# Ultimate Implementation and Analysis of the AMO Algorithm for Approximate Pricing of European-Asian Options

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# Summary of This Talk

- option：typical financial derivative
- pricing European-Asian option on binomial model

  --- difficult to compute accurately

  ⇒ approximation
- Aingworth, Motwani & Oldham (SODA00)

  time: $O(kn^2)$, absolute error $O(nX/k)$
- Our Algorithm:

  time: $O(kn^2)$, absolute error $O(X/k)$

n, X: problem parameters, k: time-error tradeoff param.

# Option


- option: right to sell (or buy)
  some financial asset (e.g., stock)
  at some point in the future (expiration date)
  for a specified price (strike price)

- gain more benefit by investment
- hedge risk from the fluctuation of stock price

# Payoff of Option

Example: option to buy a stock of Google Inc. at the year-end at $200

- **stock price goes up to $220 at the year-end**
  **⇒exercise option to buy the stock at $200**
  **⇒sell it for $220 ⇒ gain $20 (payoff)**
- **stock price goes down to $170**
  **⇒do not exercise option ⇒ payoff = $0**

Payoff of European Option:

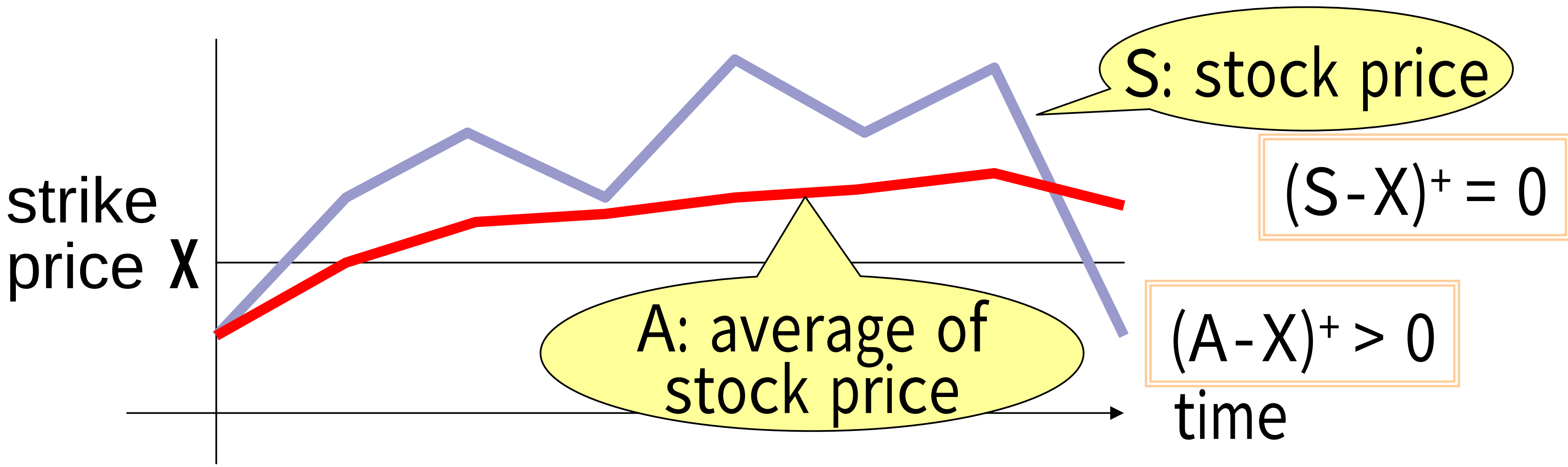
$$(S - X)^+ = \max\{S - X, 0\}$$

(S : stock price at expiration date,　X : strike price)

# European-Asian Option

- payoff of European-Asian option
  depends on average of stock price A
  during whole period

payoff: $(A - X)^+ = \max\{A - X, 0\}$



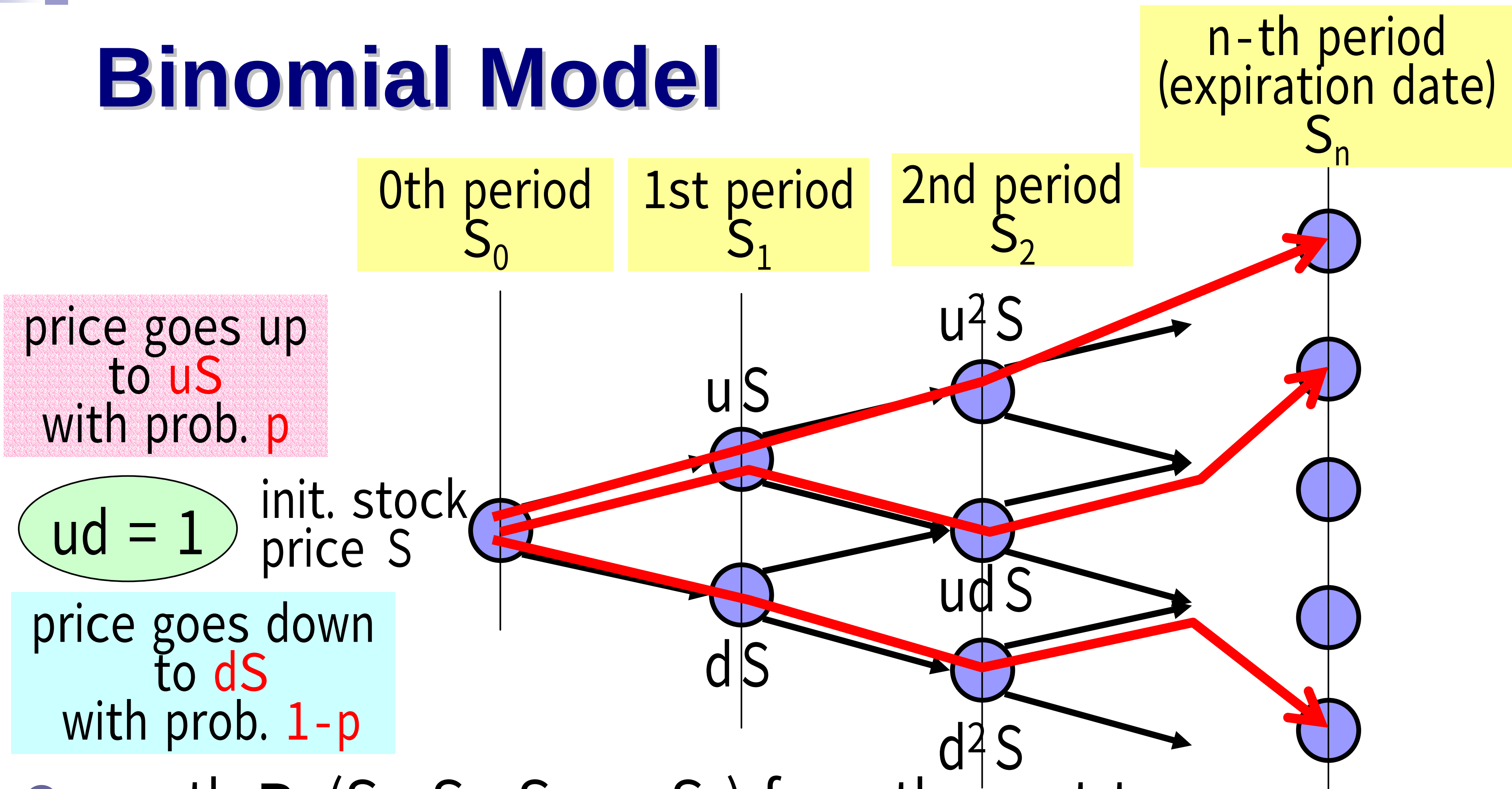

safe against fluctuation of stock price

# Computation of Option Price

- price of option = discounted expected value of payoff
  --- need to model the movement of stock price

- Our model: binomial model (discrete model)
  - **proposed by** Cox, Ross & Rubinstein (1979)
  - **represent stock price movement by a binomial tree**
  - **can compute exact option price by DP**

# Binomial Model




- a path **P**=($S_0$, $S_1$, $S_2$, …, $S_n$) from the root to a leaf represents the movement of stock price
- payoff of European-Asian option = $\left(\frac{\sum_{i=0}^{n} S_i}{n+1} - X\right)^+$

# Our Problem

compute the expected payoff
of European-Asian option
on the binomial model

$$E\left(\left(\frac{\sum_{i=0}^{n} S_i}{n+1} - X\right)^{+}\right)$$

- payoff is dependent on the path **P**=($S_0$, $S_1$, $S_2$, ..., $S_n$)
(path-dependent option)
- payoff is nonlinear w.r.t. the running total $\mathbf{\Sigma}_i S_i$
⇒ need enumeration of all the paths
⇒ exponential time
- computation of the price of path-dependent option is #P-hard

# Approximation Algorithms for Pricing European-Asian Option

- Monte Carlo Method
  - based on path sampling
  - error bound depends on the volatility of stock price
- Other methods
  - based on heuristics
  - no theoretical error bound

# AMO Algorithm and its Variants

Aingworth, Motwani & Oldham (2000)
time: $O(kn^2)$
abs. err.: $O(nX/k)$
DP + bucketing

independent of volatility

Dai, Huang & Lyuu (2002)
abs. err.: $O\left(\frac{\sqrt{n}X}{k}\right)$
adjust # of buckets

Ohta, Sadakane, Shioura & Tokuyama (2002)
abs. err.: $O\left(\frac{n^{\frac{1}{4}}X}{k}\right)$
randomization

Our Result

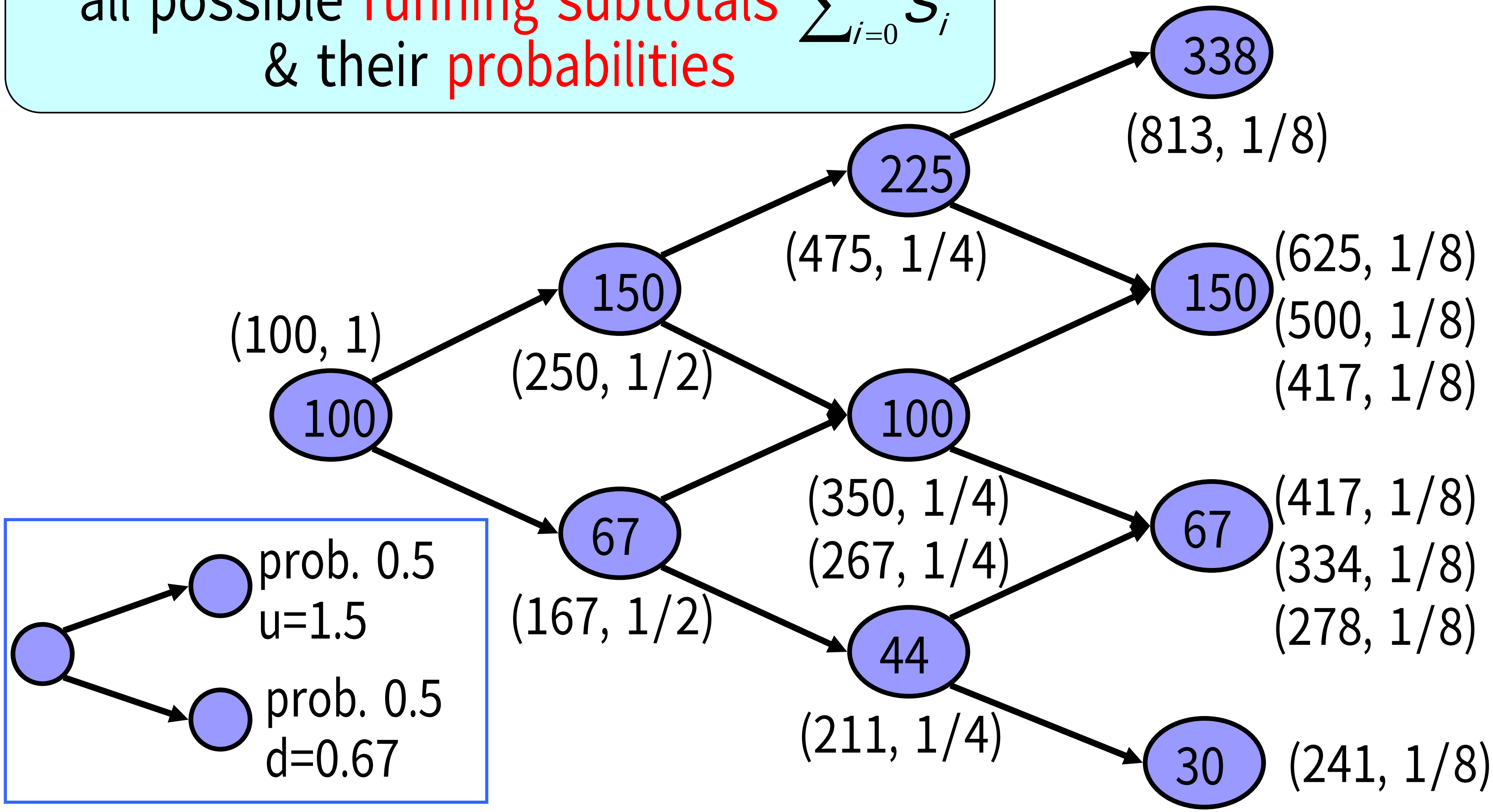
Shioura & Tokuyama (2004)
abs. err.: $O\left(\frac{X}{k}\right)$
use both ideas

n disappears!

(n：depth of binomial tree, X: strike price, k: positive integer)

# Exact Algorithm by DP

at each node of binomial tree, compute all possible running subtotals $\sum_{i=0}^{t} S_i$ & their probabilities



# AMO Algorithm (1)

- # of running subtotals can be exponential

⇒ approximate running subtotals by bucketing

| interval | running subtotal & probability |
|---|---|
| 400<br>300 | (310, 0.05) |
| 300<br><br>200 | (205, 0.15)<br>(240, 0.12)<br>(285, 0.20) |
| 200<br><br>100 | (170, 0.10)<br>(150, 0.10)<br>(110, 0.10) |
| 100<br>0 | (80, 0.05)<br>(30, 0.01) |

round up
running subtotals
&
sum up
probabilities
in each bucket

→

| | |
|---|---|
| 400<br>300 | (400, 0.05) |
| 300<br>200 | (300, 0.47) |
| 200<br>100 | (200, 0.30) |
| 100<br>0 | (100, 0.06) |

# AMO Algorithm (2)

- k: # of buckets at each node
  ⇒ error bound ≦ max. value of running subtotal/k

**Proposition:**

running subtotal $\sum_{i=0}^{t} s_i$ is ≧ (n+1)X
at the t-th period

⇓

- option will be exercised at the expiration date
- conditional expectation of the payoff can be computed easily

⇒ error bound of AMO algorithm = (n+1) X/k

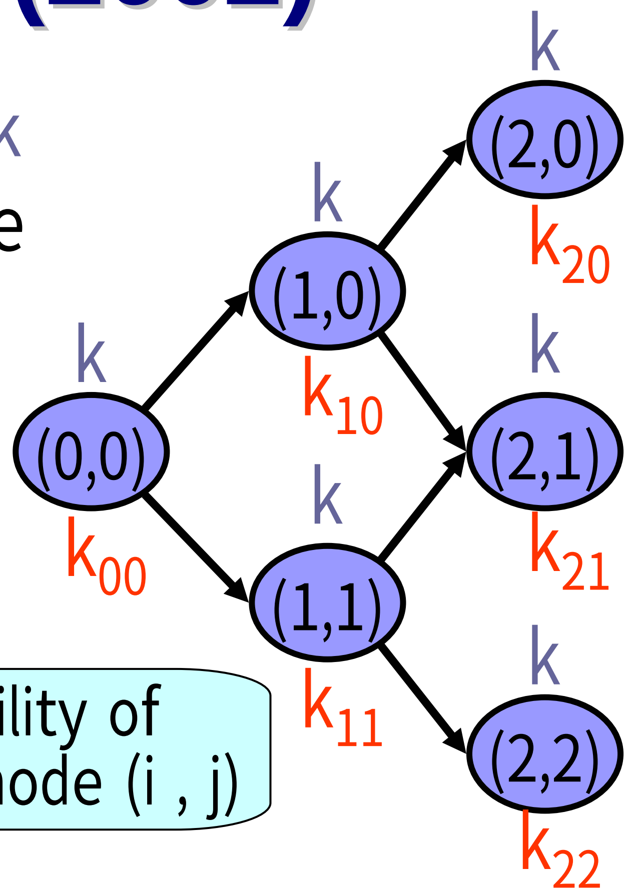
# Algorithm by Dai et al. (2002)

AMO algorithm： use the same number k of buckets at each node

set the number of buckets $k_{ij}$ at the node (i, j) flexibly

error bound $\displaystyle\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{X}{k_{ij}}\omega(i,j)$

probability of reaching node (i , j)




- **adjust # of buckets $k_{ij}$ to minimize error bound under the condition $\Sigma k_{ij} = kn^2$**

error bound $O\left(\frac{\sqrt{n}X}{k}\right)$

# Algorithm by Ohta et al. (2002)

AMO algorithm：approximate running subtotals
in a bucket by rounding-up

choose a running subtotal randomly
as approximate value

| interval | running subtotal & probability |
|---|---|
| 200 | (170, 0.30) |
| | (150, 0.10) |
| 100 | (110, 0.20) |

(200, 0.60)
(170, 0.60)

prob. 1/2

(150, 0.60)

prob. 1/6

prob. 1/3

(110, 0.60)

# Analysis of Ohta et al. (2002)

- regard the behavior of randomized algorithm as stochastic process ⇒ Martingale
  expectation of the error by random choice of running totals at a node = 0

⇒ apply Azuma's inequality (1967)

error bound $O\left(\frac{n^{\frac{1}{4}}X}{k}\right)$ (with high probability)

analysis is difficult

# Our Algorithm

- set the number of buckets $k_{ij}$ at node (i, j) flexibly
- random choice of running subtotal

⇒ error bound $O\left(X\sqrt{\sum_{i=1}^{n}\sum_{j=0}^{i}\left[\frac{\omega(i,j)}{k_{ij}}\right]^2}\right)$

- **adjust # of buckets $k_{ij}$**
  **to minimize error bound**
  **under the condition $\Sigma k_{ij} = kn^2$**

⇒ error bound $O\left(\frac{X}{k}\right)$

**analysis is quite easy!**

# Open Problems

- derandomization of our algorithm with the same error bound
- approximation of American-Asian option
- analysis of error bound compared to exact price