Computing Nash Equilibria

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We first derive conditions for equilibria for zero sum games, then generalize them for non-zero sum games. Let 1_n denote a column vector of n ones. Let A be a $m \times n$ payoff matrix for Alice. Let (x, y), with $x \in R^m, y \in R^n$, be the strategies used by Alice and Bob respectively. The LP formulation for a zero sum game based on A is:

$$max\ z \tag{1}$$

$$1_n z \le x^T A$$

$$1_m^T x = 1 \tag{2}$$

$$x \ge 0$$

$$min\ w \tag{3}$$

$$1_m w \ge Ay$$

$$1_n^T y = 1 \tag{4}$$

$$y \ge 0$$

Note that these are a dual pair of LPs. In the first LP, if Alice chooses strategy x, Bob's best repsonse is to select from the minimum components of the vector $x^T A$. Bob's payoff is $-z$. In the second LP, if Bob chooses strategy y Alice's best response is to choose from the maximum components of the vector Ay. Alice's payoff is w. By the duality theorem of LP, at optimality $z = w$, and this is the *value* of the game. The corresponding strategies (x, y) are equilibrium strategies: no player can improve the outcome by deviating unilaterally from his/her equilibrium strategy. The complementary slackness conditions for LP dictate that this occurs when the following additional conditions hold:

$$y^T(x^T A - 1_n z) = 0 \tag{5}$$

$$x^T(1_m w - Ay) = 0 \tag{6}$$

For example, if $w > A_i y$ (A_i is the i-th row of A) then Alice will not choose row i, so $x_i = 0$.

We move to non-zero sum games. Now there is no notion of a value of the game, but the idea of equilibrium strategies does generalize to this case. Let B be a $m \times n$ payoff matrix for Bob, and let v be his payoff. By substituting $v = -z$ in (1), we may rewrite the first LP:

$$min\ v \tag{7}$$

$$\begin{aligned} 1_n v &\geq x^T(-A) \\ 1_m^T x &= 1 \\ x &\geq 0 \end{aligned} \tag{8}$$

Letting B replace $-A$ as Bob's payoff matrix and dropping the objective functions, we get the constraint sets:

$$\begin{aligned} 1_n v &\geq B^T x \\ 1_m^T x &= 1 \\ x &\geq 0 \end{aligned} \tag{9}$$

$$\begin{aligned} 1_m w &\geq Ay \\ 1_n^T y &= 1 \\ y &\geq 0 \end{aligned} \tag{10}$$

Then (x, y), with $x \in R^m, y \in R^n$, is a Nash Equilibrium pair of strategies if:

$$x^T(1_m w - Ay) = 0 \tag{11}$$

$$y^T(1_n v - B^T x) = 0 \tag{12}$$

A simple procedure for computing equilibrium pairs is as follows. We observe that **(9)** defines a polyhedron in R^{m+1} and **(10)** defines a polyhedron in R^{n+1}. We first compute the vertices of these two polyhedra, using a package such as *lrs* (http://cgm.cs.mcgill.ca/˜avis/). For each pair of vertices $(v, x; w, y)$ from the polyhedra **(9)** and **(10)** respectively, we check the conditions **(11)** and **(12)**. The pair is an equilibrium pair if the conditions **(11)** and **(12)** are satisfied.

EXAMPLE:

$$A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 3 \end{bmatrix}$$

We first prepare the lrs input files. The format is a list of the coefficients of inequalities in the format $b + cx \geq 0$. The linearity option is used to specify an equation. The inputs are:

for (9):	**for (10)**:
H-representation	H-representation
linearity 1 6	linearity 1 6
begin	begin
6 5 integer	6 4 integer
0 1 -1 0 -4	0 1 0 -6
0 1 0 -2 -3	0 1 -2 -5
0 0 1 0 0	0 1 -3 -3
0 0 0 1 0	0 0 1 0
0 0 0 0 1	0 0 0 1
-1 0 1 1 1	-1 0 1 1
end	end

Using these inputs, we run lrs and we get the vertex lists:

v	x_1	x_2	x_3
1	1	0	0
2/3	2/3	1/3	0
2	0	1	0
8/3	0	1/3	2/3
4	0	0	1

and

w	y_1	y_2
3	1	0
3	2/3	1/3
4	1/3	2/3
6	0	1

Thus there are 20 pairs $(v, x; w, y)$ to test conditions **(11)** and **(12)** on. But things can be done a bit more simply by fixing x and then listing vectors y (if any) that form a pair with x. We rewrite **(11)** and **(12)** as

$$x_i = 0 \text{ or } A_i y = w \qquad i = 1, ..., m \tag{13}$$

$$y_j = 0 \text{ or } B_j^T x = v \qquad j = 1, ..., n \tag{14}$$

where A_i is row i of A, B_j^T is column j of B.

Take $x = (0, 0, 1), v = 4$. We compute:

$$B_1^T x = (1, 0, 4)(0, 0, 1) = 4 = v$$

$$B_2^T x = (0, 2, 3)(0, 0, 1) = 3 \neq v.$$

By **(14)**, any Nash Equilibrium (x, y) with this x must have $y_2 = 0$. The only candidate is $y = (1, 0)$ with $w = 3$. We need to check **(13)**. Since only $x_3 > 0$, we check $A_3 y = (3, 3)(1, 0) = 3 = w$. This means that $(0, 0, 1), (1, 0)$ is a Nash Equilibrium, and this is the only pair with $x = (0, 0, 1)$. Proceeding for each x we get a complete list of Equilibria: $(0, 0, 1), (1, 0); (0, 1/3, 2/3), (2/3, 1/3)$; and $(2/3, 1/3, 0), (1/3, 2/3)$.

Exercise: check this list is correct!