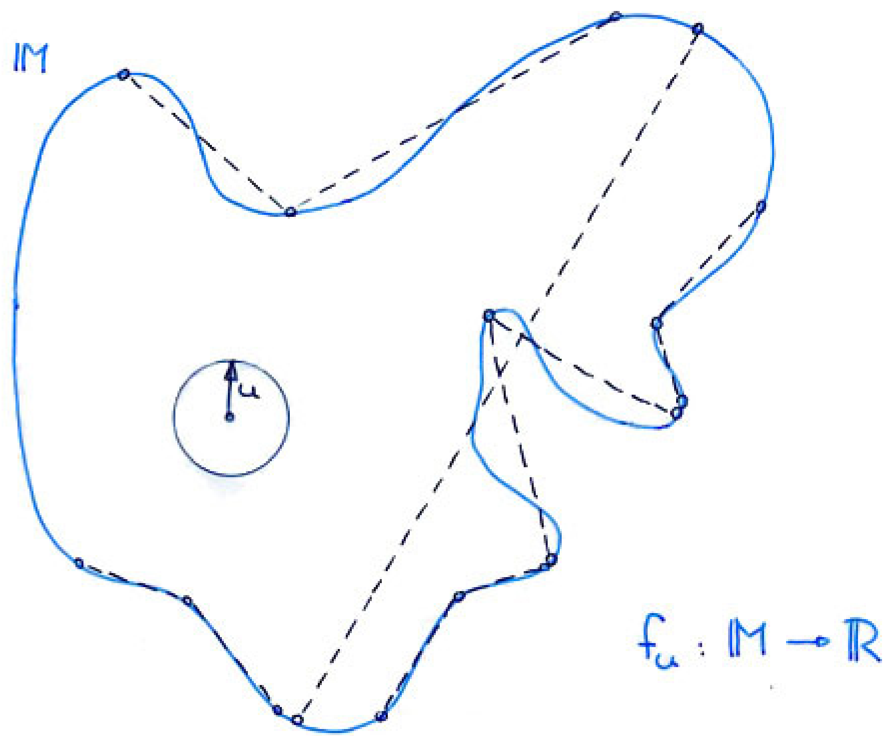

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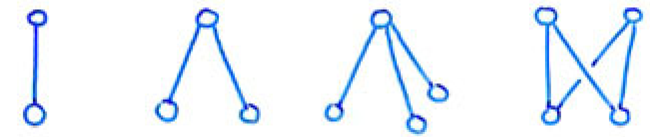
III.7 Maxima in $\mathbb{R}^3$

four types



1- 2- 3- 4-legged

To align $\mathbb{M}$ with $\mathbb{N}$,
fit legs in pairs to define
rigid motions.