Lecture 6

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1 Point/Hyperplane duality

In this lecture we discuss a very basic and important relation between points and hyperplanes. Thus far two problems arised in this course, the vertex enumeration (of a polyhedron) and the facet enumeration (of the convex hull of a set of points). We will see how these two are connected by using a duality relation. Consider the two definitions we are using:

Definition. Let $P = \{x \in \mathbb{R}^d \mid a_i x \leq b_i \text{ for all } 1 \leq i \leq m\}$ be a polytope and $v \in \mathbb{R}^d$. The vector v is called a *vertex* of the polytope P if and only if there exists a set $S \subseteq \{1, ..., m\}$ with $|S| = d$ such that v is the unique solution to the system $a_i x = b_i$ for all $i \in S$.

Definition. Let Q be the convex hull of a given set of points $v_1, v_2, ..., v_n \in \mathbb{R}^d$. Then the set $\{x \in Q \mid ax = b_0\}$ ($ax = b_0$ in short) is called a *facet* of Q if and only if $av_i \leq b_0$ for all $1 \leq i \leq n$ and there exists a set $S \subseteq \{1, ..., n\}$ with $|S| = d$ such that $av_i = b_0$ for all $i \in S$ and $\{v_i \mid i \in S\}$ is linear independent.

Lets consider some data and the different ways to look at it.

<u>Data:</u> Let $v = (v_1, \ldots, v_d)$ and $w = (w_1, \ldots, w_d)$ be two non-zero vectors in $\mathbb{R}^d$.
We could interpret this as,
Viewpoint 1: $v \in \mathbb{R}^d$ being a point, and $wx = 1$ being a hyperplane, or
Viewpoint 2: $w \in \mathbb{R}^d$ being a point, and $vx = 1$ being a hyperplane.

Case 1, $vw = 1$: In veiwpoint 1, point v lies on the hyperplane $wx = 1$ and the and in viewpoint 2, point w lies on the hyperplane $vx = 1$.
Case 2, $vw \leq 1$: In viewpoint 1 point v lies in the halfspace $wx \leq 1$ and in viewpoint 2, point w lies in the halfspace $vx \leq 1$.
So while this is just some data for the PC, it is up to the human imagination to interpret this data, for example as points or inequalities. We now look at the same two viewpoints, with more input data.

<u>Data:</u> Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ be non-zero vectors in the $\mathbb{R}^d$. Let $P := \{x \in \mathbb{R}^d \mid w_i x \leq 1, \text{ for all } 1 \leq i \leq m\}$ and $P' := \{x \in \mathbb{R}^d \mid v_i x \leq 1, \text{ for all } 1 \leq i \leq n\}$ be two polyhedra.
How are P and P' related?
In general we cannot hope that these polyhedra have anything in common, since they have totally different input data, but the case we are interested in behaves differently. As we have seen in the case of two vectors, we can interpret the same set of vectors once

as points and once as a system of inequalities. So let P be defined as above, and let $Q := conv.hull\{w_1, \dots, w_m\}$ be the convex hull. Now we have a vertex enumeration instance and a facet enumeration instance arising from the same set of input data. Are these two problems related?
Note that $0 \in P$ strictly satisfies every inequaltiy and therfore 0 is an interior point of P and P has full dimension.

So suppose $v \in \mathbb{R}^d$ is a vertex of P. From the definition of vertex follows, that there exists a set $S \subseteq \{1, \dots, m\}$ with $|S| = d$ such that the system $w_i x = 1$ for all $i \in S$ has the unique solution v.
So lets change our point of view, and think about the w_i as points in the $\mathbb{R}^d$. We know that the system $w_i = 1$, $i \in S$ has a unique solution v, and therefore the set $\{w_i \mid i \in S\}$ is linear independent. We also know that $w_i v \leq 1$ for all $1 \leq i \leq m$, since $v \in P$. This is excatly the definition of $vx = 1$ being a facet of Q. Starting with a facet $ax = b_0$ from Q we also get that $\frac{1}{b_0} a$ is a vertex of P, so we have a one-to-one relation between the vertices of P and the facets of Q. Thus the two problems considered (vertex and facet enumeration) are the same.

But are we looking at the general case here? P seems to be special, since not every polyhedron has full dimenson and the origin as an interior point. What assumptions did we make?

1. no zero vectors as input
2. origin is an interior point
3. $b = \mathbb{1}$

We can always make a polyhedron fully dimensional by elimination of redundant variables using Gaussian elimination. So given a full dimensional polyhedron we have some interior point and we can archieve 2. (and 1.) by a linear translation if necessary. After this step, if necessary, the origin is feasible and all b_i must be greater than 0. Just divide by b_i to get $b = \mathbb{1}$ without changing the polyhedron.

So we can use the same program to solve vertex and facet enumeration, for example the graph search method.

Example. *Problem: We want to enumerate the facets of the convex hull of a given set of points S. In this case let the input be some points in the plane, $S = \{(1,1), (1,0), (-2,-2), (-1,1)\}$. We will use the insight gained by the duality and therefore not start by computing the convex hull, but by interpreting the points as inequalities. This gives us the following set of inequalities:*

$$\begin{array}{lr} a) & x_1 + x_2 \leq 1 \\ b) & x_1 \leq 1 \\ c) & -2x_1 - 2x_2 \leq 1 \\ d) & -x_1 + x_2 \leq 1 \end{array}$$

We can now compute the vertices of this system, for example with the graph search method from Lecture 2. This gives us the vertices $\{(1,0),(0,1),(1,-\frac{3}{2}),(-\frac{3}{4},\frac{1}{4})\}$*. See Fig. 1.*

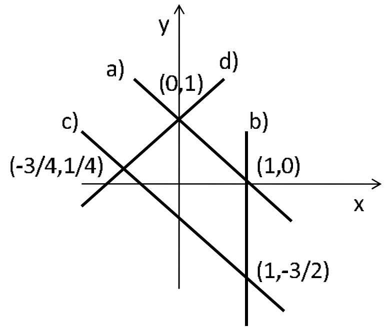

Figure 1: The Polytope defined by a),b),c) and d)

Writing these vertices again as equations will deliver us the facets a), b), c) and d) we were looking for.

$$
\begin{aligned}
&a) && x_1 \leq 1 \\
&b) && x_2 \leq 1 \\
&c) && x_1 - \frac{3}{2}x_2 \leq 1 \\
&d) && -\frac{3}{4}x_1 + \frac{1}{4}x_2 \leq 1
\end{aligned}
$$

Indeed, these equations describe the facets of the convex Hull of the set S*, see Fig. 2.*

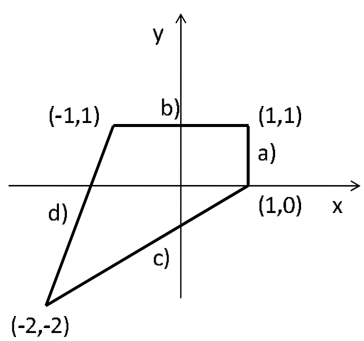

Figure 2: Convex Hull of the Set S, Facets a), b) c) and d)

2 Degeneracy

Lets see what kind of degeneracy can occur in the two problems discussed and how we can deal with it. A fundamental method to remove degeneracy known in *linear programming* since the 1950s, is *pertubation*. By adding small random values to the input data, it is possible to avoid degenercy and hence streamline our algorithm.

Viewpoint 1: Vertex enumeration

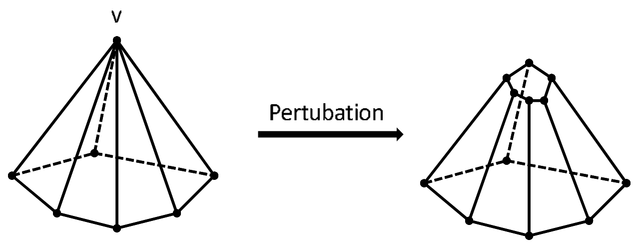

Figure 3: pertubation splits v in 6 new vertices

The vertex v lies on 6 planes so there are $\binom{6}{3}$ ways to define v. For the graph search algorithm this means, that starting from v we may need $\binom{6}{3} - 1$ pivots, to find a new vertex. Pertubation, i.e. adding small random numbers, splits v into 6 new vertices so that there are only 3 planes intersecting at each vertex. This means that we find a new vertex for every pivot in our algorithm. Of course we are going to find the vertex v 6 times, but this is a vast improvement compared to $\binom{6}{3}$ pivots.

Viewpoint 2: Here the same problem as in viewpoint one arises, multiple vertices lying on the same facet of polytop results in multiple defintions of the facet and therefore a lot of (redundant) pivots. Pertubation solves this problem, so no 4 points lie on the same plane, i.e. every facet is triangulated. See Fig. 4.

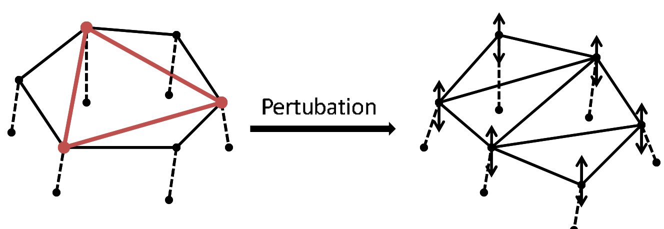

Figure 4: facet of a higher dimensional polytop

On the left we see a case where no single pivot can leave the facet, on the right the same facet is triangulated, thanks to pertubation. So again, we will be able to find a new facet with every pivot.

Changing the input data is always risky, since the output produced may no longer be correct. We could find, for example, a totally different number of facets, which would make pertubation useless. So its good to know that it is actually possible to do this perbuation "symbolically" without any change in the original data ("Simulation of pertubation").

3 Volume of Polytopes

As we have seen in the previous section, pertubation in viewpoint 2 not only removes degeneracy, but also gives a triangulation of the facets of the polytope. We can use this fact to compute its volume. In general its complicated to compute the volume of an arbitrary polytope, but its considerably easy to compute the volume of a simplex, since then the volume is a determinant.

$$\det \begin{bmatrix} v_1 & 1 \\ v_2 & 1 \\ \vdots & \vdots \\ v_{d+1} & 1 \end{bmatrix} = \text{volume of simplex}$$

The triangulation of the facets helps, because given a triangulation, we can compute the volume by adding the volume of each simplex.
In facet enumeration by graph search, we visit every facet and if they are all simplices, we can compute the volume of all the facets of the polytope. Of course we are not interested in the volume of the facets, but lifting our polytope takes care of that, making our polytope a facet in a $d+1$ dimensional polytope. So this approach gives us an possibility to compute the volume of a polytope.