Polyhedral Computation	2010/11/19
Lecture 8	
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1 Voronoi Diagram

In this lecture we formally define the Voronoi diagram and Delaunay triangulation, which were introduced in Lecture 1, and relate them to the vertex and facet enumeration problems.

We consider two points in d-dimension $a = (a_1, a_2, \cdots, a_d)$, and $b = (b_1, b_2, \cdots, b_d)$. Let $L(a, b)$ be the perpendicular bisector of a and b,

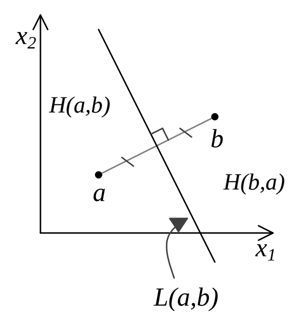
$$L(a, b) : (b - a)^T x = \frac{b^T b}{2} - \frac{a^T a}{2}$$

and let $H(a, b)$ be the set of points for which a is a closest point. (b is not closer than a.)

$$H(a, b) : (b - a)^T x \leq \frac{b^T b}{2} - \frac{a^T a}{2}$$
$$H(b, a) : (b - a)^T x \geq \frac{b^T b}{2} - \frac{a^T a}{2}$$

It can be also written as $H(a, b) = \{x : d(x, a) \leq d(x, b)\}$, where $d(x, y)$ is the Euclidian distance of x and y.

Example (2-dimension)
$a = (a_1, a_2)$
$b = (b_1, b_2)$
$L(a, b) : (b_1 - a_1)x_1 + (b_2 - a_2)x_2 = \frac{b_1^2 + b_2^2}{2} - \frac{a_1^2 + a_2^2}{2}$



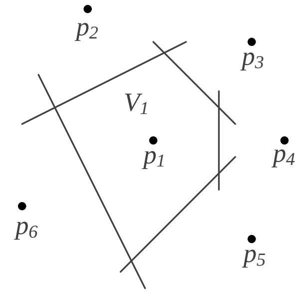
1.1 Problem

Input $P = \{p_1, p_2, \cdots, p_n\} \subset \mathbb{R}^d$ in general position, that is, no $d + 2$ points lie on a hypersphere.

Output Voronoi diagram of input points, P.

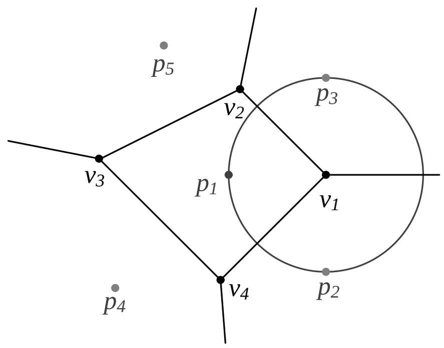
$$\begin{aligned} V_i &= \{x \in \mathbb{R}^d : d(x, p_i) \leq d(x, p_j), \text{for } j \neq i\} \\ &= \bigcap_{j \neq i} H(p_i, p_j) \end{aligned}$$

Voronoi diagram is the set of V_i, for $i = 1, 2, \cdots, n$.



Brute force algorithm is easy. Just compute n vertex enumeration problems for V_i. However, our goal is to compute the Voronoi diagram by using one vertex enumeration problem in $d+1$ dimensions.

General Position Assumption $P = \{p_1, p_2, \cdots p_n\} \in \mathbb{R}^d$ has no subset of $d+2$ points lying on hyper sphere. (For example $d = 2$, no 4 points are on a circle.)



The Voronoi diagram is a graph on vertex set of Voronoi points with some unbounded edges. In this figure, v_1 is the center of a circle containing p_1, p_2, p_3 in its boundary. This circle cannnot contain any input point on its interior.

1.2 Delaunay Triangulation

Input $P = \{p_1, p_2, \cdots, p_n\} \in \mathbb{R}^d$ in general position.

Output Set of simplicies on P such that the hypersphere on its vertices is empty.

We claim that there is essentially the same information in Delaunay triangulation and Voronoi diagram.

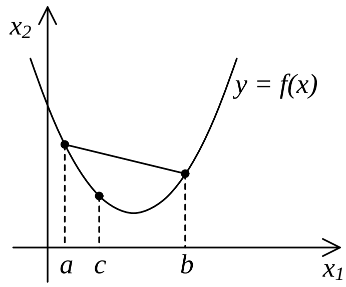
- (V.D.→D.T.) Take each Voronoi vertex, and it defines a simplex in Delaunay triangulation given by its nearest neighbors.
- (D.T.→V.D.) Each simplex in Delaunay triangulation defines a hypersphere the center of which is a Voronoi point.

1.3 Strictly convex function

Function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is a strictly convex, if for any $a, b \in \mathbb{R}^d$ and for any λ, $0 < \lambda < 1$,

$$f(\lambda a + (1-\lambda) b) < \lambda f(a) + (1-\lambda) f(b)$$

Example $f(x) = x^2 \quad (c = \lambda a + (1 - \lambda) b)$



It can be seen that $f(x) = x^T x = x_1^2 + x_2^2 + \cdots + x_d^2$ for $x \in \mathbb{R}^d$ is also strictly convex function.

1.4 Computing Delaunay Triangulation

We define the lifting operation $\mathbb{R}^d \mapsto \mathbb{R}^{d+1}$. We define $x_{d+1} = x^T x = x_1^2 + x_2^2 + \cdots + x_d^2$.

$$\begin{aligned} x = (x_1, x_2, \cdots, x_d) &\mapsto \bar{x} = (x_1, x_2, \cdots, x_d, x^T x) \\ P &\mapsto \bar{P} \end{aligned}$$

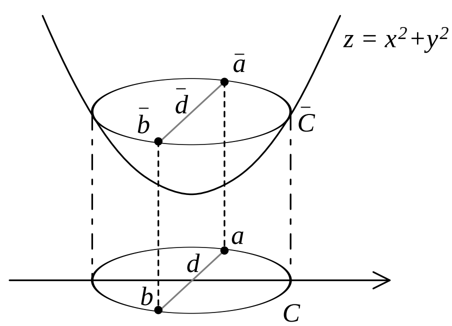
The magic is essentially circles turning into planes. $(P \mapsto \bar{P})$

$$C : x^2 + y^2 + ax + ay + c = 0$$

Lift up $(x, y) \mapsto (X, Y, Z)$, where $X = x, Y = y, Z = x^2 + y^2$.

$$\bar{C} : Z + aX + bY + c = 0$$

Idea is computing the convex hull facets of the lifted point set of $\bar{P}$.



Downward facing facets of $CH(\bar{P})$, *Lower Hull* : $LH(\bar{P})$, correspond to the simplices in the Delaunay triangulation.

Claim C is an empty circle on three points $a, b, c \in P$, if $\bar{C}$ defines a facet of the $LH(\bar{P})$

Proof. By contrapositive. Let d be the point that lies in the interior of C, $d \neq a, b, c$. The triangle of abc is not in the Delaunay triangulation. Lift d to $\bar{d}$. By strict convexity, $\bar{d}$ lies below $\bar{C}$ which is the plane through $\bar{a}, \bar{b}, \bar{c}$. Because $d = \lambda p + (1 - \lambda)q$, where p, q are boundary of C. $\bar{p}, \bar{q}$ are on plane $\bar{C}$, by strict convexity $\bar{d}$ is below $\bar{C}$. So $\bar{C}$ is not a facet. $\square$

1.5 Computing Voronoi diagram directly

Compute half spaces of Voronoi diagram directly. Lift points a, b to tangent planes of paraboloid $L(a), L(b)$.

$$\begin{aligned} a = (a_1, a_2) &\quad\mapsto\quad L(a) : z = 2a_1x + 2a_2y - a_1^2 - a_2^2 \\ b = (b_1, b_2) &\quad\mapsto\quad L(b) : z = 2b_1x + 2b_2y - b_1^2 - b_2^2 \end{aligned}$$

We compute $L(a) \cap L(b)$ and project back to $\mathbb{R}^2$.

$$\begin{aligned} 2a_1x + 2a_2y - a_1^2 - a_2^2 &= 2b_1x + 2b_2y - b_1^2 - b_2^2 \\ (b_1 - a_1)x + (b_2 - a_2)y &= \frac{b_1^2 + b_2^2}{2} - \frac{a_1^2 + a_2^2}{2} \end{aligned}$$

We see that this is just the bisector of a, b. Therefore the edges of the convex hull of the lifted set correspond to the Voronoi edges, and the vertices of the lifted set correspond to Voronoi vertices.