Polyhedral Computation, Dec 16, 2011, Kyoto U.

Ham-Sandwich Cuts

Stefan Langerman
Université Libre de Bruxelles

FOOD!

SAY WE HAVE A SANDWICH

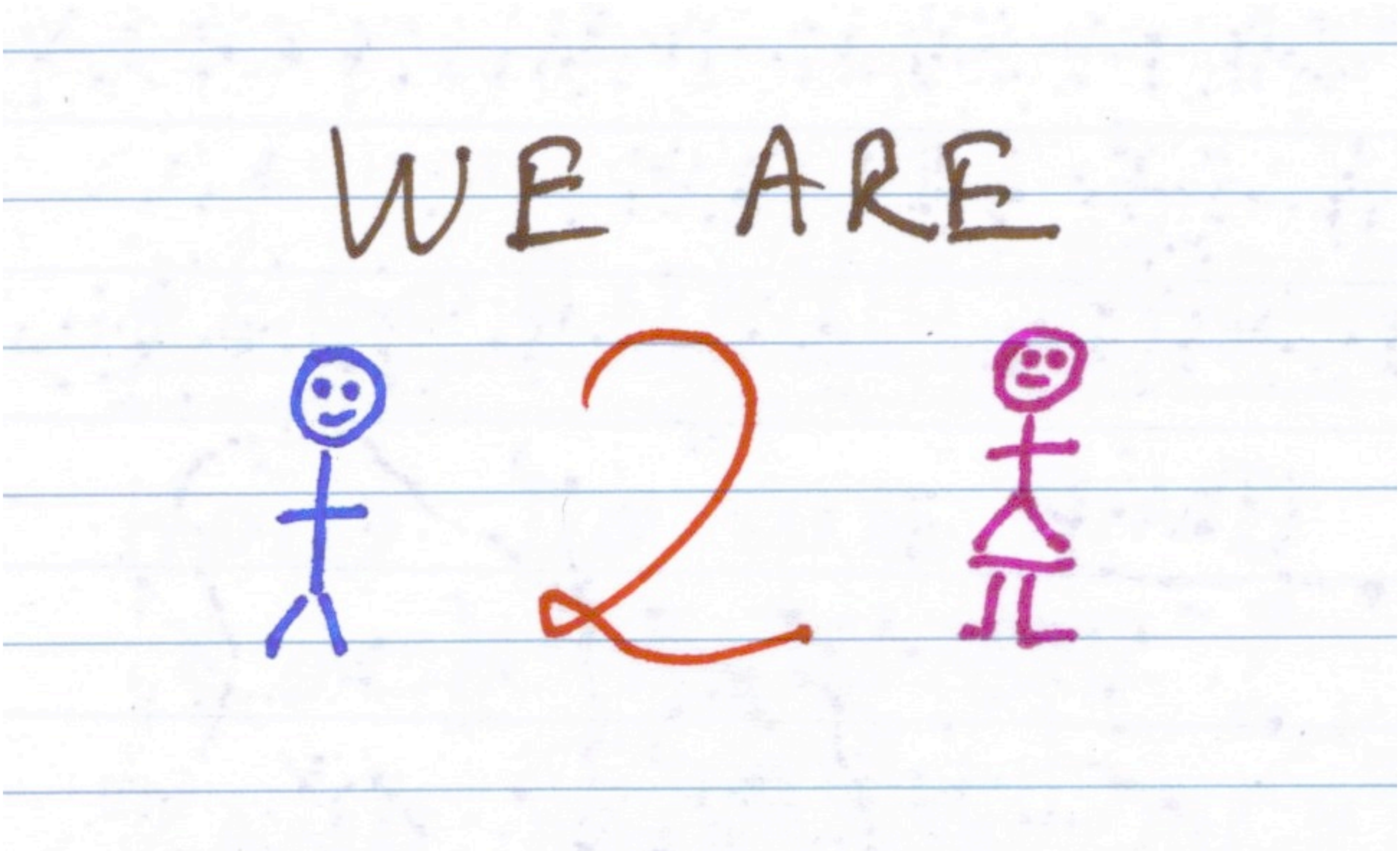
SAY WE HAVE A

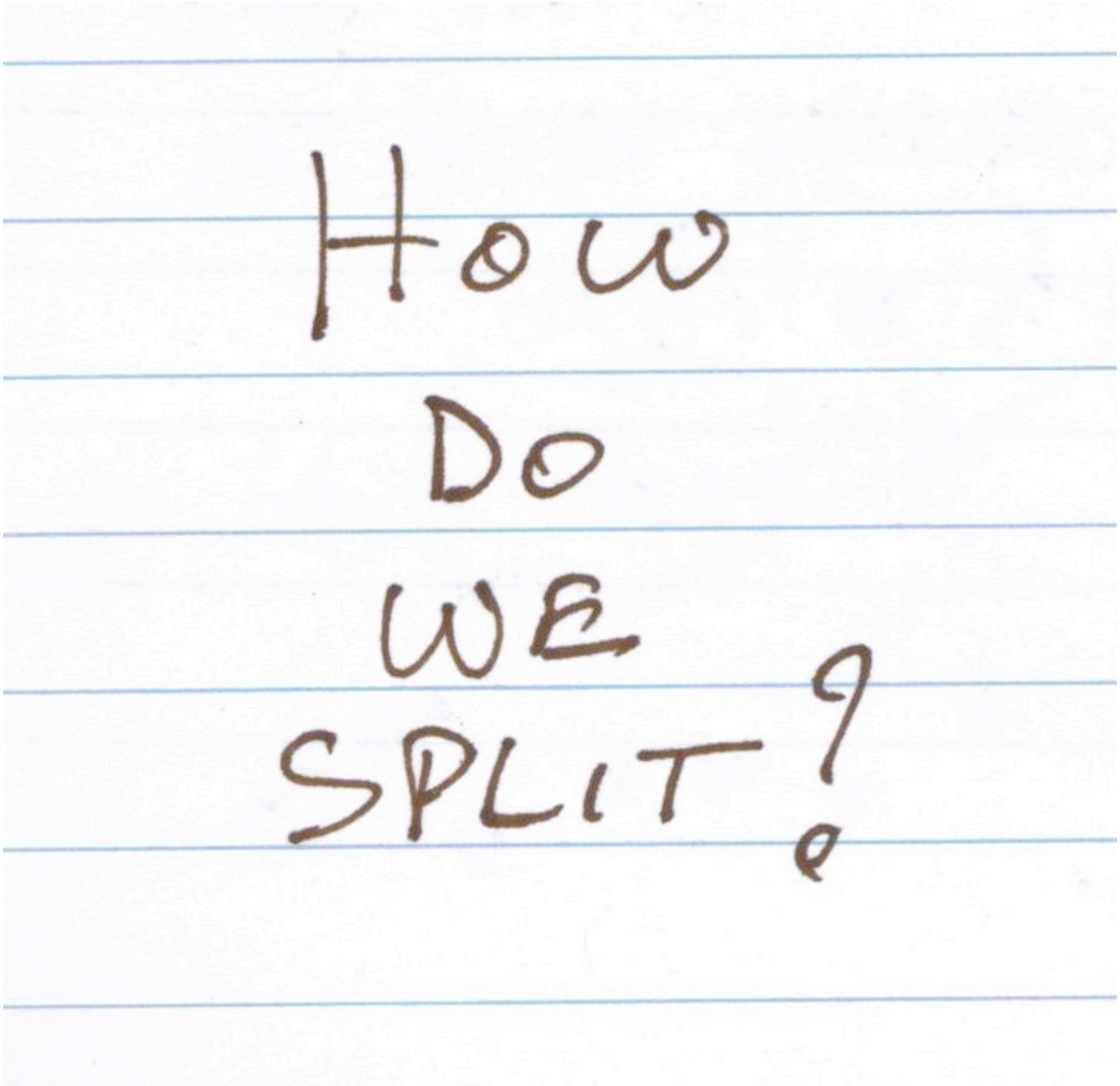
SANDWICH

WITH

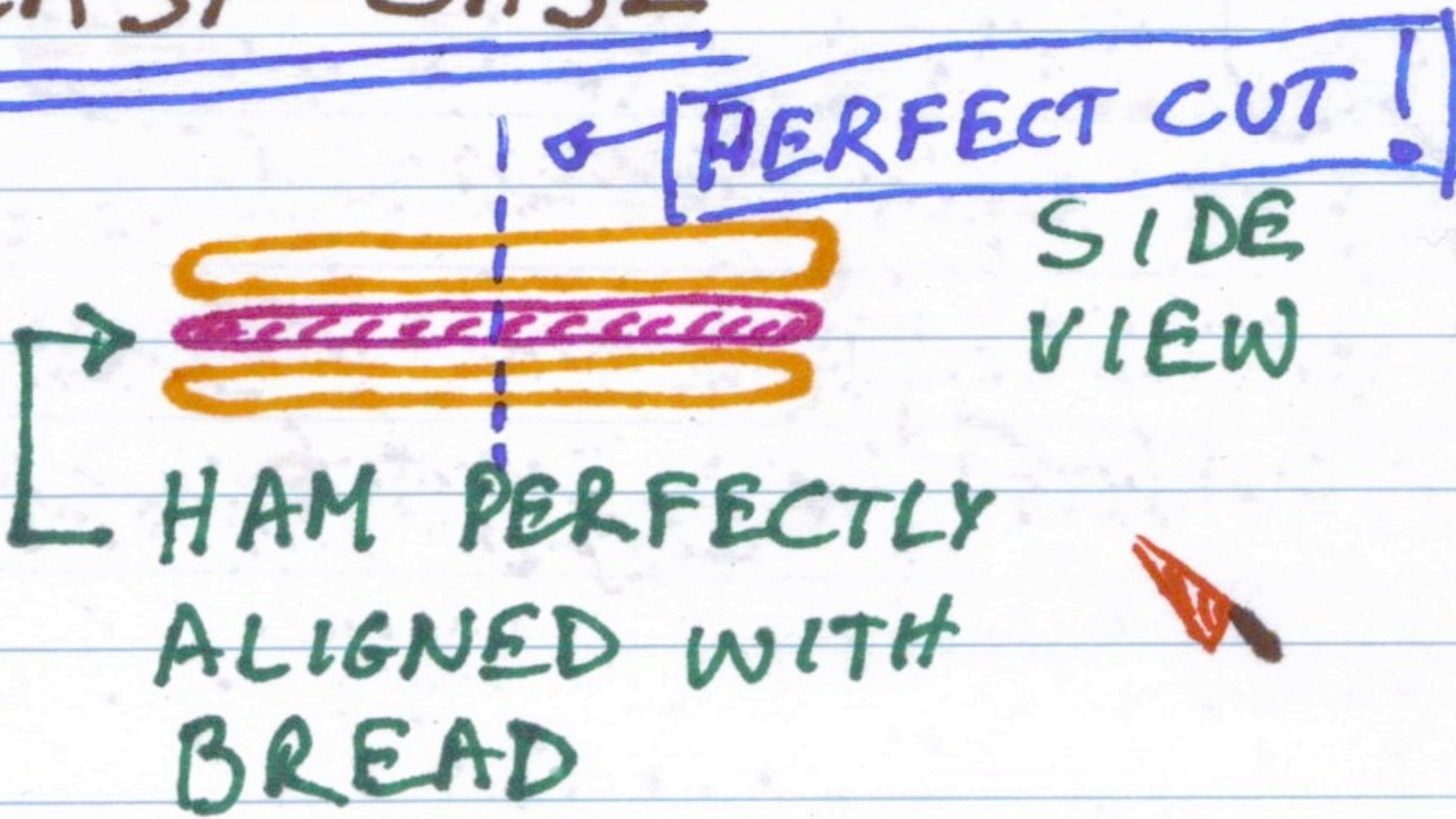
HAM

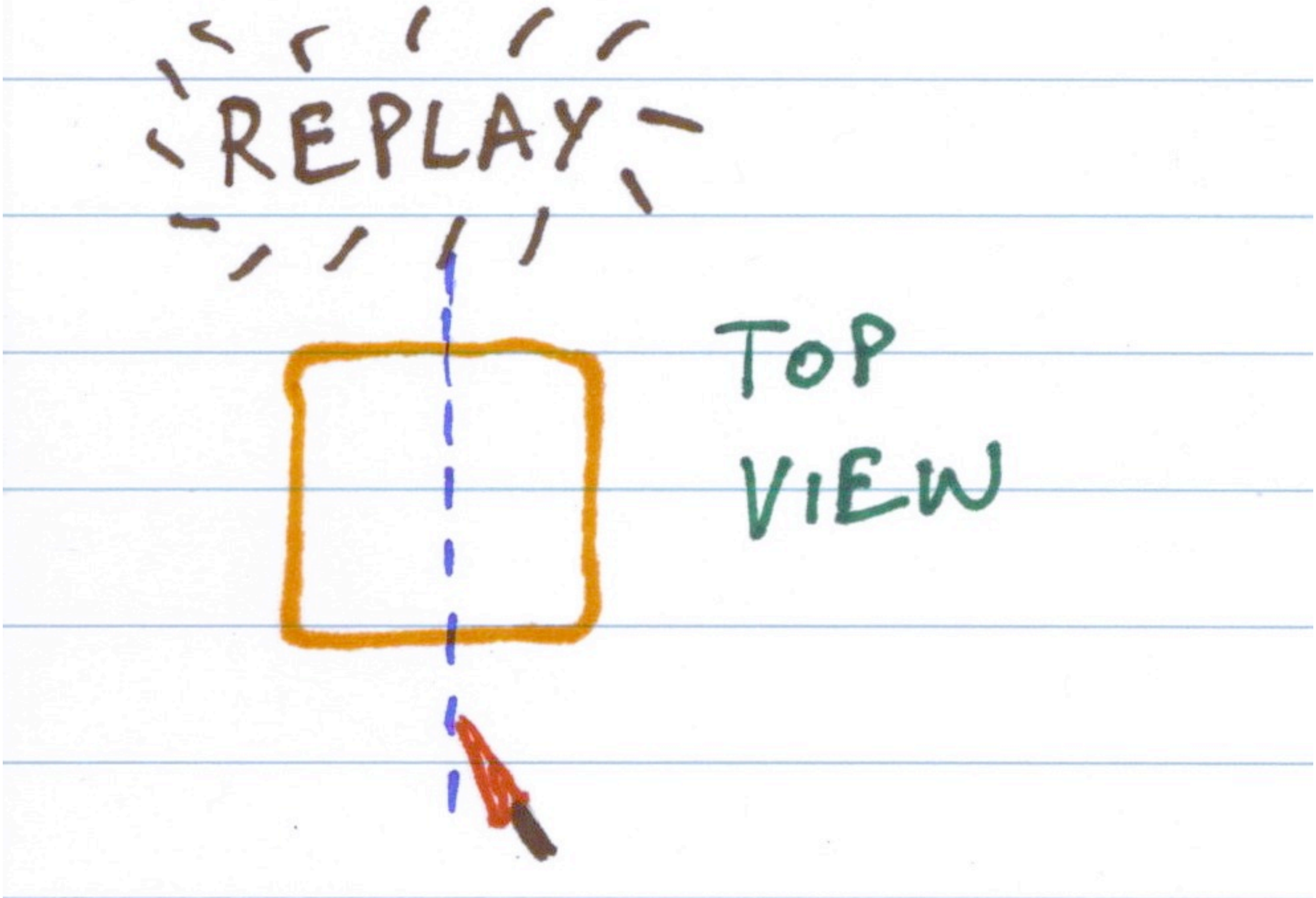
BUT...


WE ARE
2

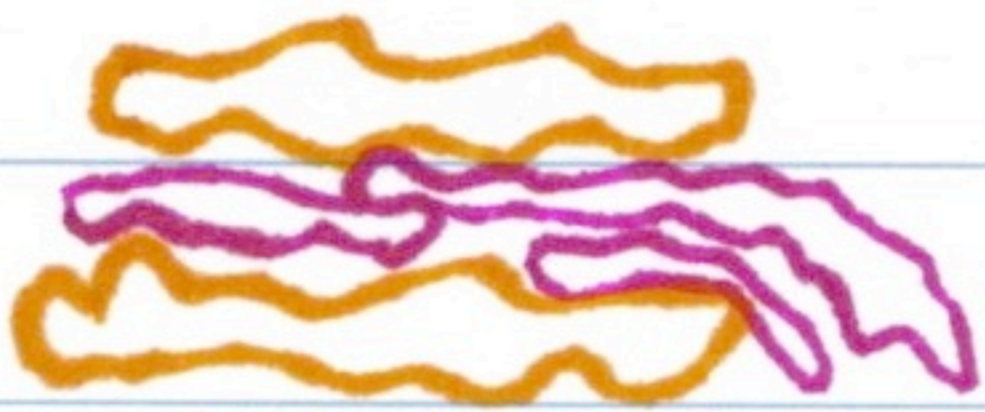

How
Do
We
Split?

EASY CASE

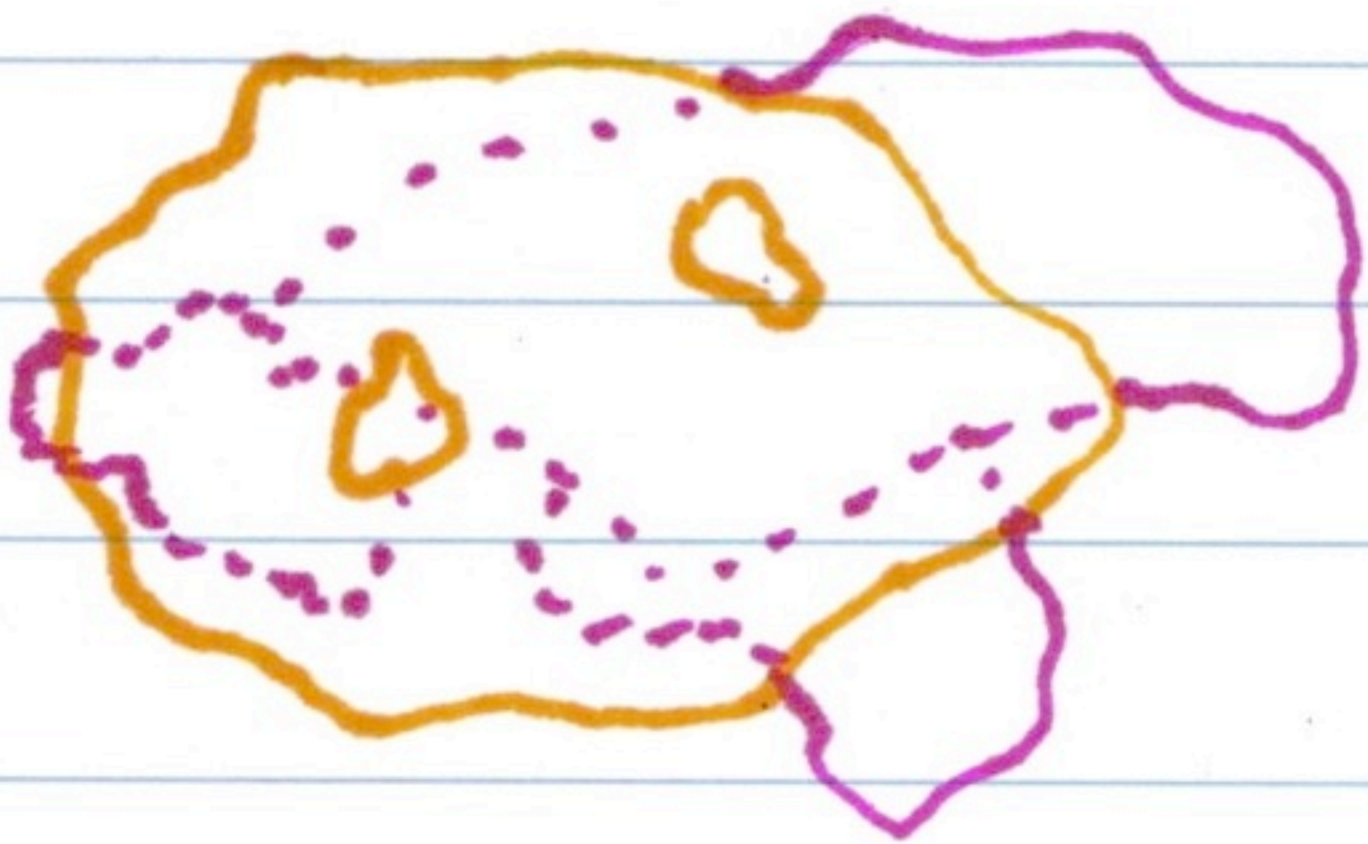



REPLAY
TOP
VIEW

TRICKY CASE !!



SIDE VIEW



TOP VIEW

HAM-SANDWICH CUT THEOREM

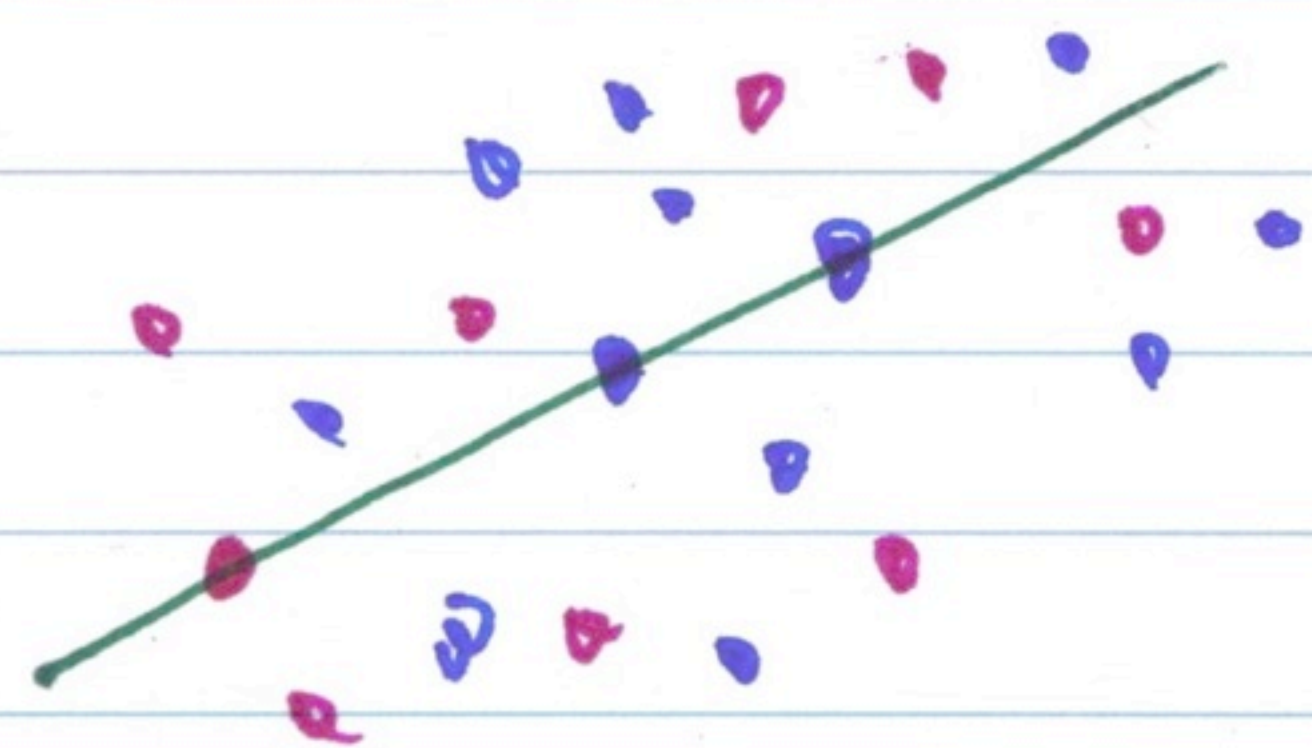
IT IS ALWAYS POSSIBLE TO CUT THE SANDWICH SO THAT BOTH PIECES HAVE ½ OF THE HAM AND ½ OF THE BREAD

DISCRETE, 2D

GIVEN A SET OF n BLUE POINTS AND n RED POINTS

DISCRETE, 2D

GIVEN A SET OF n BLUE POINTS AND n RED POINTS THERE IS A LINE L WITH $\leq \frac{n}{2}$ POINTS OF EACH COLOR IN BOTH OPEN HALFPLANES

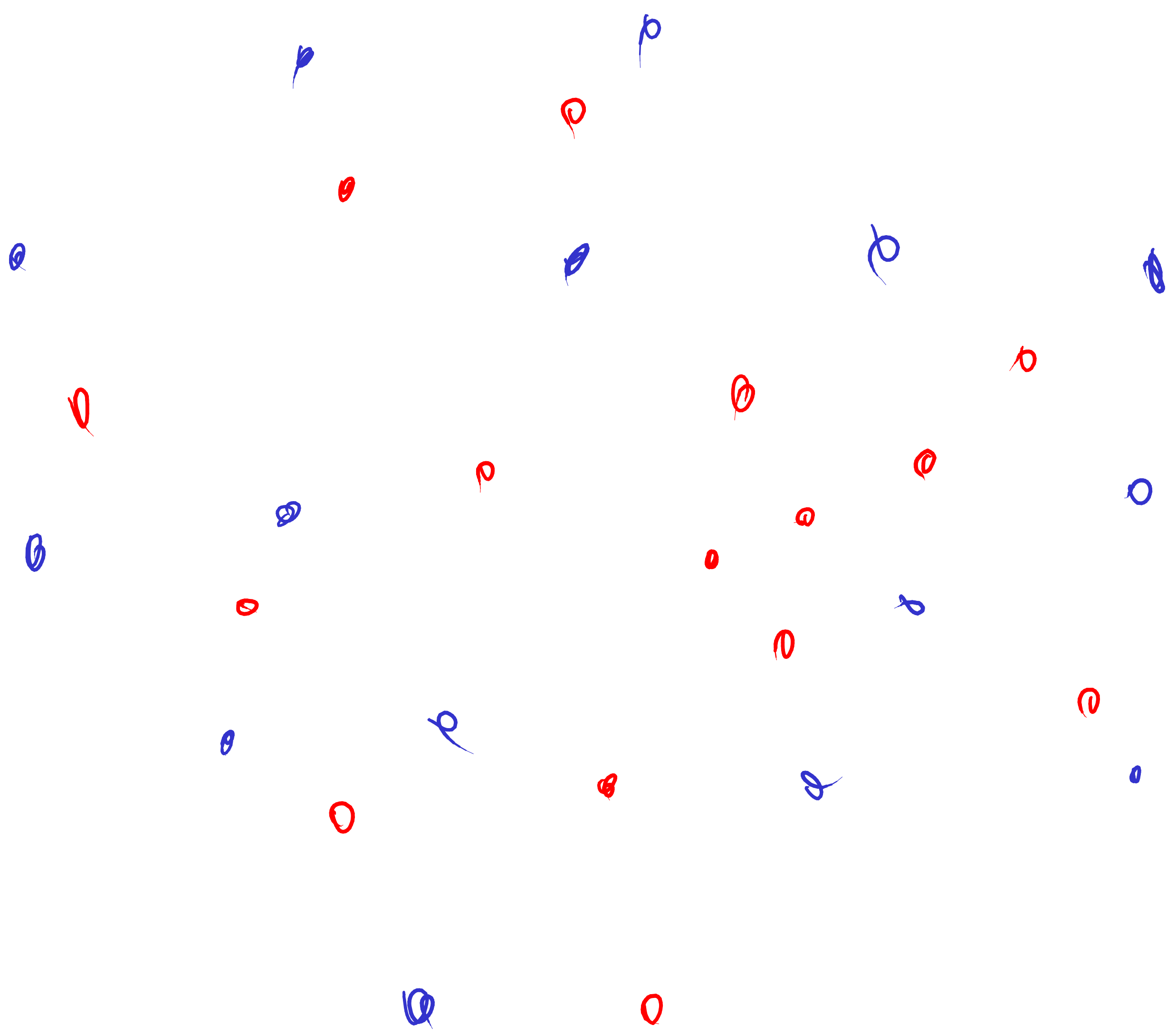


Why??

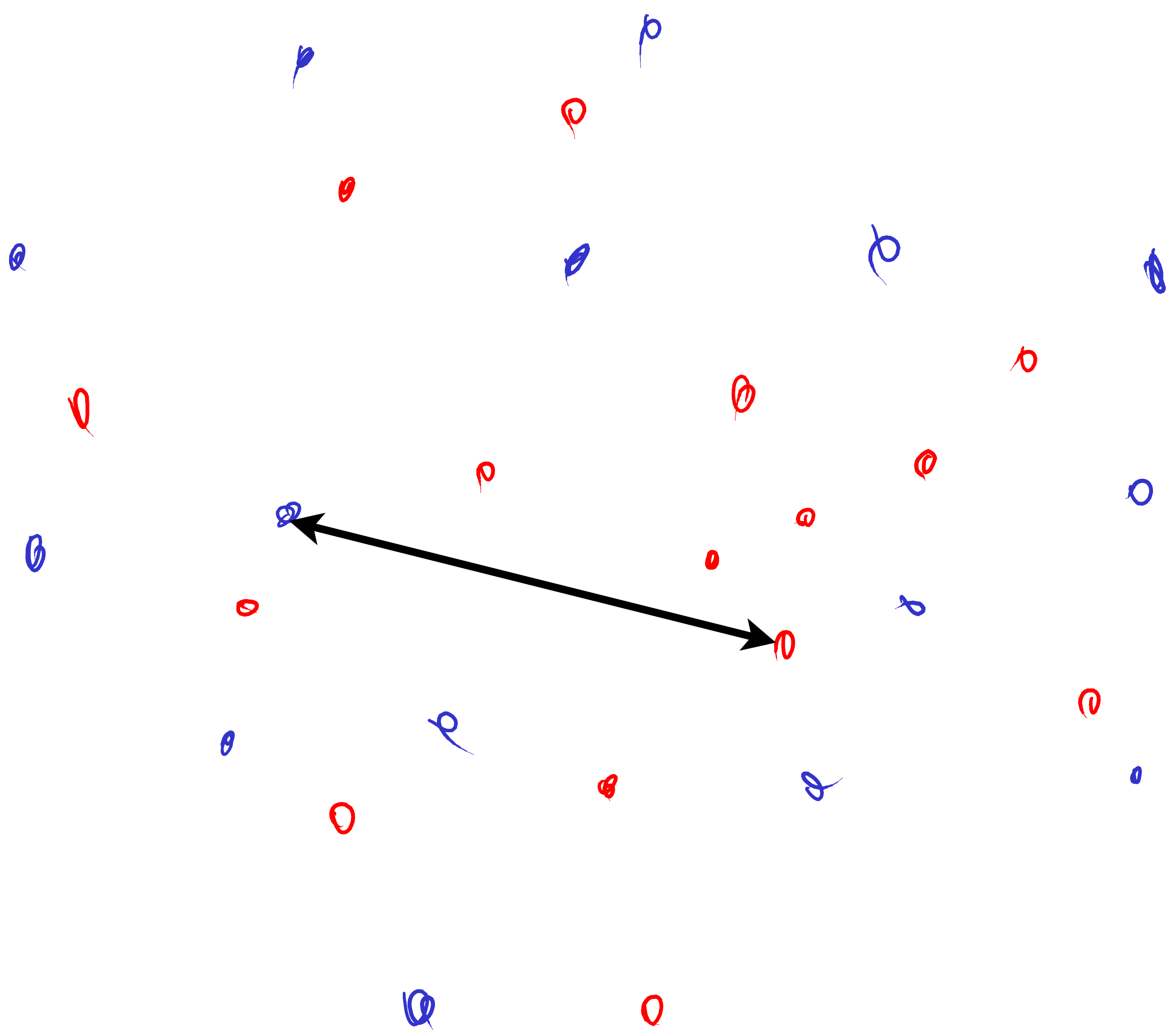
WHY?

- DIVIDE & CONQUER → ALGORITHMIC TOOL
- ≡ BORSÜK - ULAM / BROWER'S FP
- FUNDAMENTAL & FUN

Application: Let's dance



Application: Let's dance



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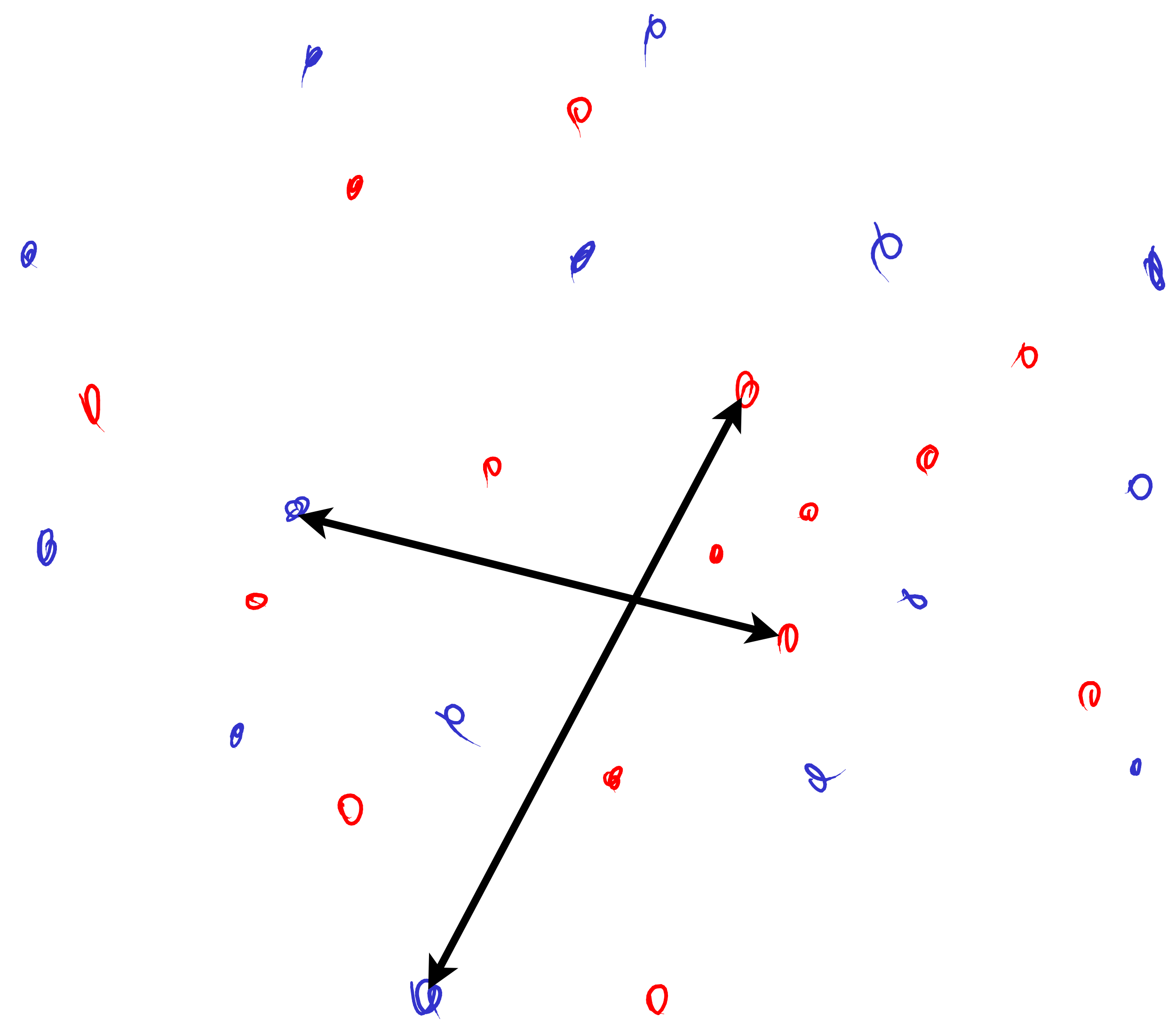
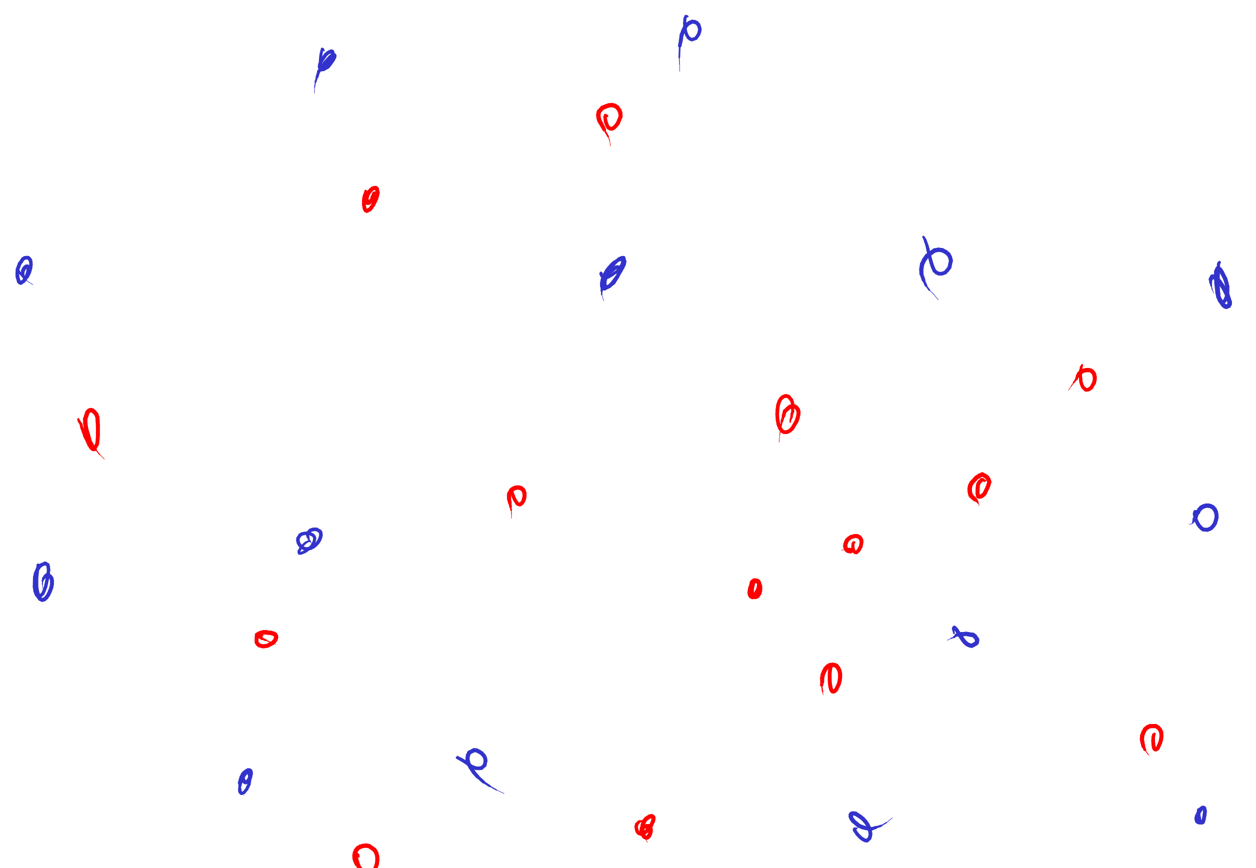
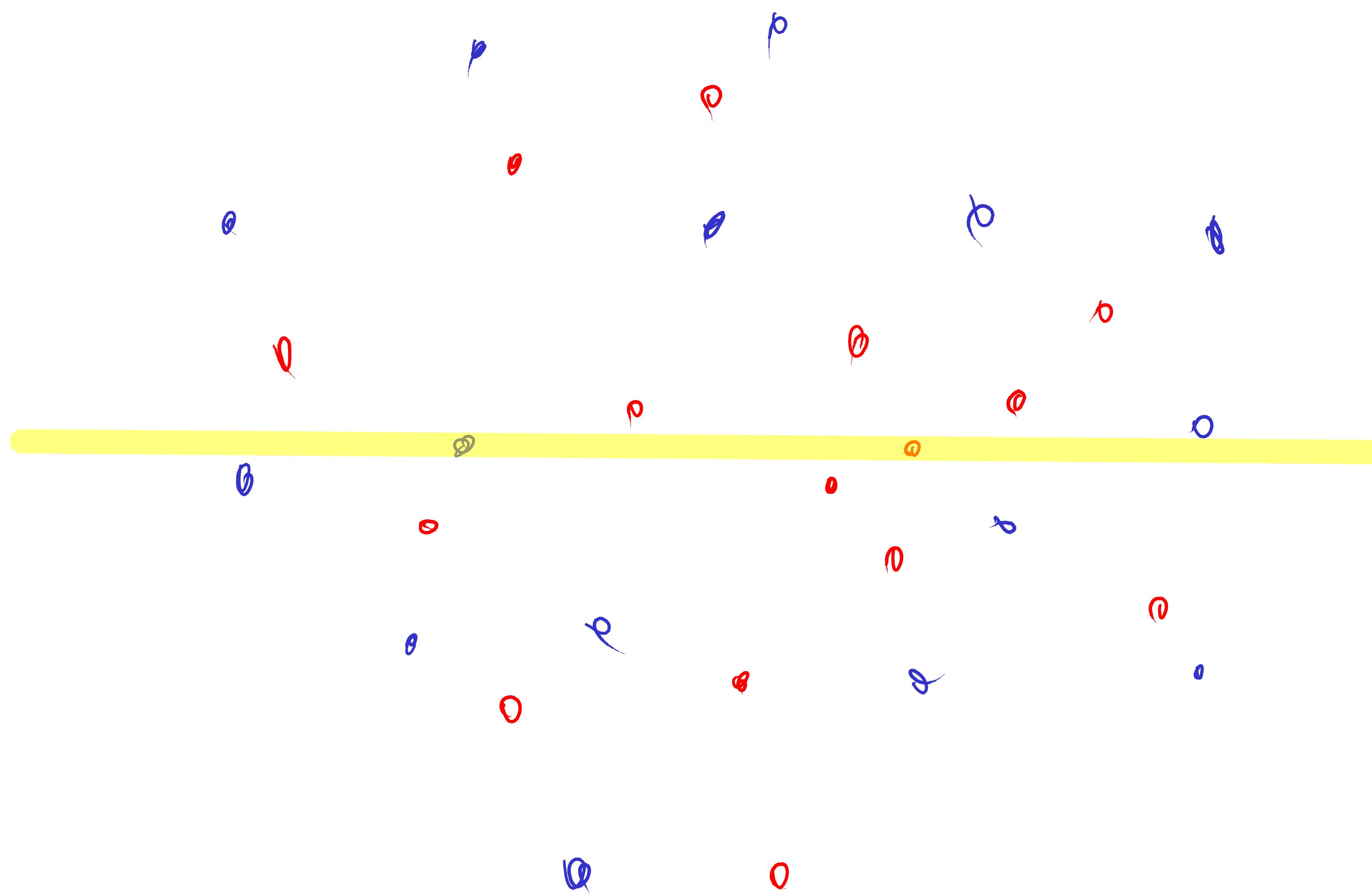


Application: Red-Blue matching

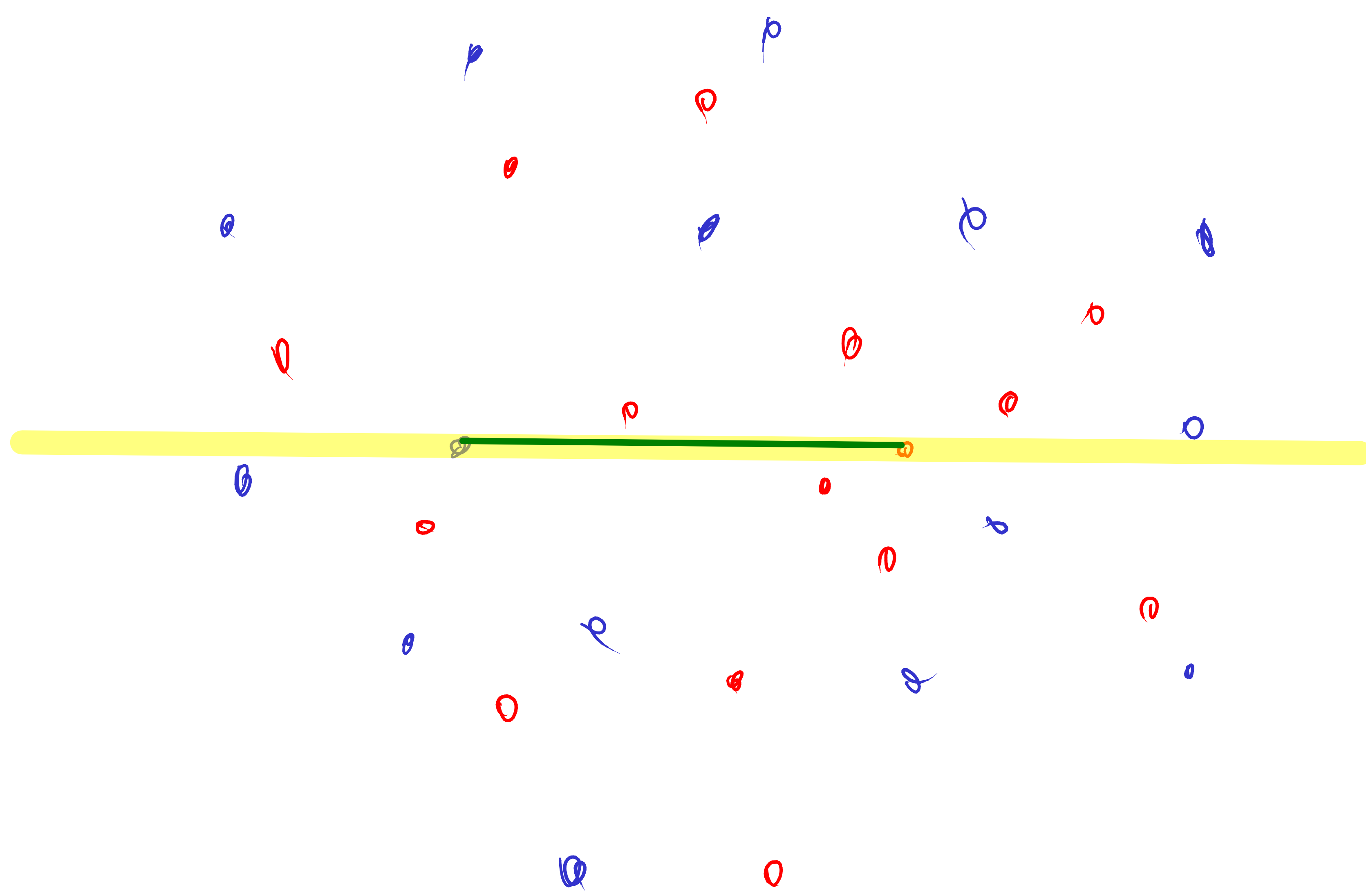


Given n red points and n blue points, find a non-crossing matching where every edge is red-blue

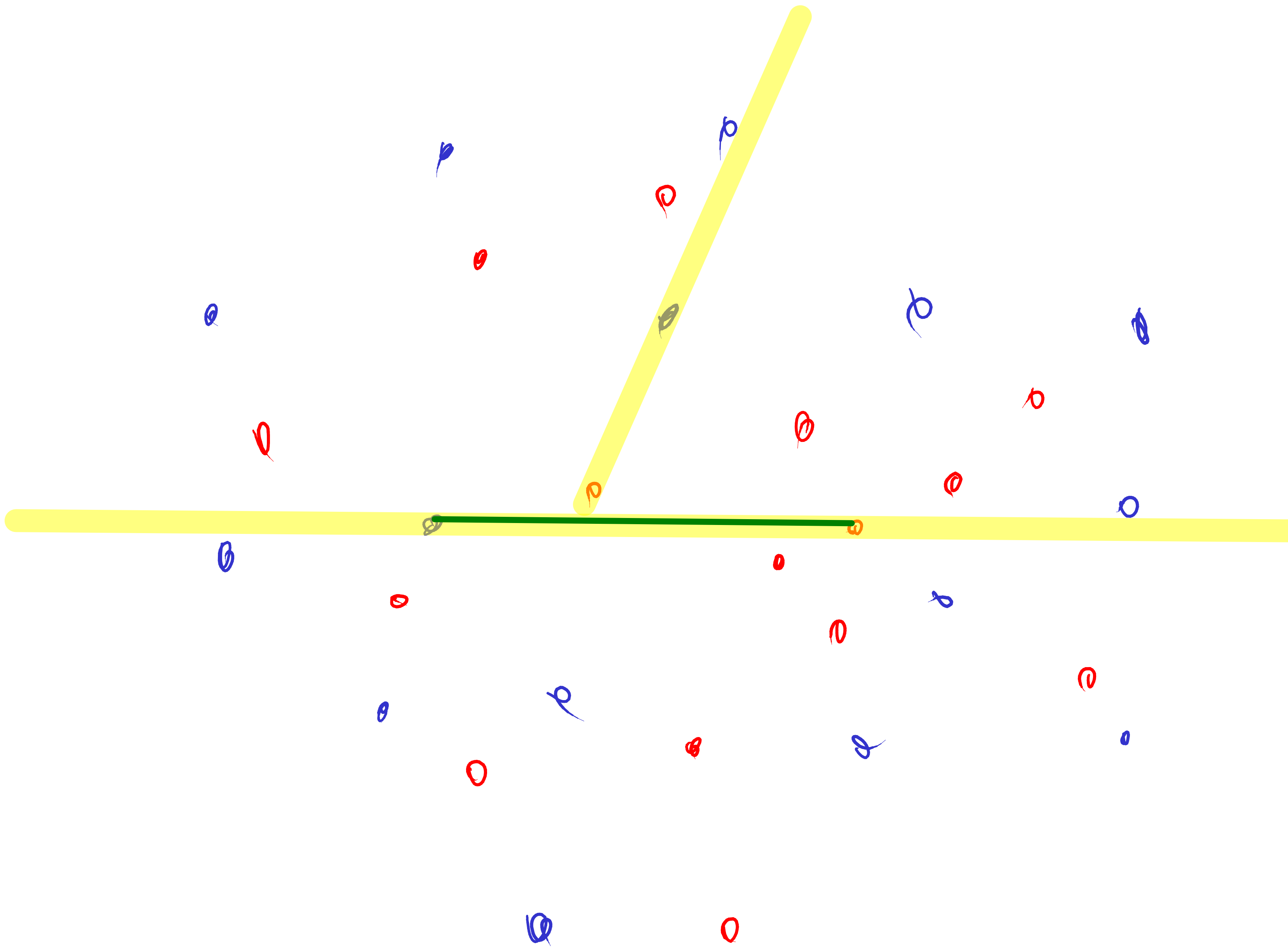
Application: Red-Blue matching



Application: Red-Blue matching



Application: Red-Blue matching



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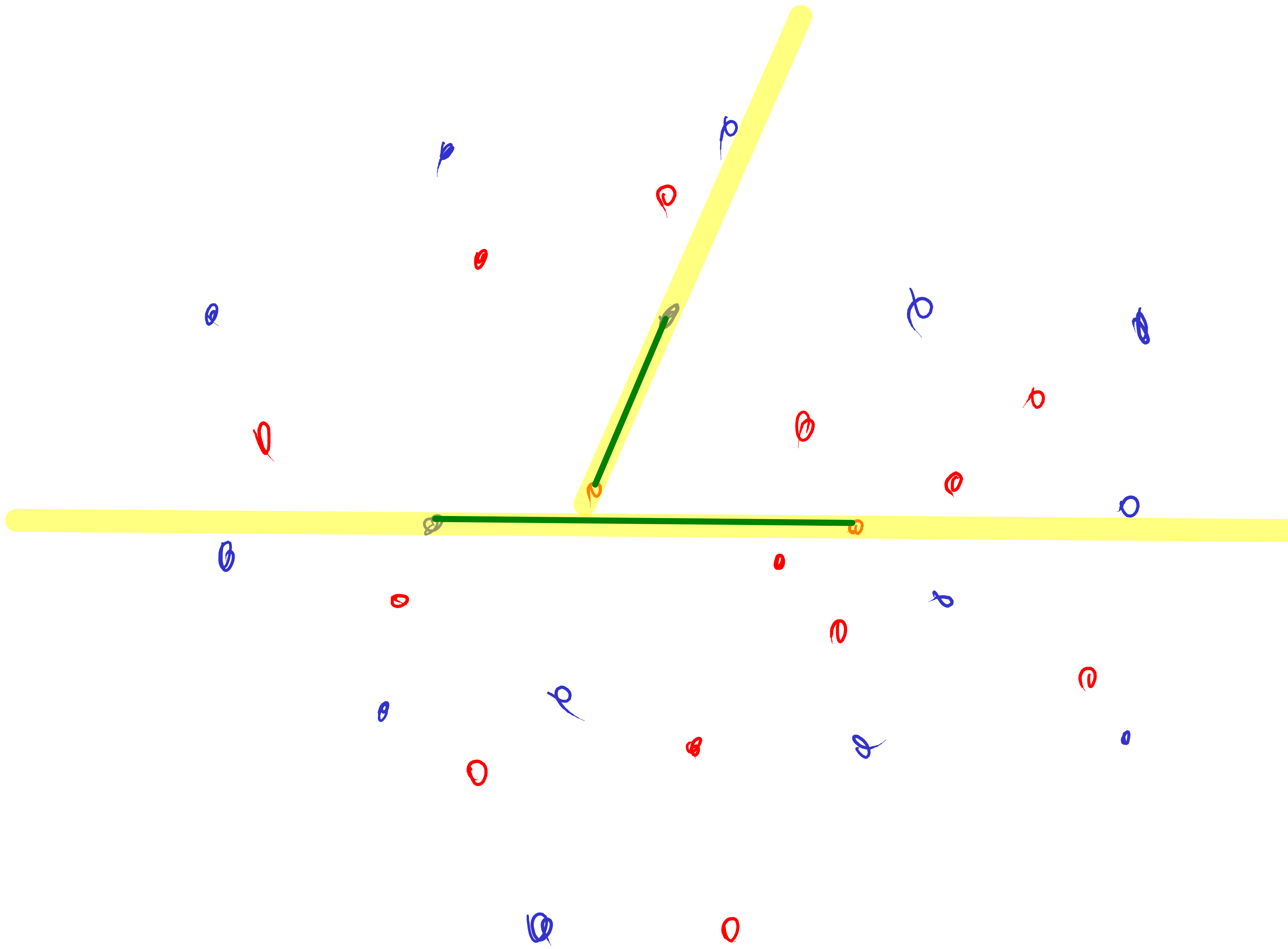

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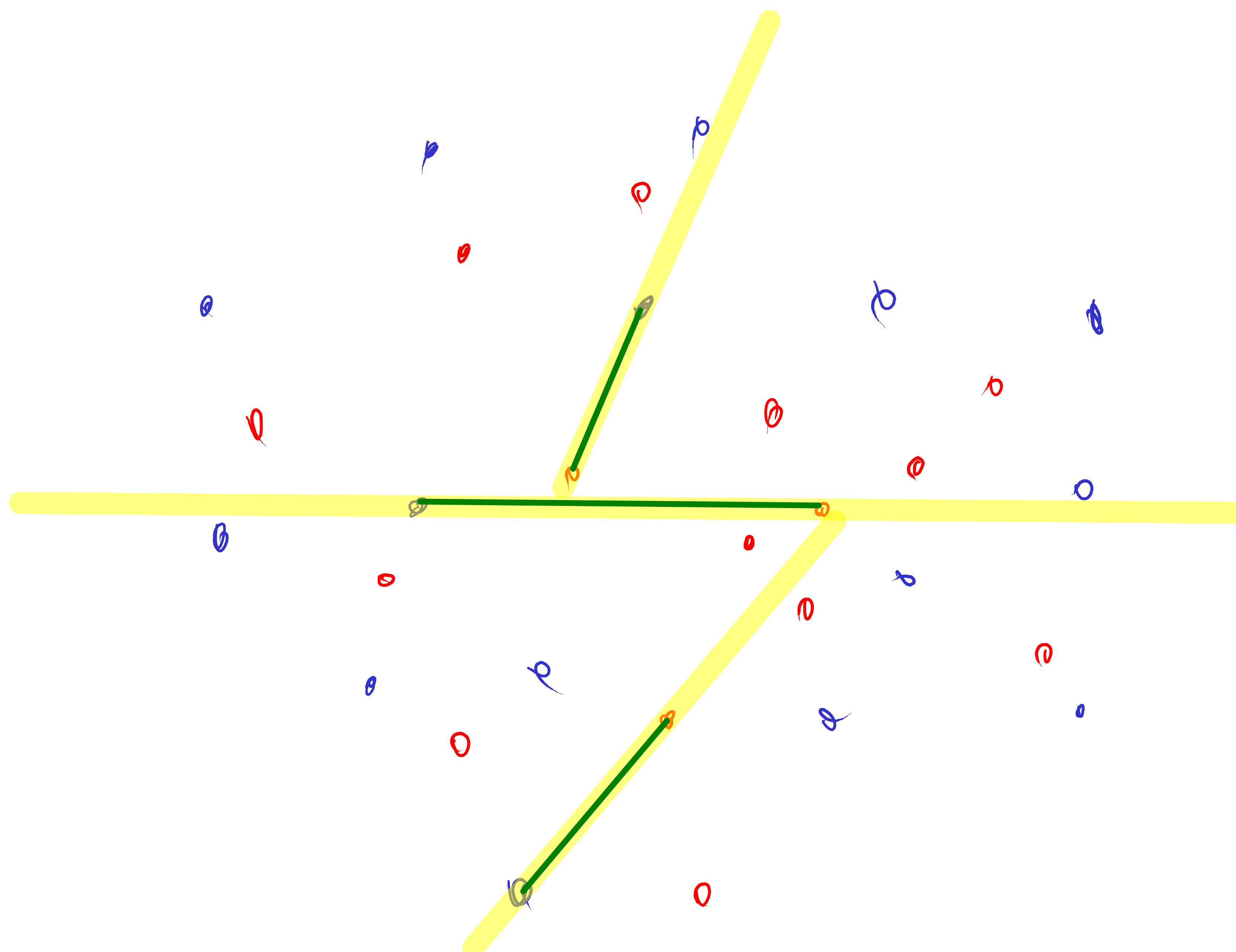
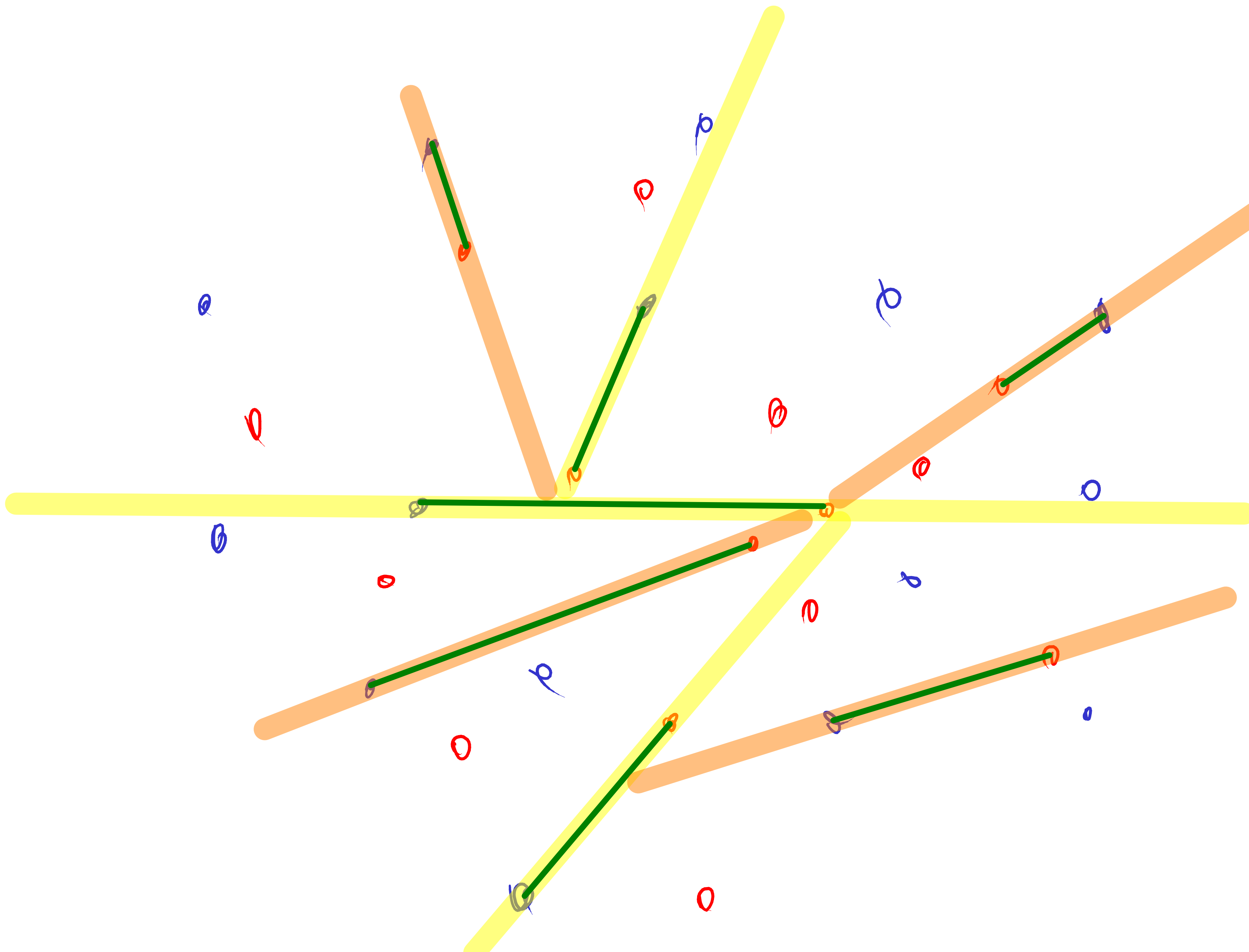


Application: Red-Blue matching



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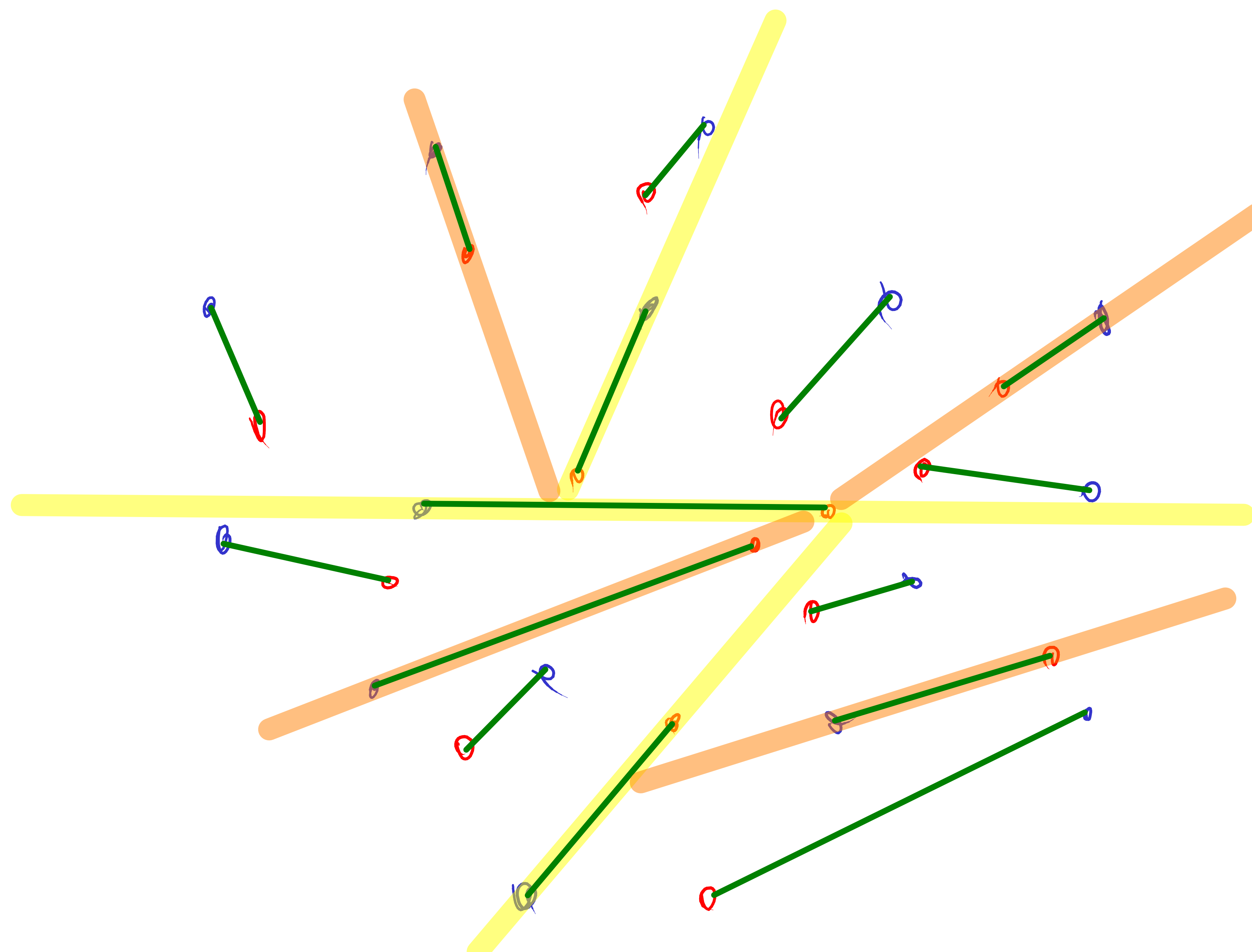


Combinatorial Problem:

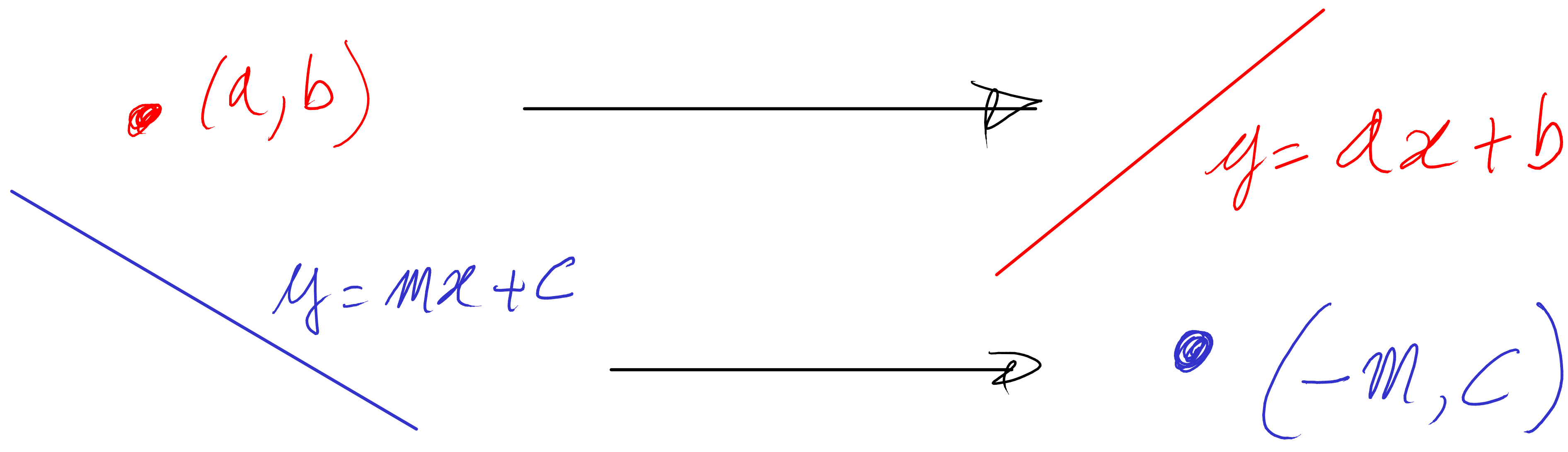
Prove the ham-sandwich cut always exists

Algorithmic Problem:

How do we find such a line?

(and how quickly?)

Duality Transform



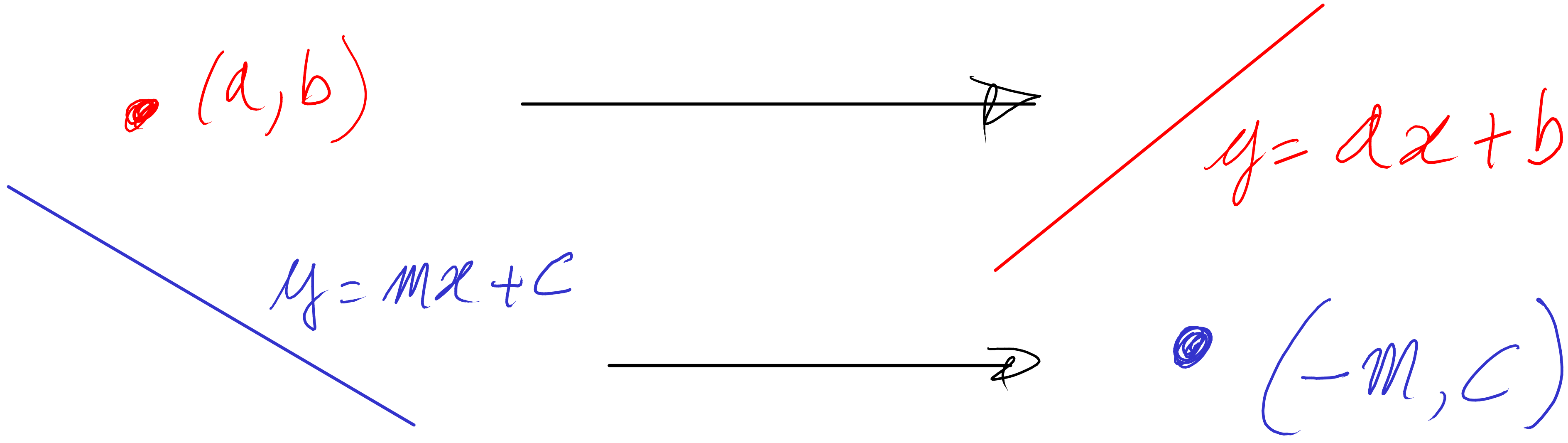
POINT ABOVE LINE

$b \geqslant ma + c$

LINE ABOVE POINT

$c \leqslant a(-m) + b$

Duality Transform



POINT ABOVE LINE

$$b \geqslant ma + c$$

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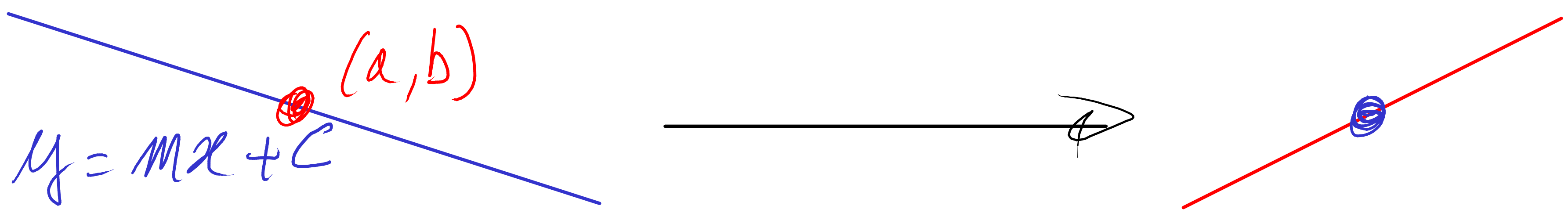


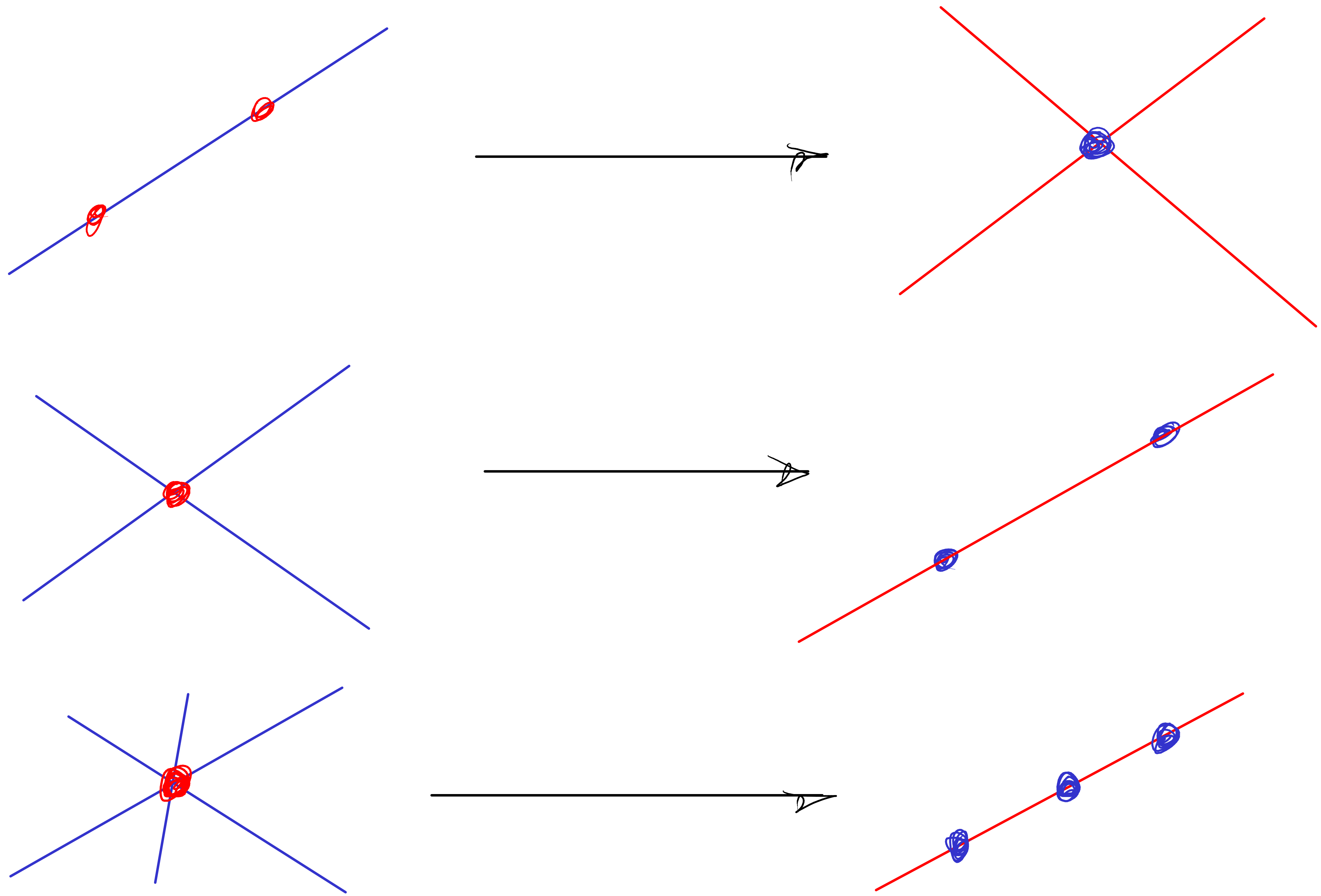
POINT ON LINE

$$b = ma + c$$

LINE ON POINT

$$c = a(-m) + b$$

Duality Transform



DISCRETE, 2D

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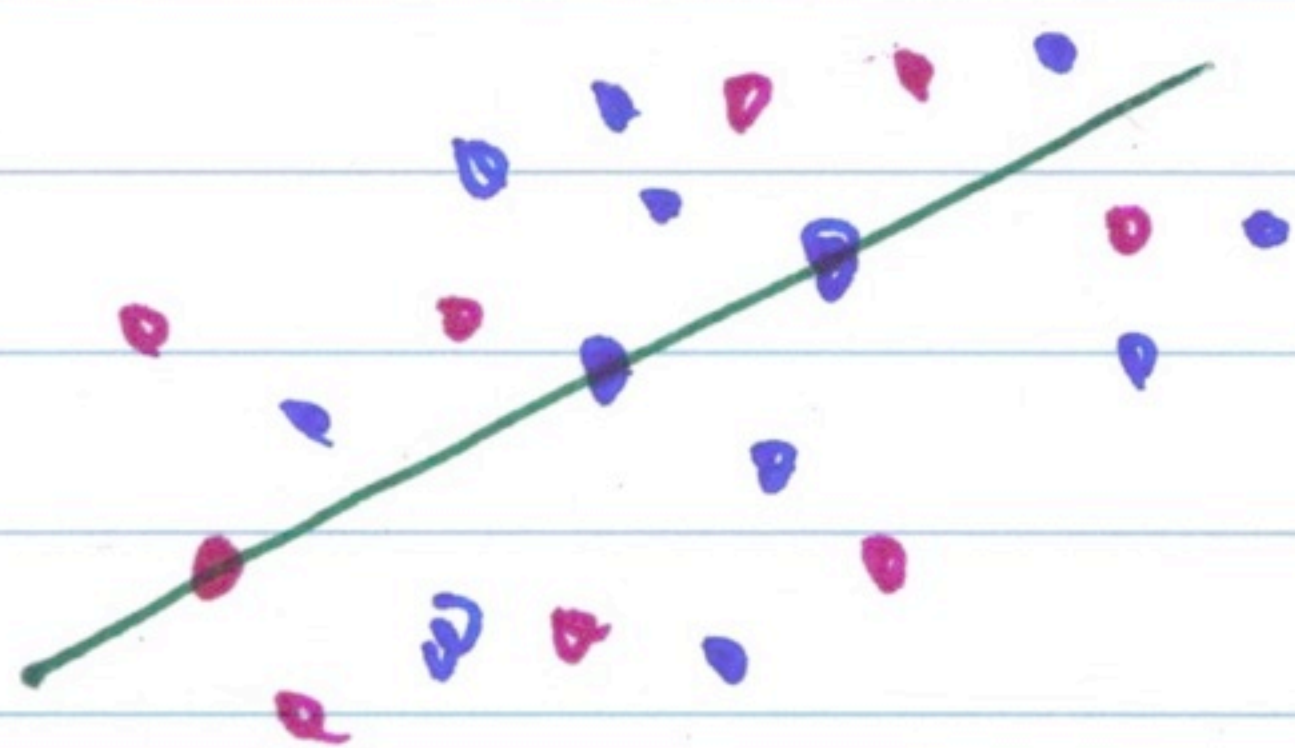
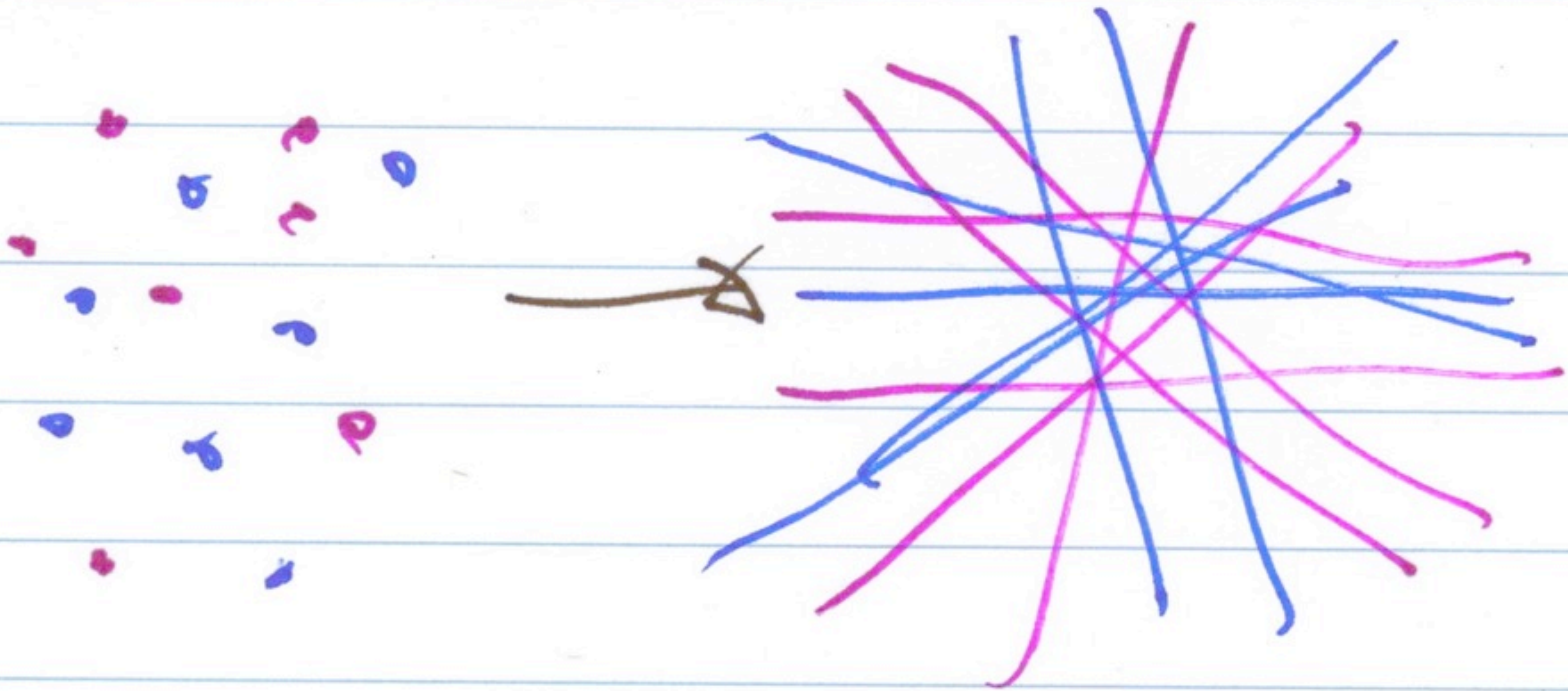
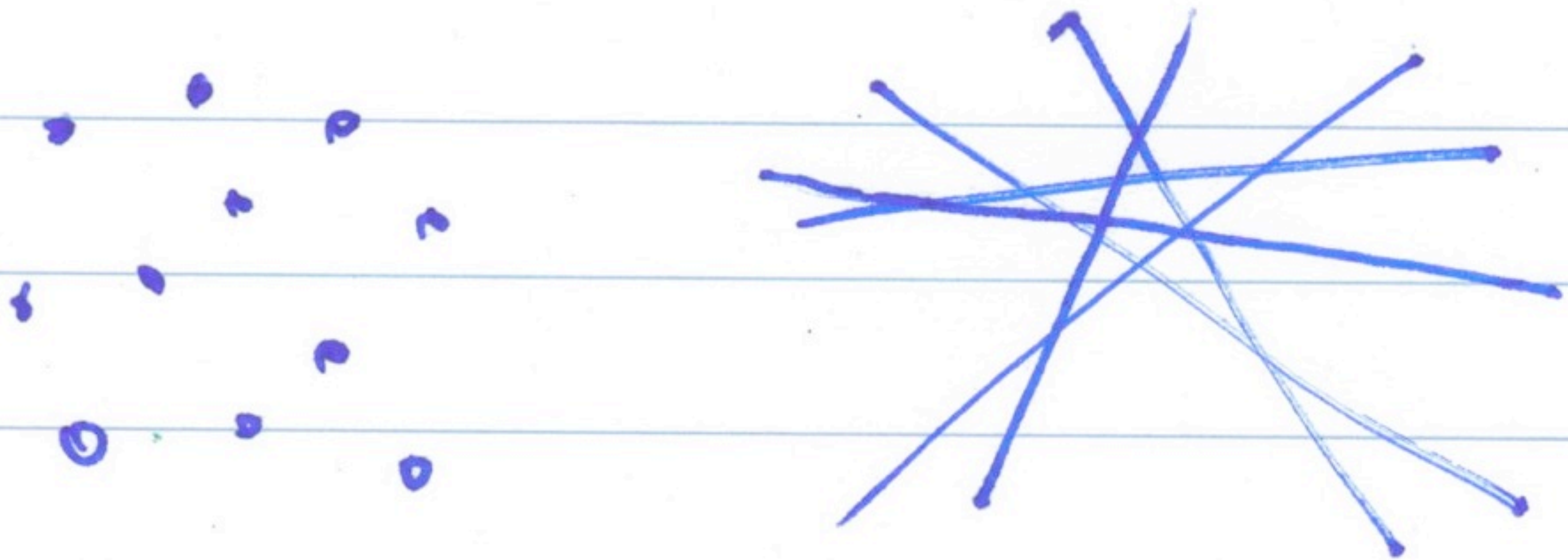


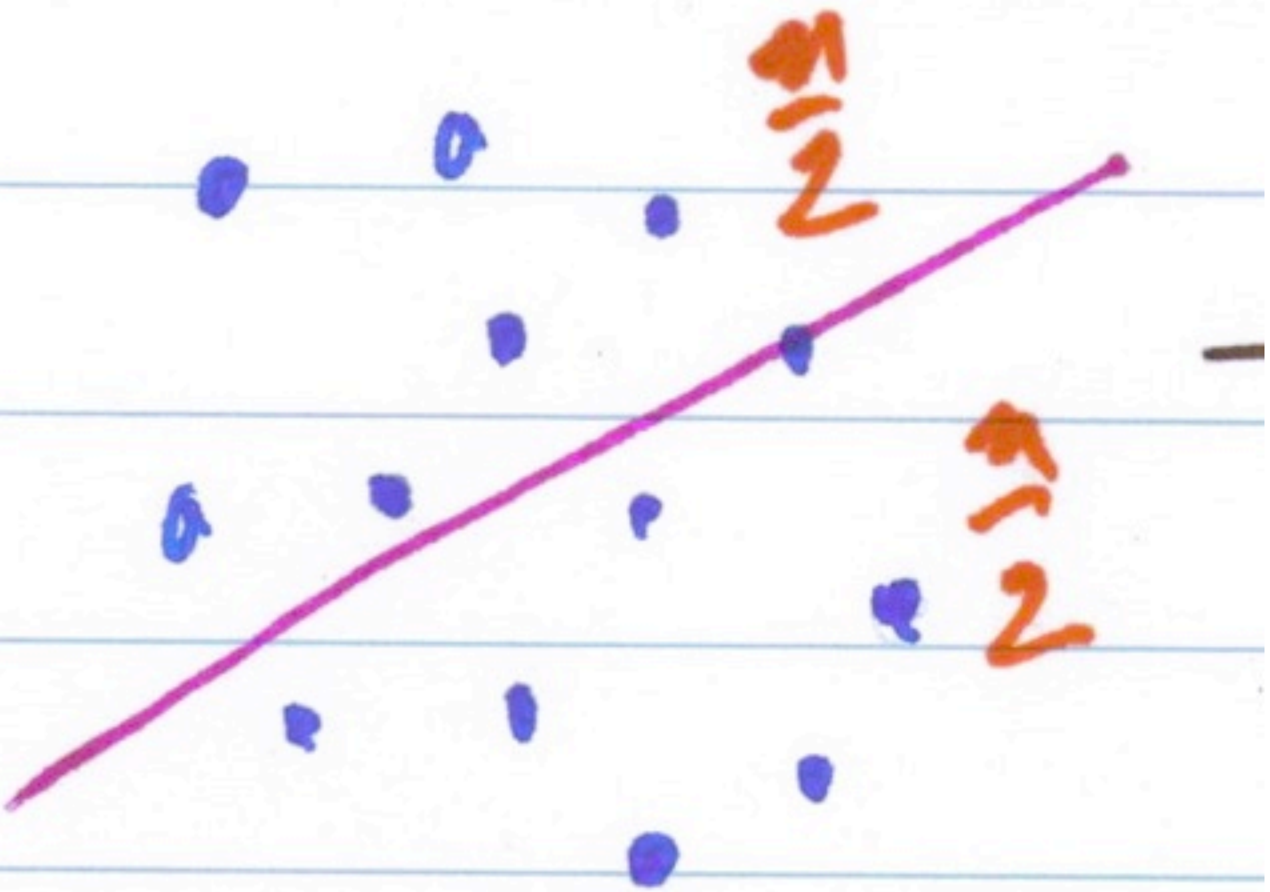
RED & BLUE POINTS → LINES



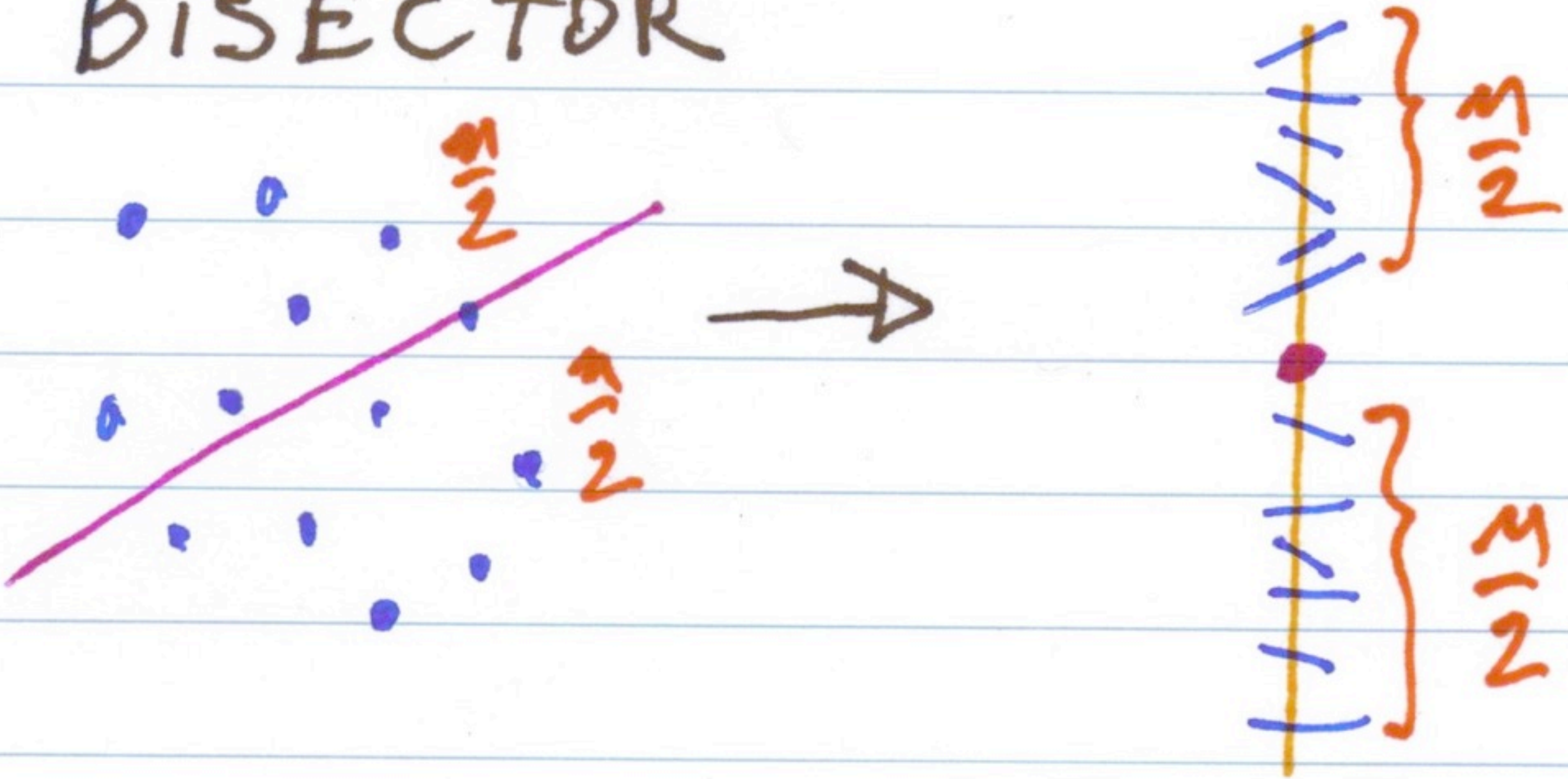
POINTS → LINES



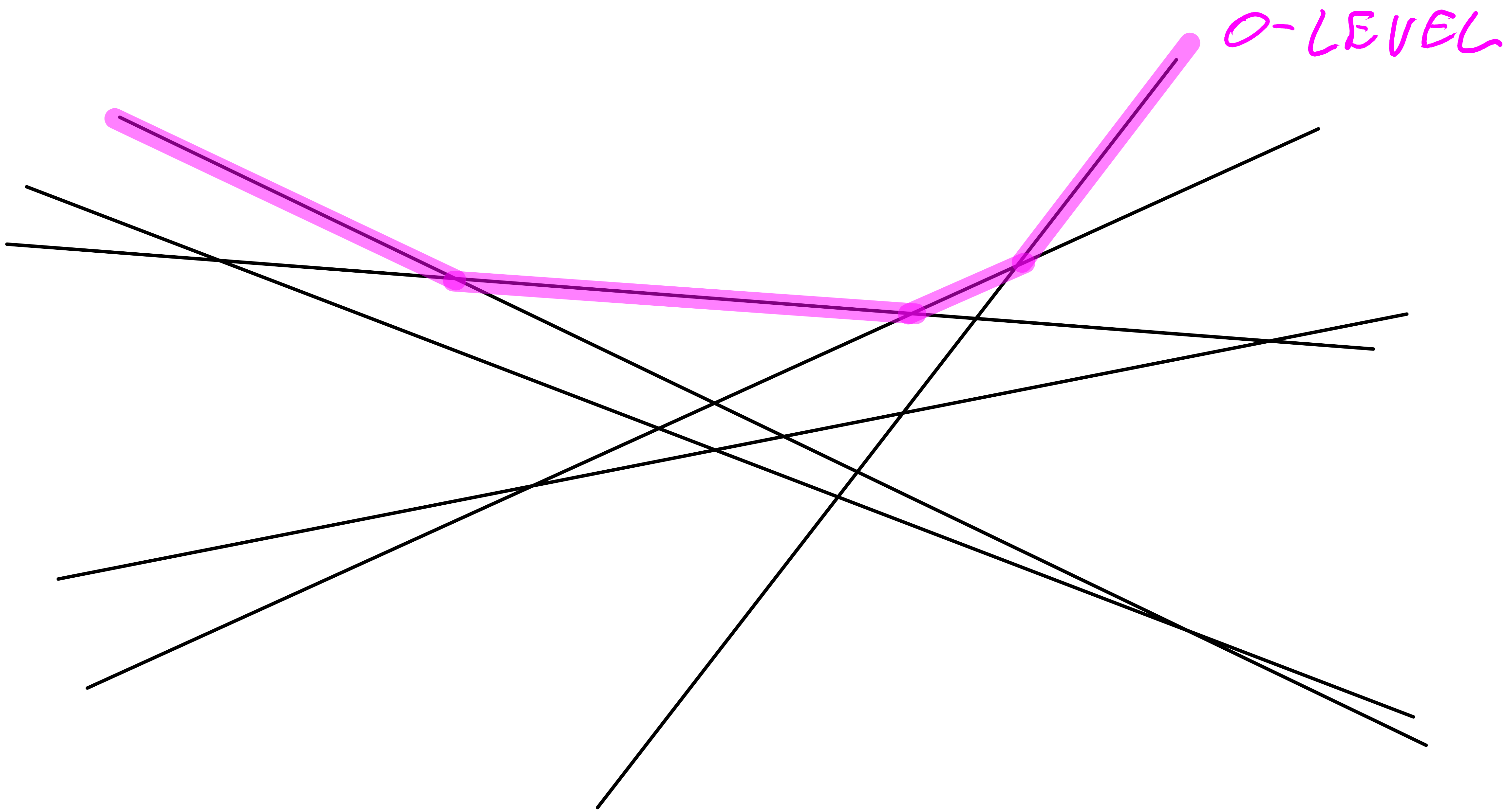
BISECTOR



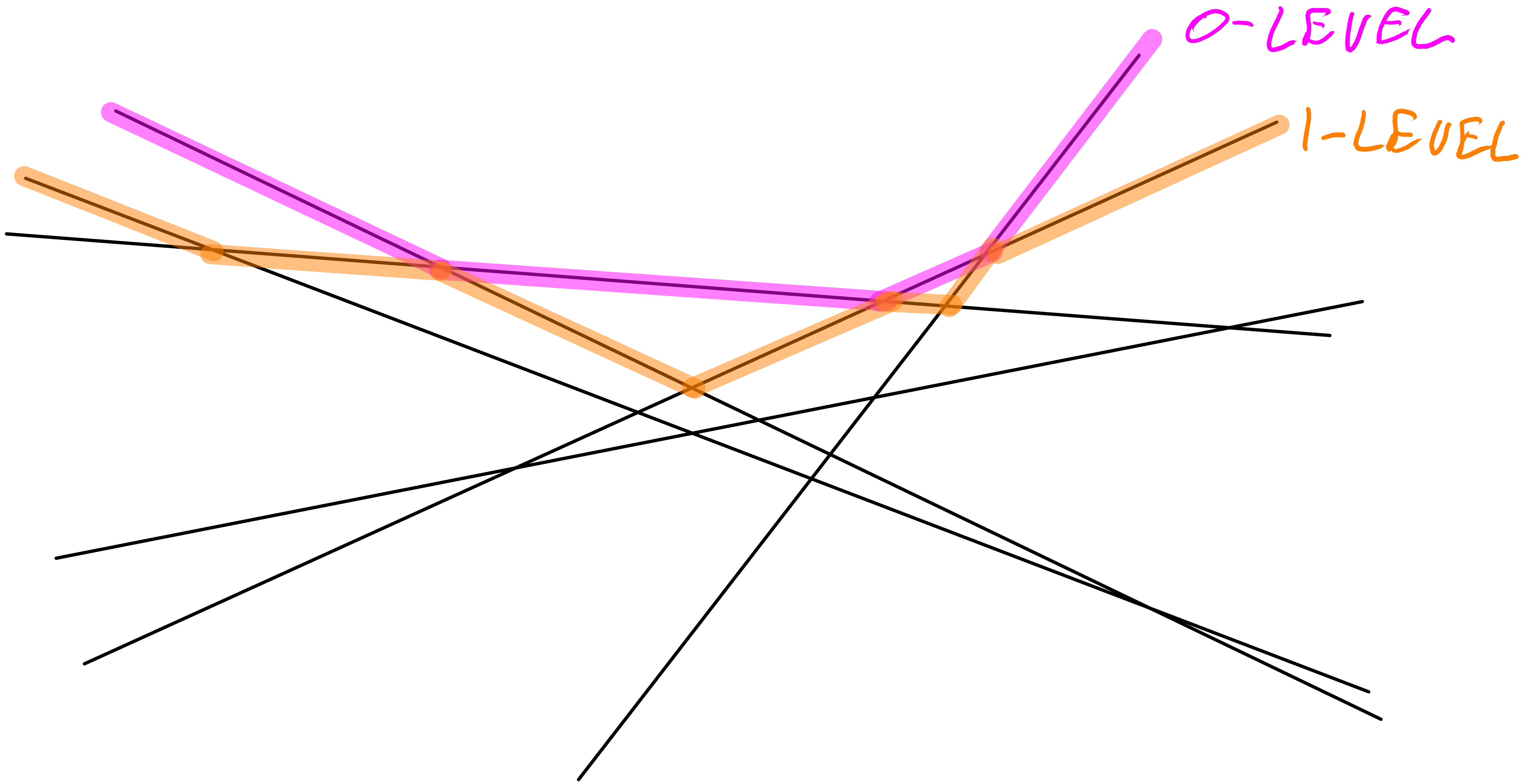
BISECTOR



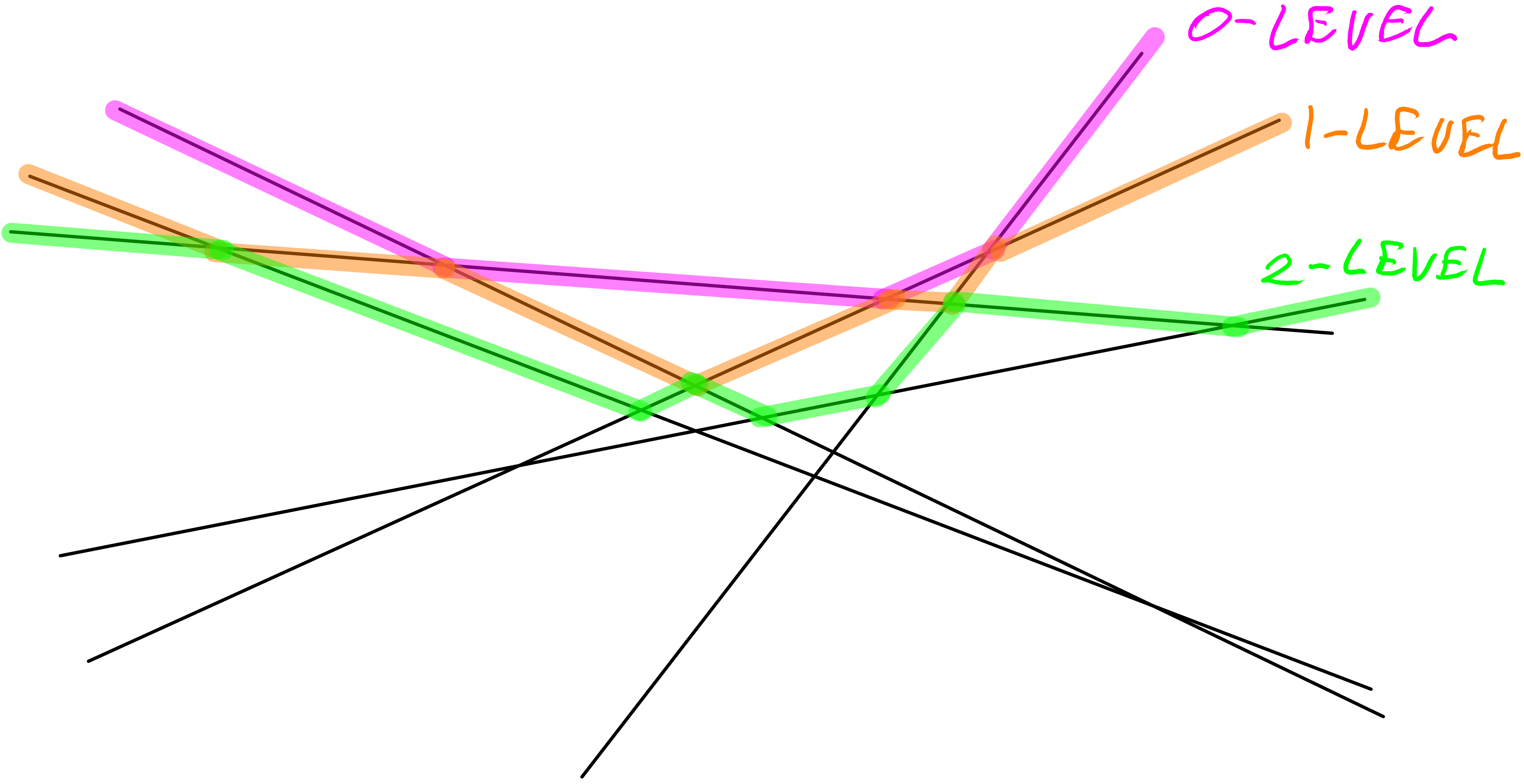
Levels



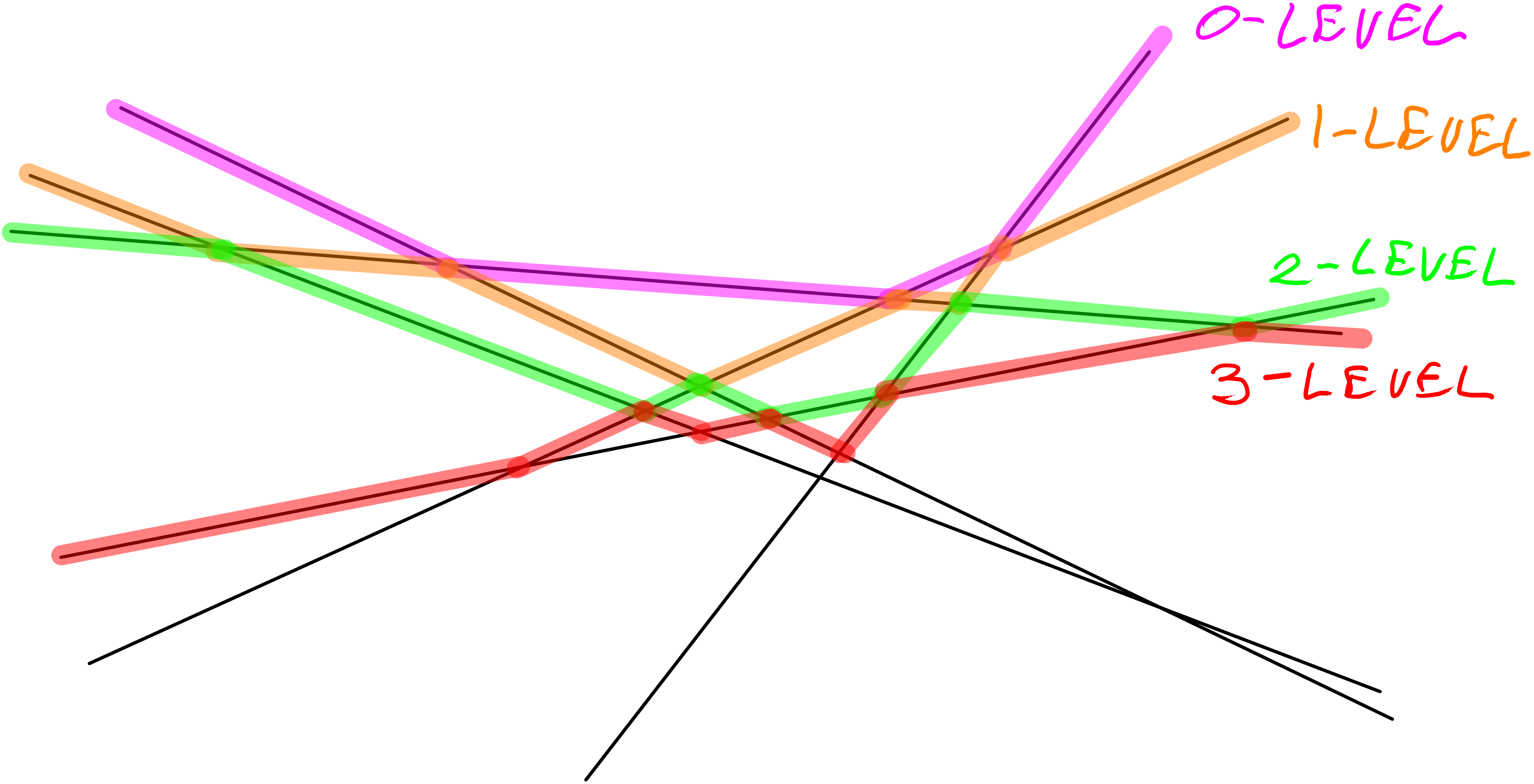
Levels



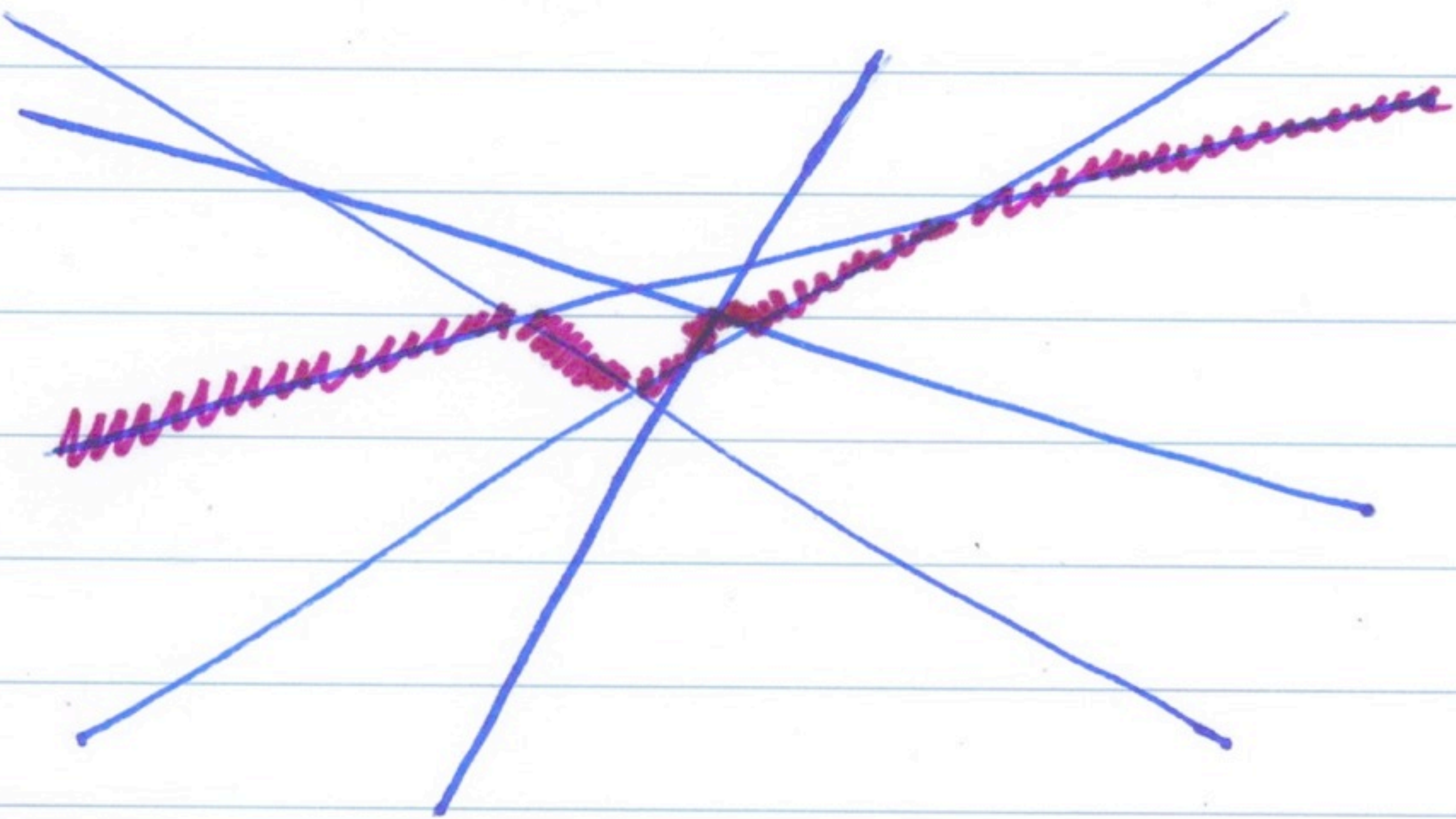
Levels



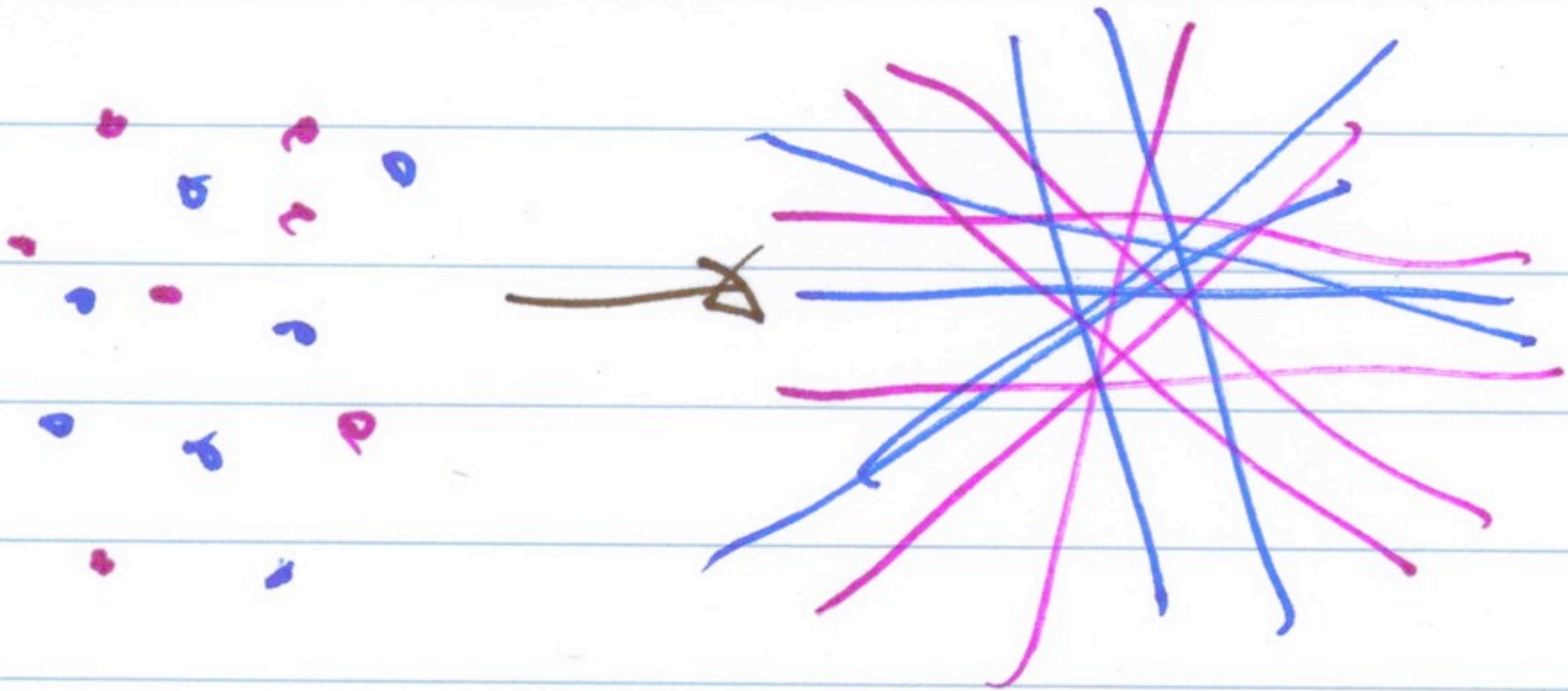
Levels



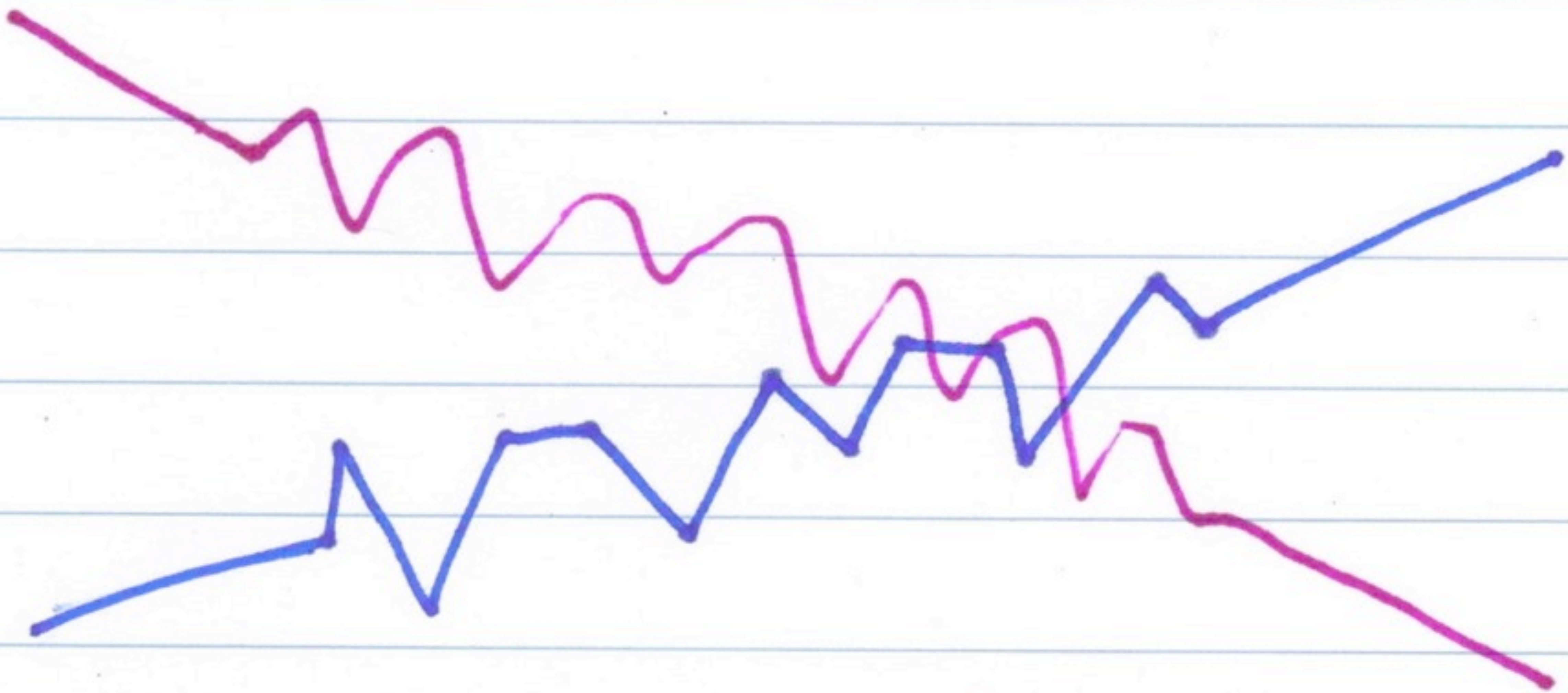
BISECTORS = MEDIAN LEVEL



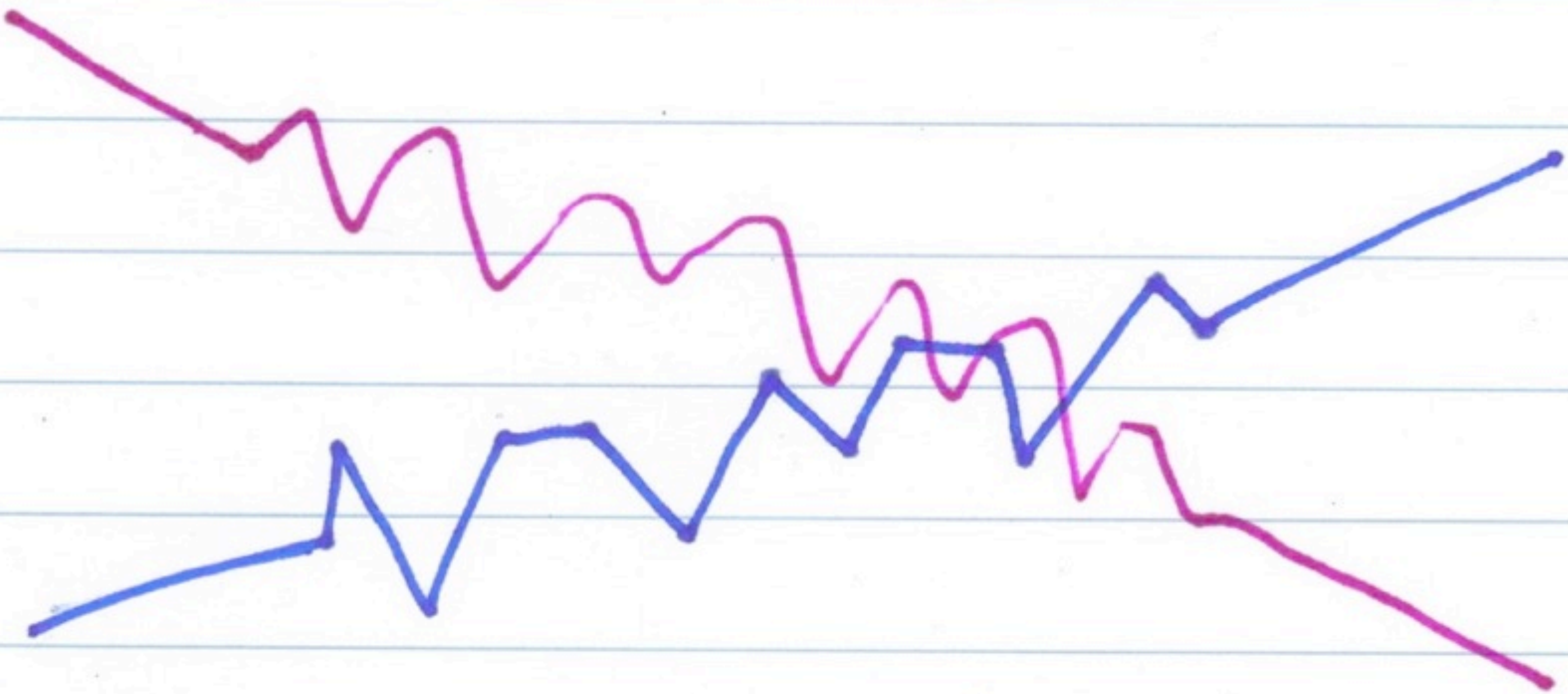
RED & BLUE POINTS → LINES



RED & BLUE MEDIAN LEVELS

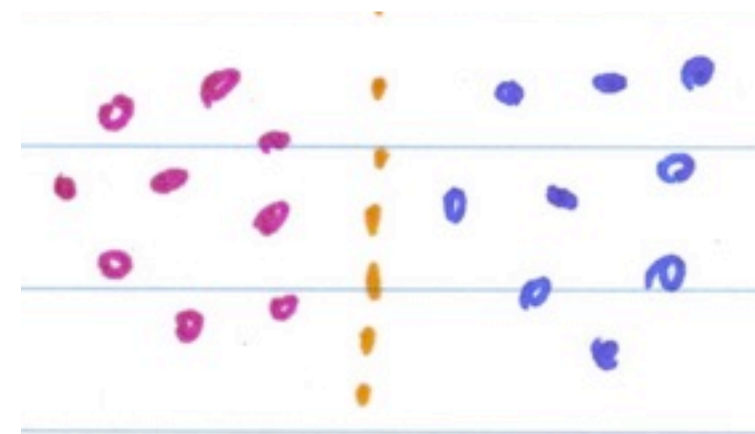


RED & BLUE MEDIAN LEVELS



INTERSECTION →
= HAM-SANDWICH CUT

Algorithms



Algorithms

- O(n) for separated case
  [Megiddo 1985]

Algorithms

- O(n) for separated case [Megiddo 1985]
- O(n log n) general case [Edelsbrunner & Waupotitsch 1986]

Algorithms

- O(n) for separated case
  [Megiddo 1985]

- O(n log n) general case
  [Edelsbrunner & Waupotitsch 1986]
- O(n) general case
  [Lo & Steiger 1990]

ALGO I (DISCRETIZE)

IF n IS ODD THEN
L MUST TOUCH ONE
RED AND ONE BLUE POINT.

⇒ TRY EACH PAIR AND
VERIFY $O(n^3)$

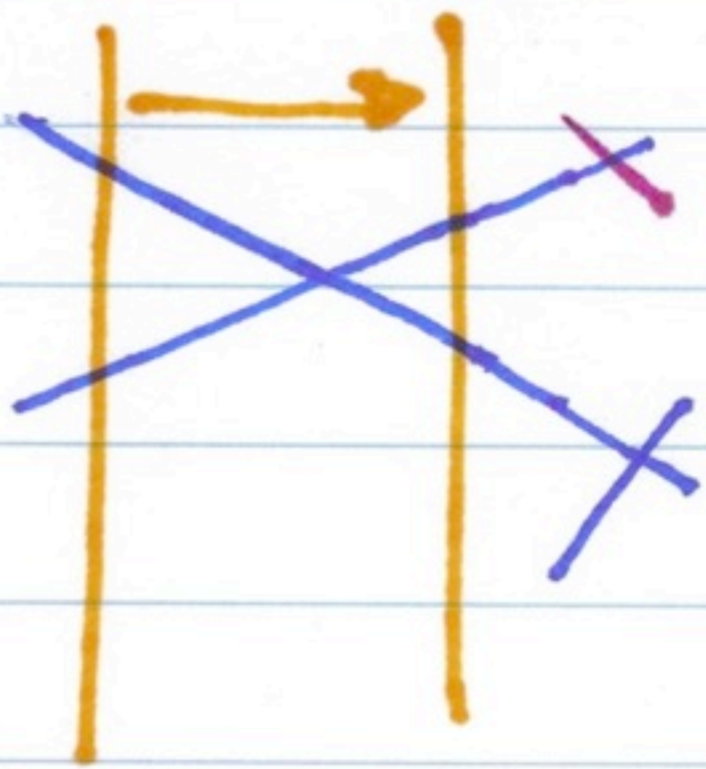
ALGO II SWEEPLINE

SWEEP THE PLANE (DUAL)
WITH A VERTICAL LINE L

MAINTAIN THE ORDER OF THE LINES WITH L IN A TREE

ALGO II SWEEP LINE

STORE THE UPCOMING X EVENTS IN A HEAP



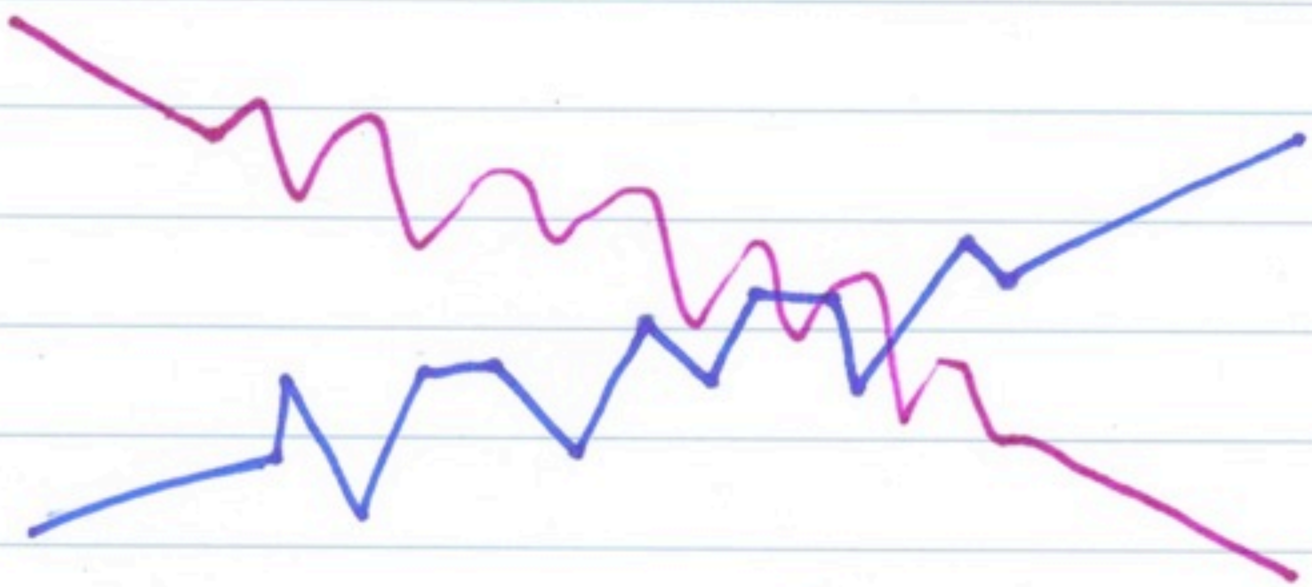
- GET NEXT EVENT
- SWAP LINES IN TREE
- STORE NEXT EVENTS IN HEAP

→ $O(n^2 \log n)$

ALGO III

- CONSTRUCT RED & BLUE MEDIAN LEVELS
- FIND THEIR INTERSECTIONS

RED & BLUE MEDIAN LEVELS



INTERSECTION ⤴
= HAM-SANDWICH CUT

THEY INTERSECT AN ODD NUMBER OF TIMES!
(⇒ > 0)

HOW BIG IS THE MEDIAN LEVEL?

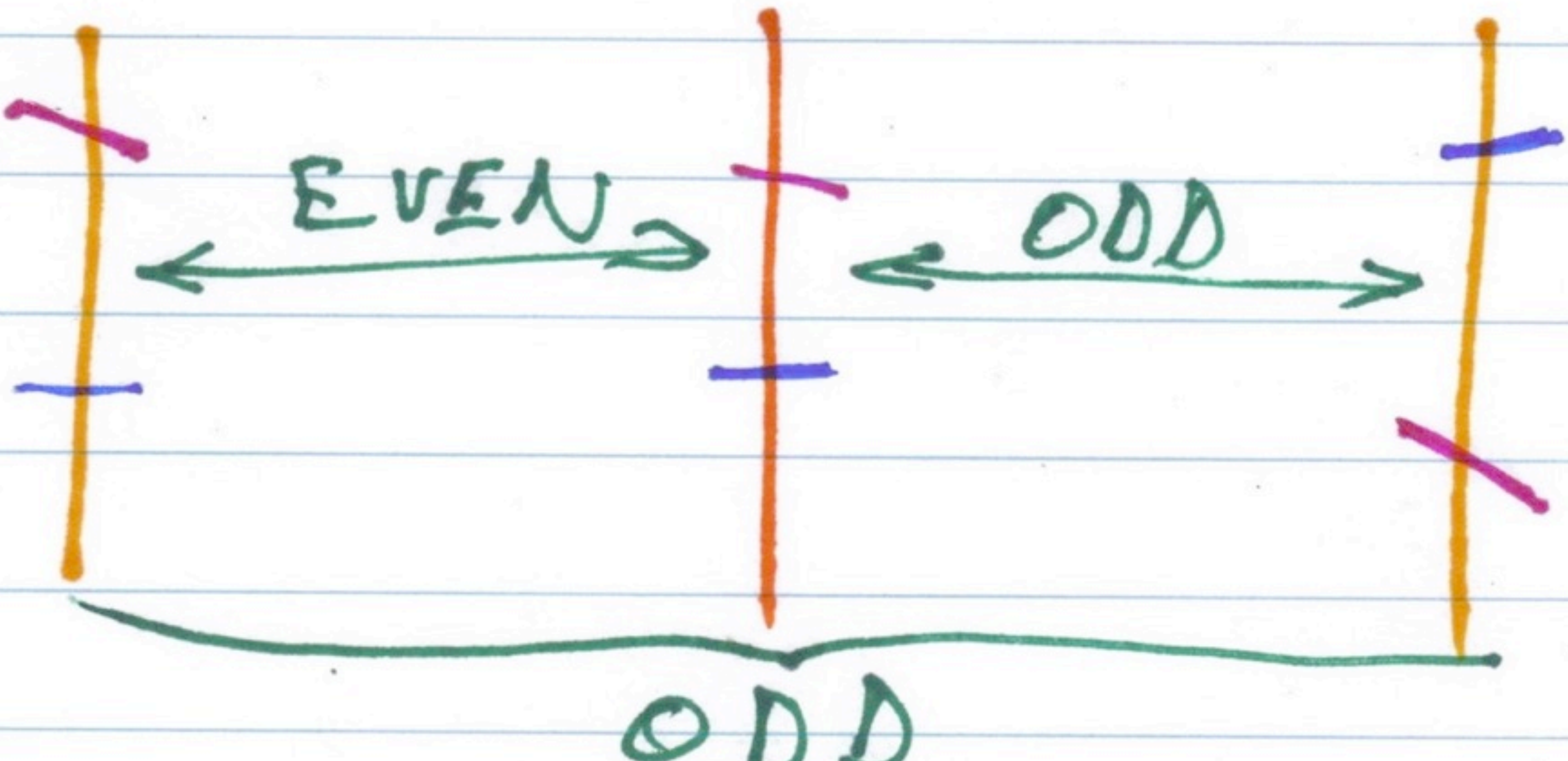
$O(n^{3/2})$ [ERDŐS, LOVASZ,
$\Omega(n \log n)$ SIMMONS, STRAUS '73]

$O(n^{3/2}/\log^* n)$ [PACH, STEIGER, SZEMEREDI '89]

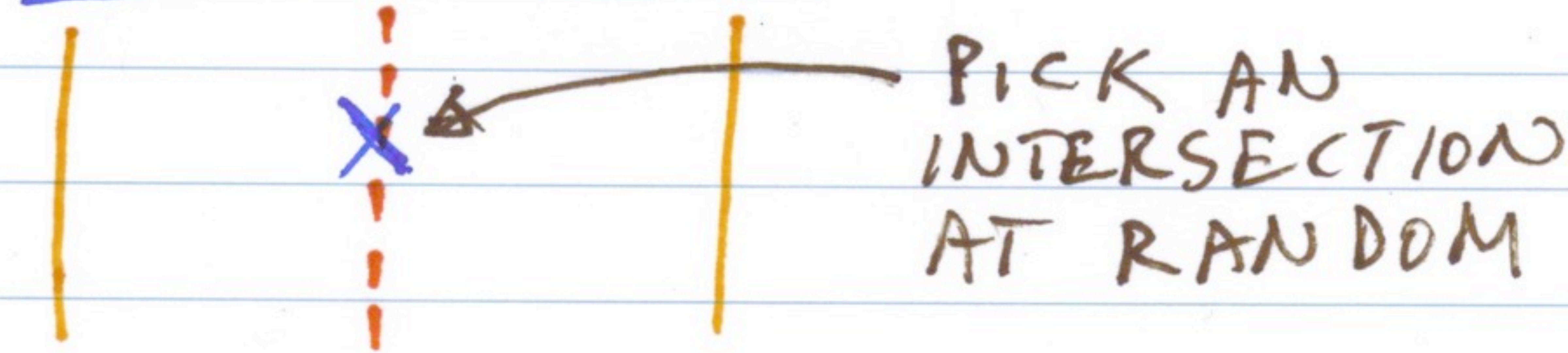
$O(n^{4/3})$ [DEY '97]

$n \cdot 2^{\Omega(\sqrt{\log n})}$ [TOTH '00]

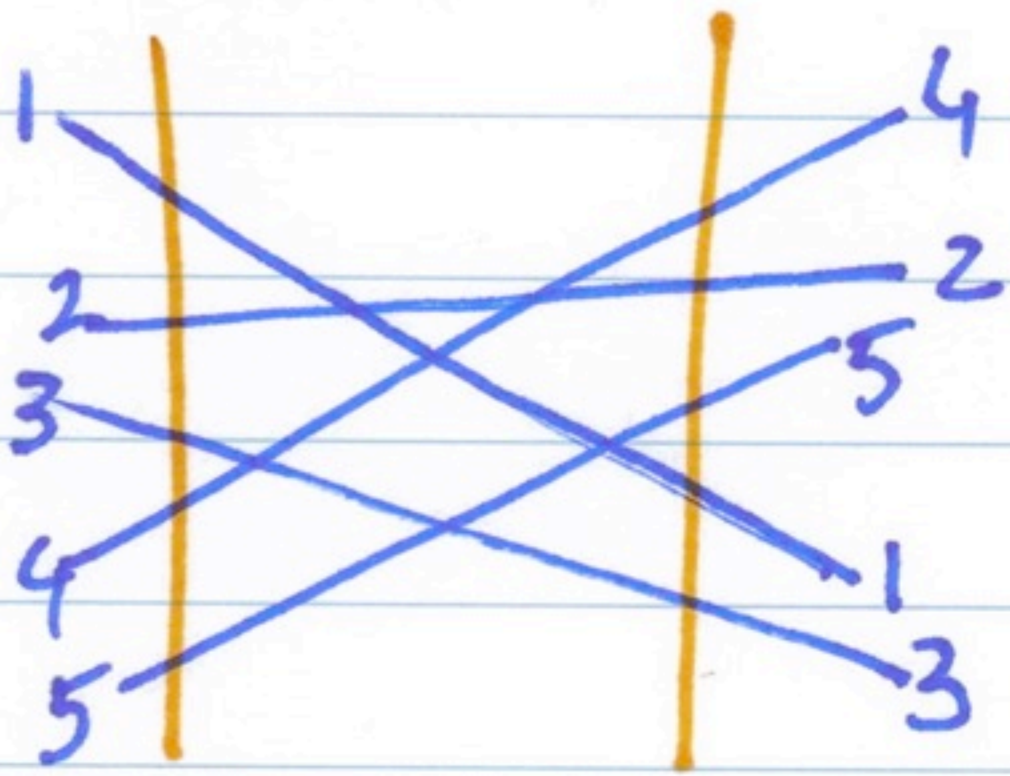
ALGO IV BINARY SEARCH



HOW TO SPLIT?

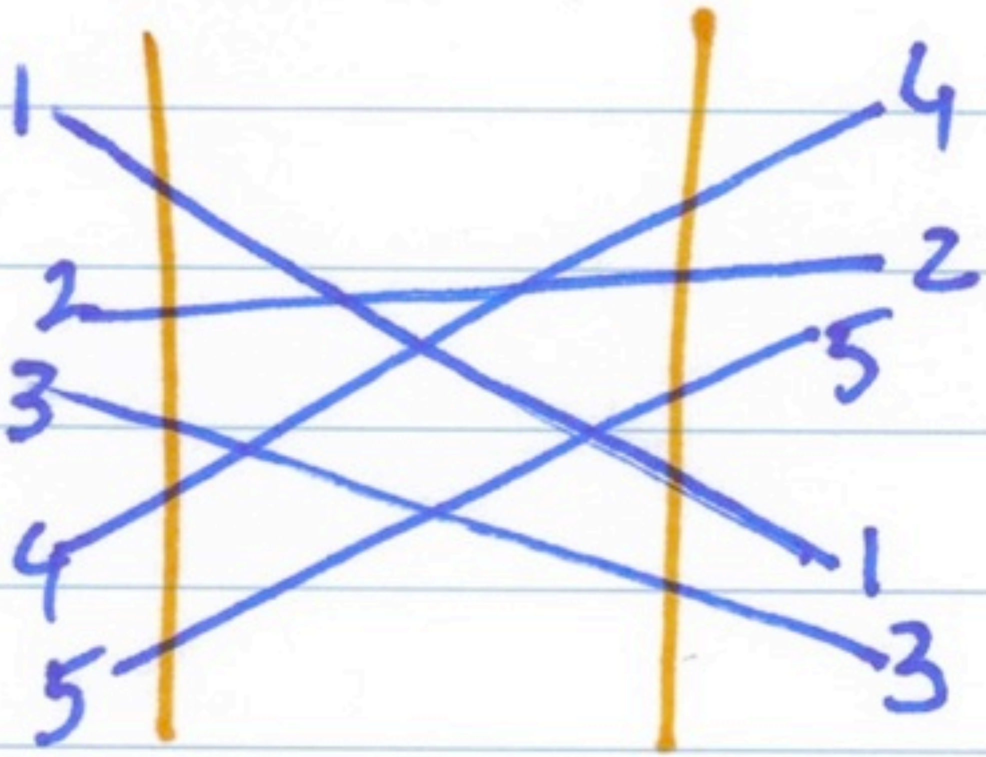


COUNTING INTERSECTIONS



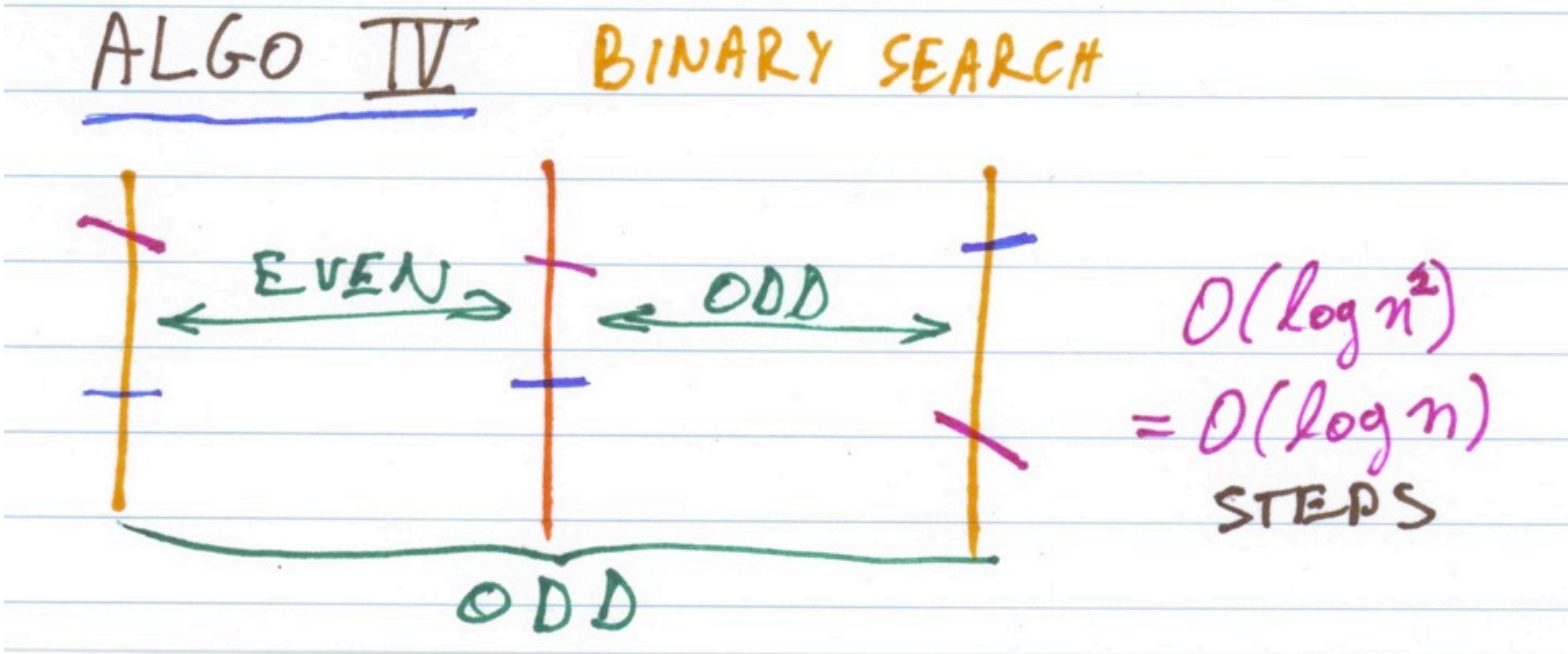
# of INTERSECTIONS
= # INVERSIONS
⇒ SORT & COUNT
$O(n \log n)$

COUNTING INTERSECTIONS



\# of INTERSECTIONS
= # INVERSIONS
⇒ SORT & COUNT
$O(n \log n)$

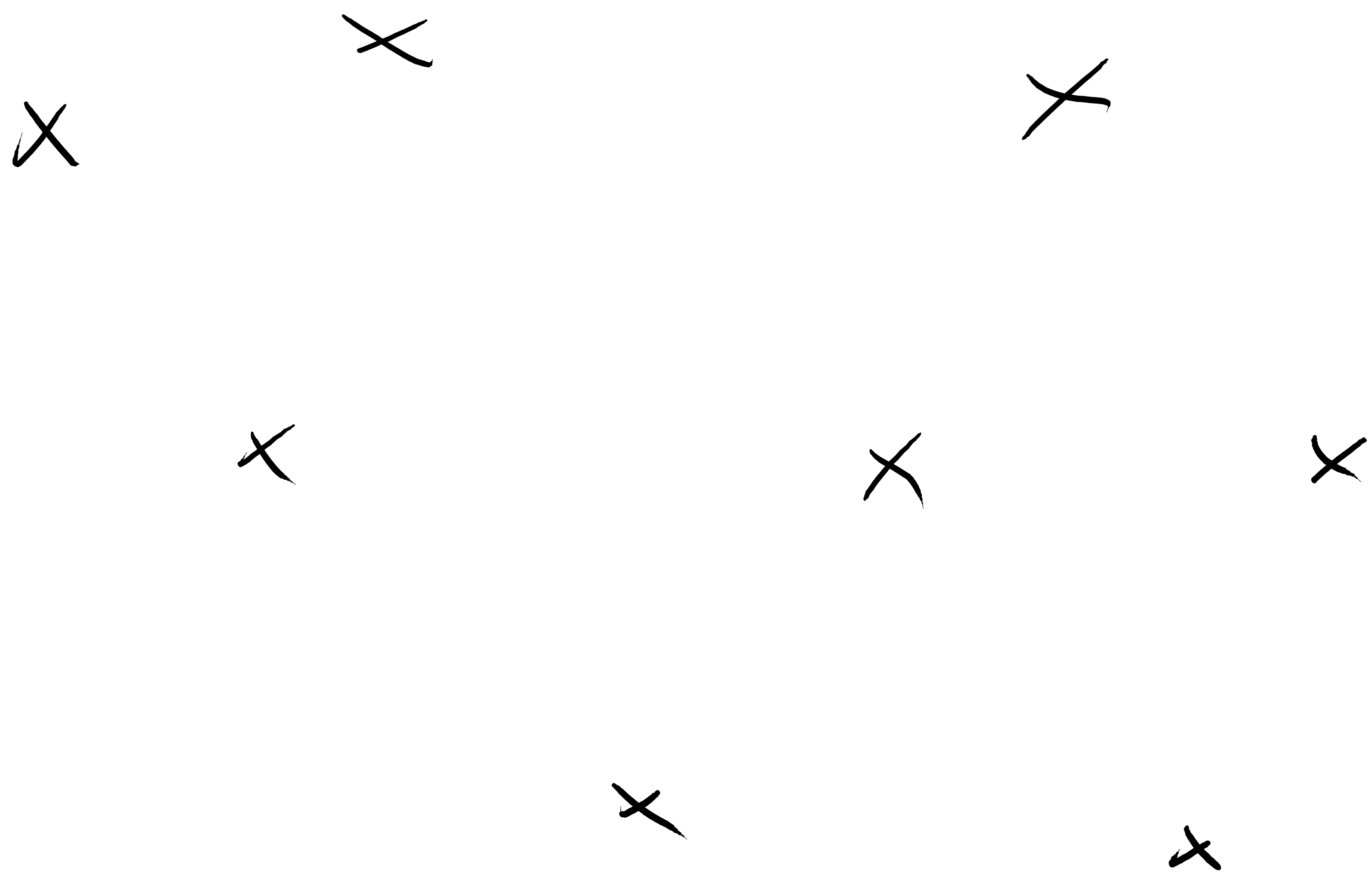
NOTE: COUNTING
⇒ PICKING AT
RANDOM



Each step: O(n), O(log n) steps -> O(n log n)

Faster?

SAMPLE THE VERTICES



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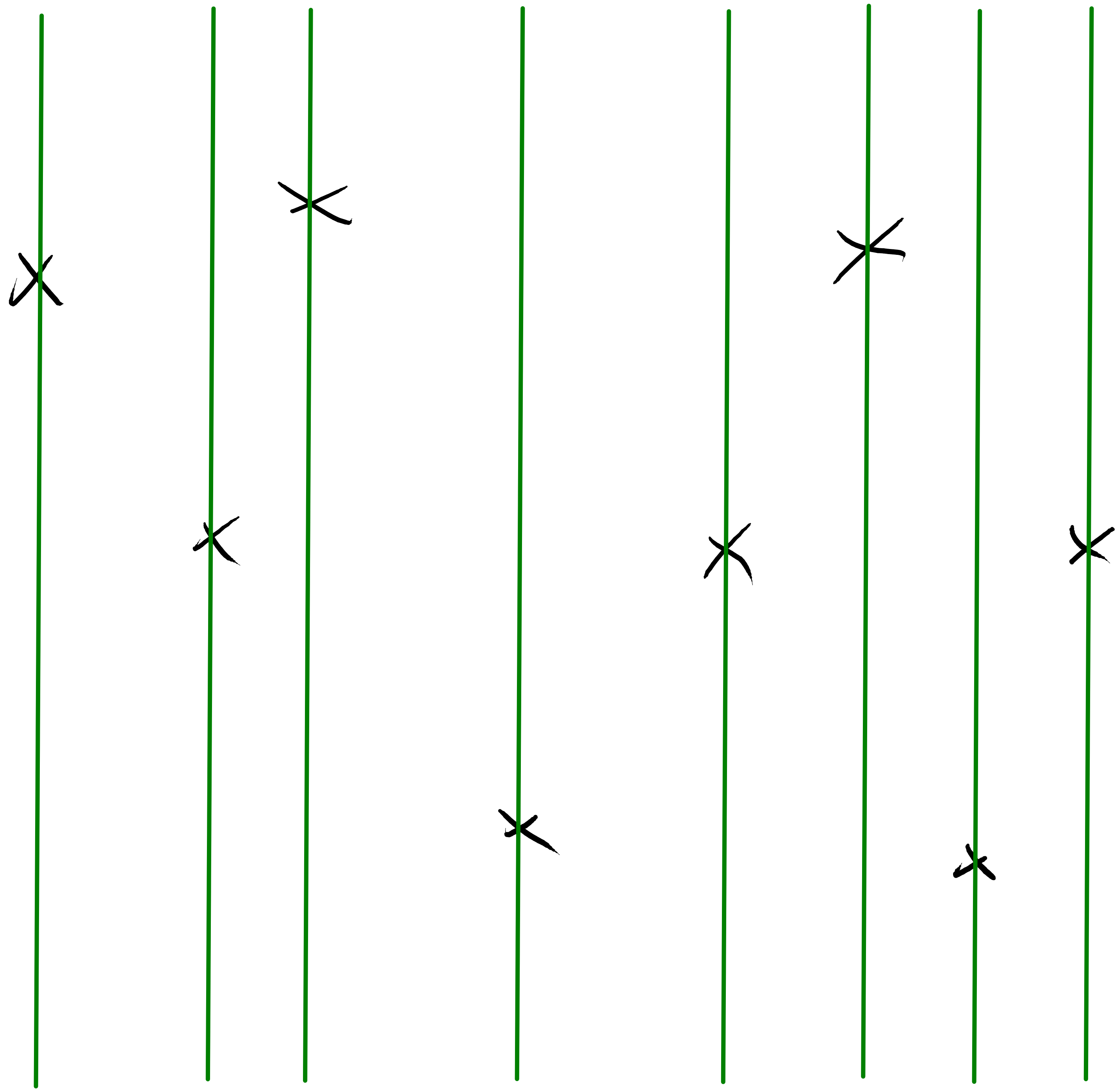

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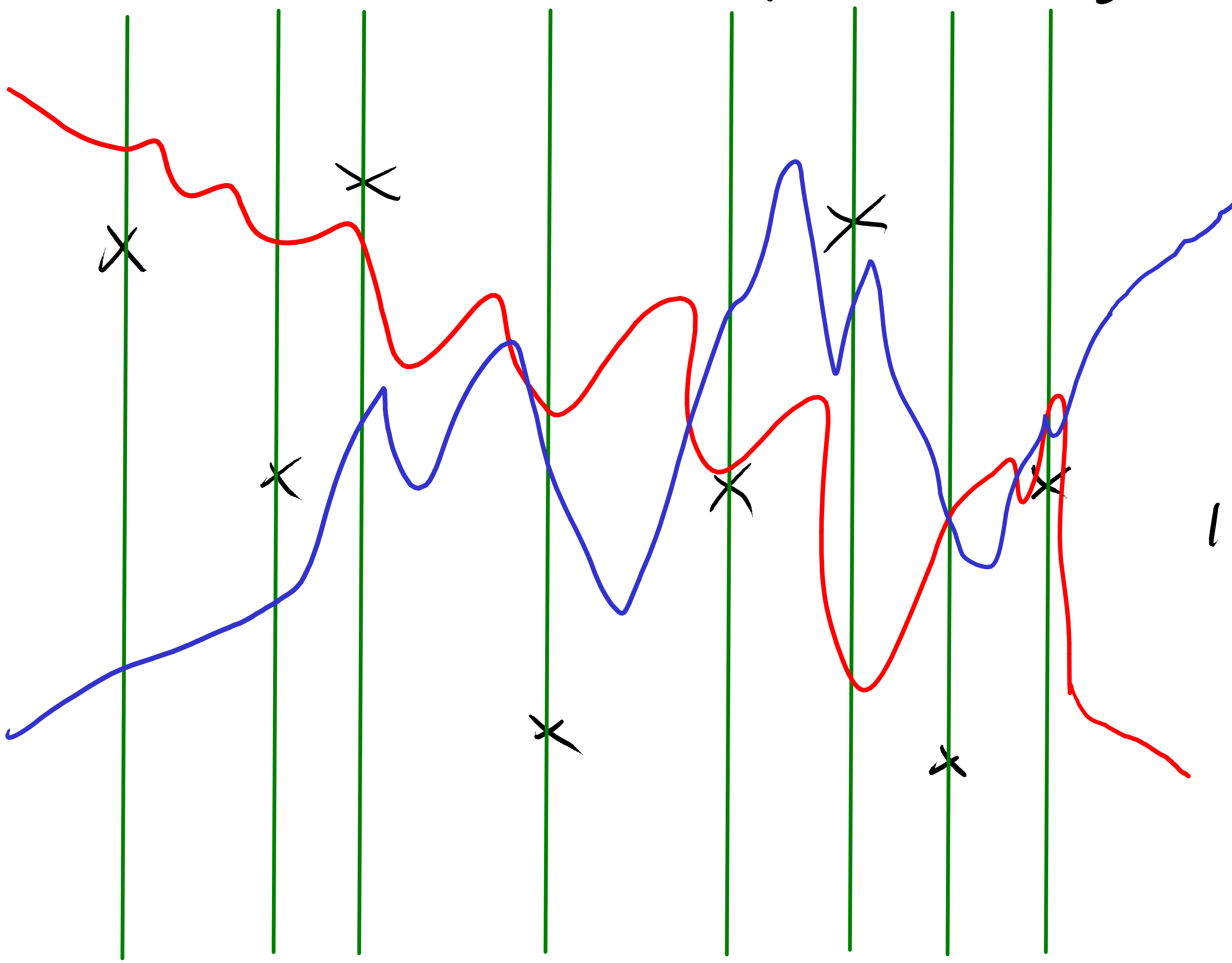
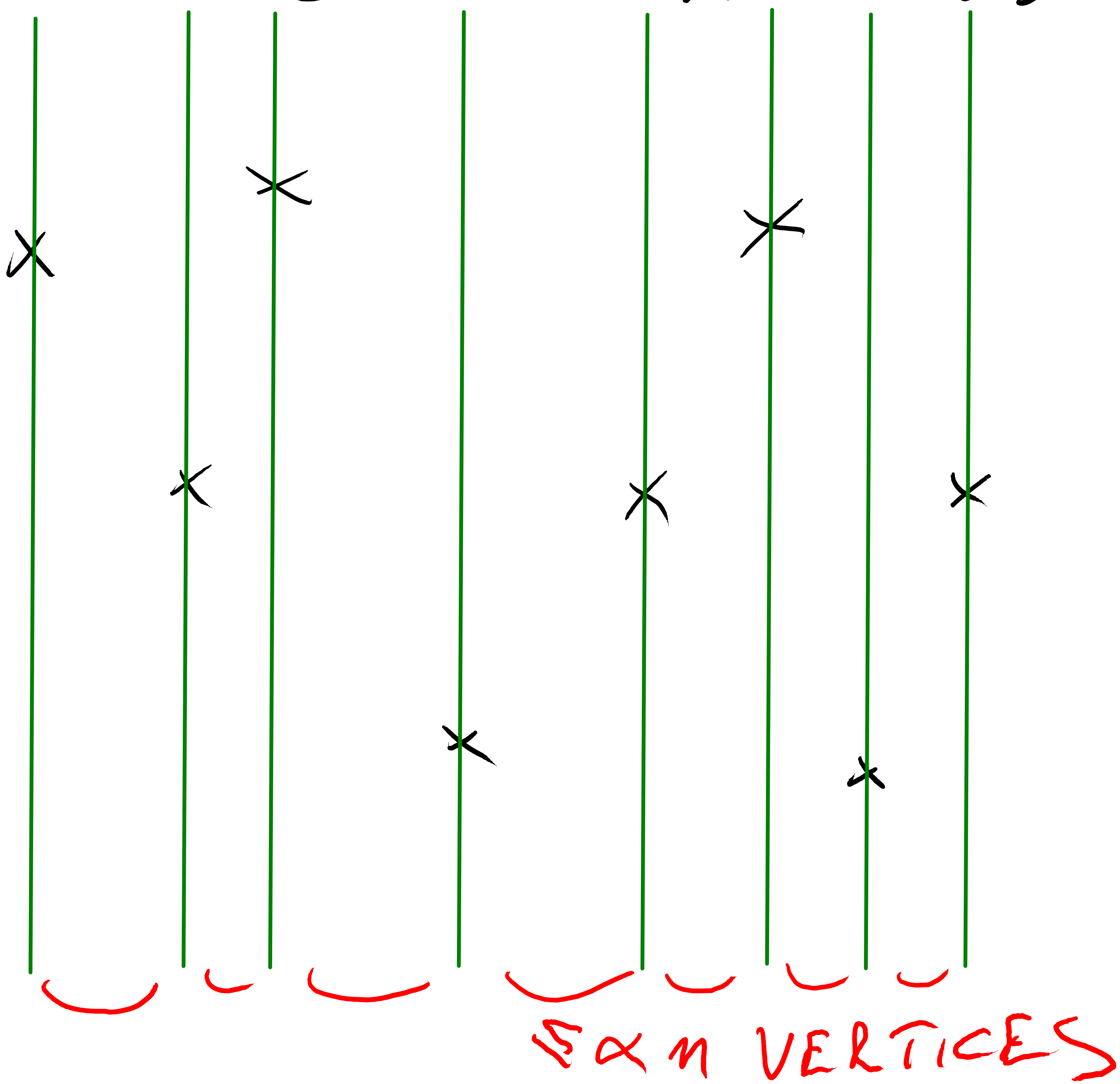
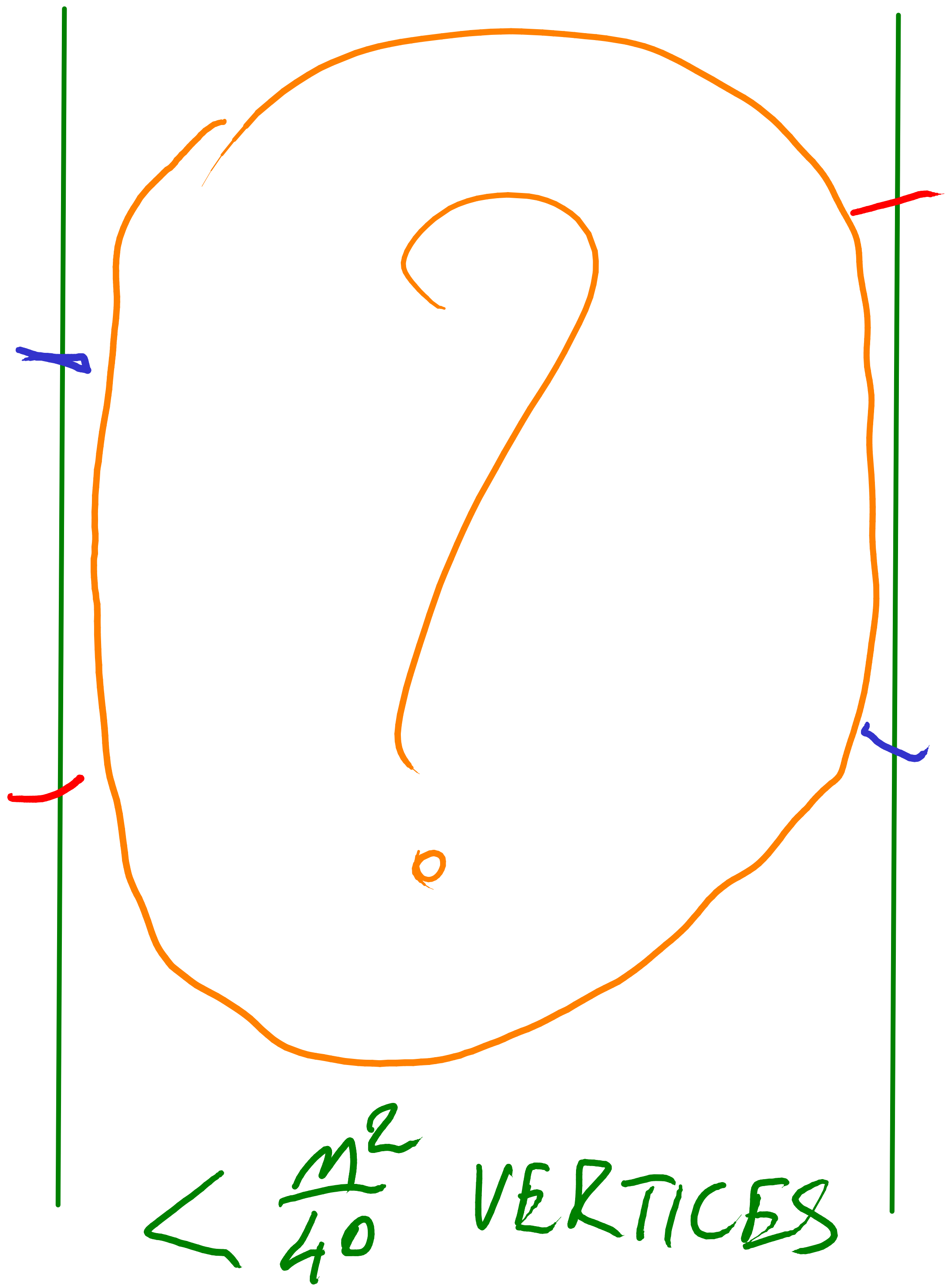


≥ 1 SLAB CONTAINS AN ODD # OF INTERSECTIONS

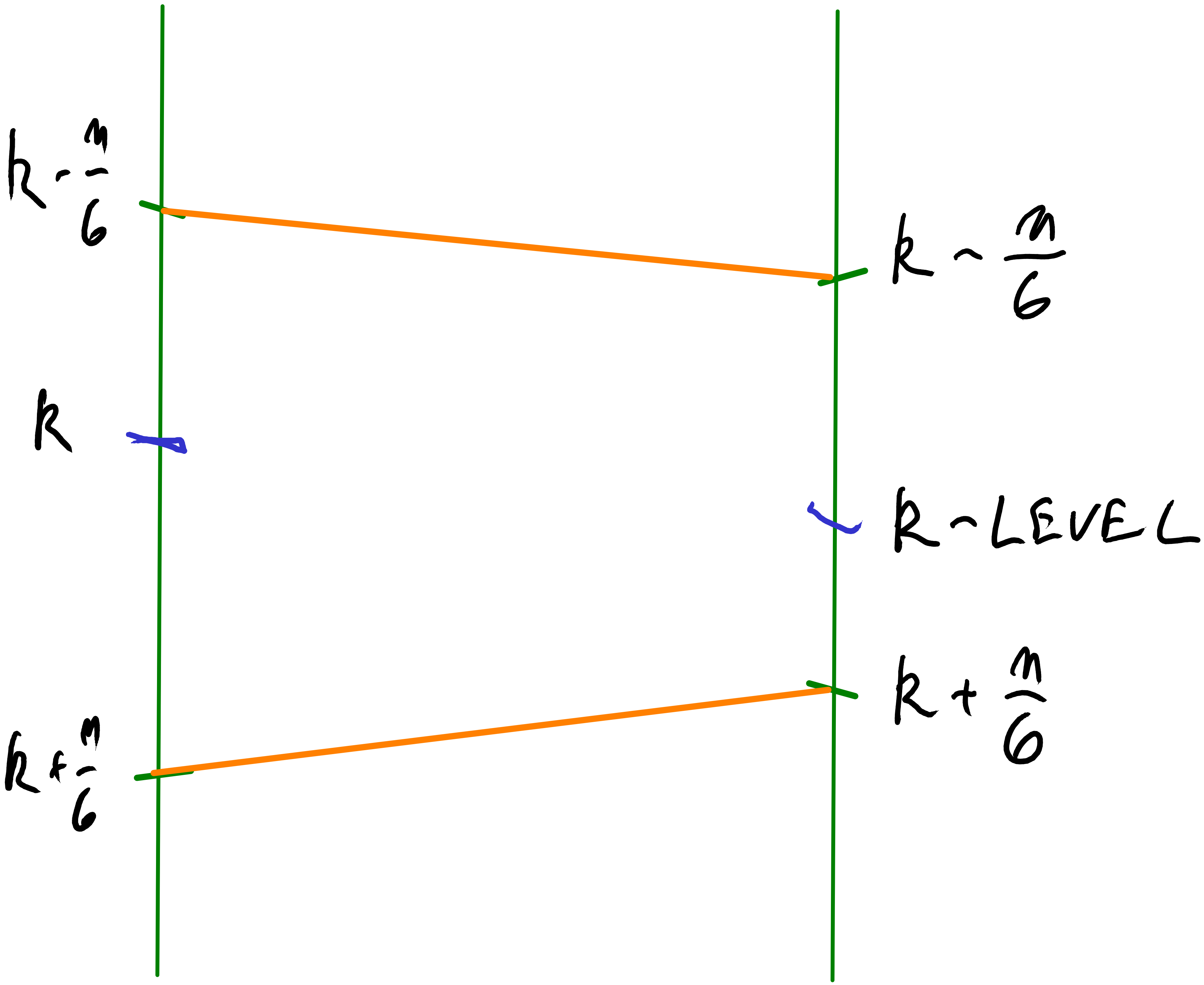
SAMPLE THE VERTICES



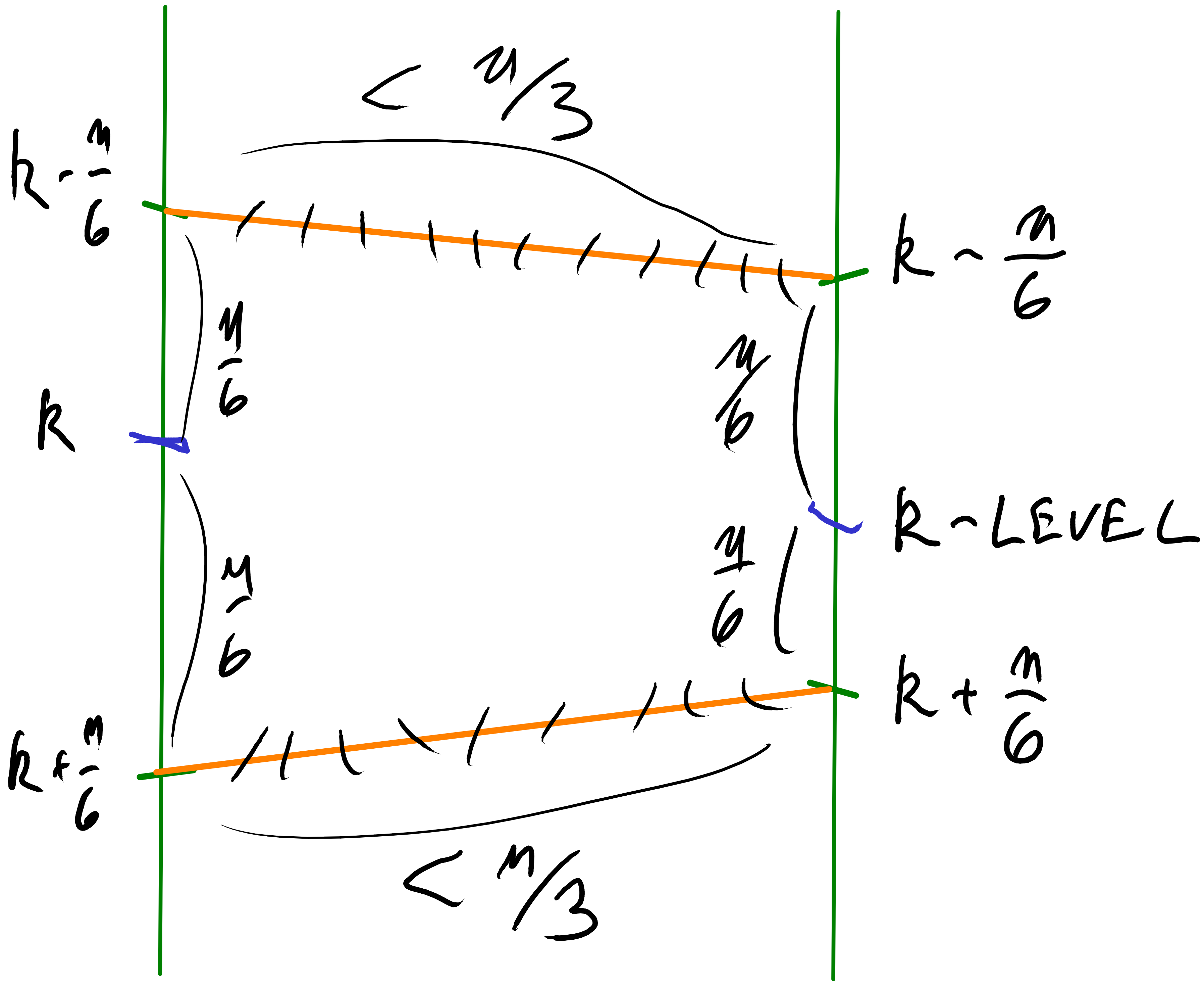
GOOD SLAB



GOOD SLAB



GOOD SLAB



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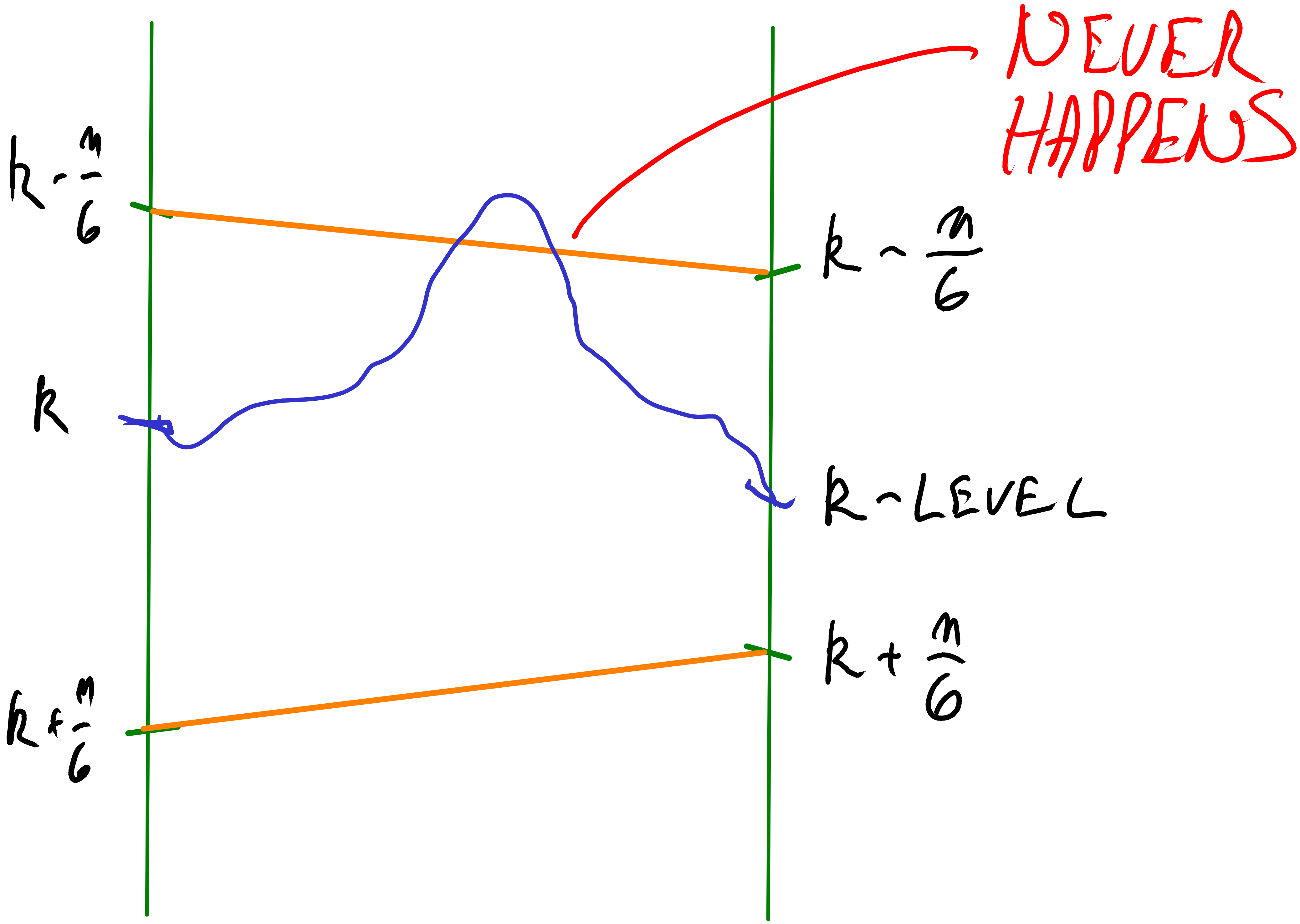
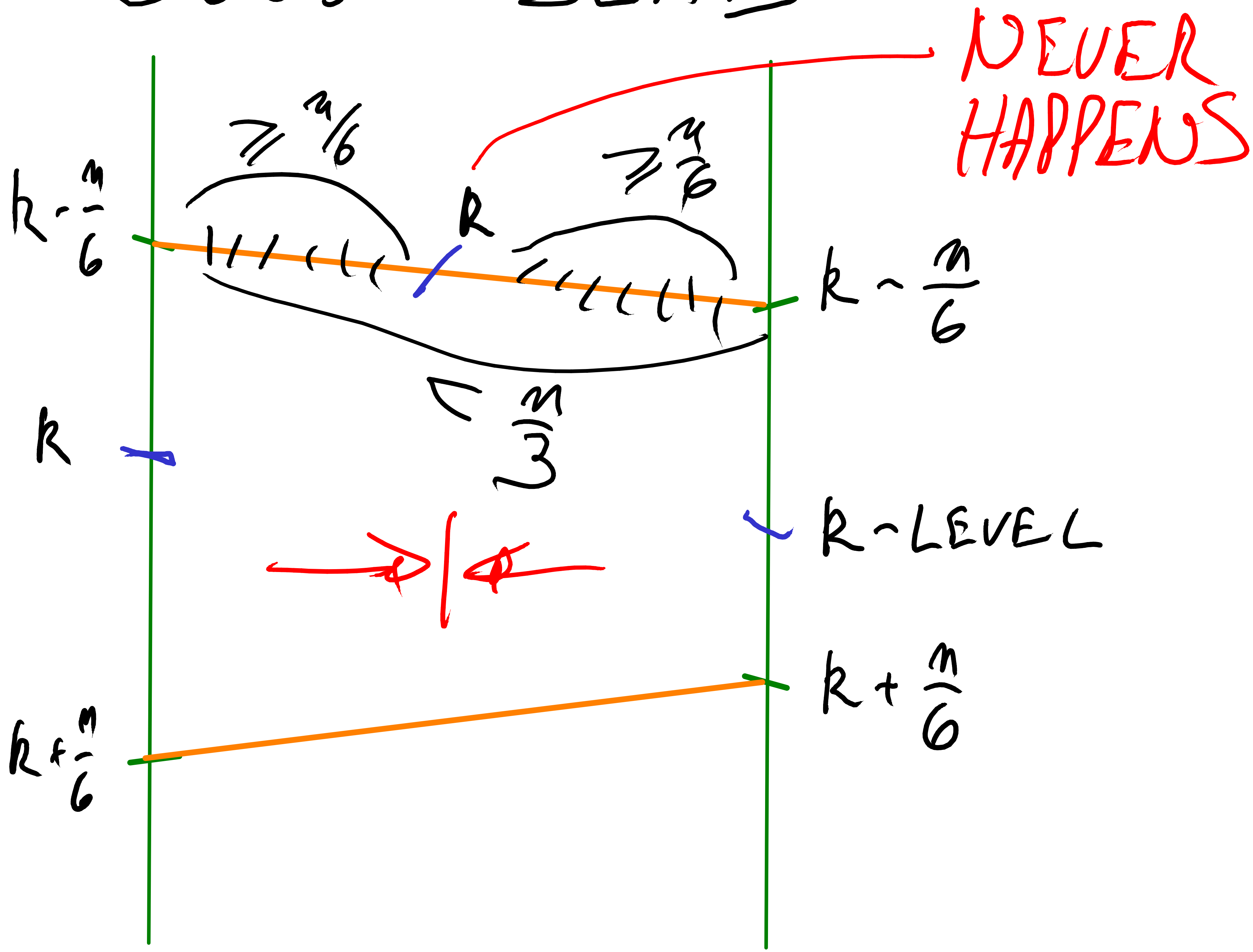
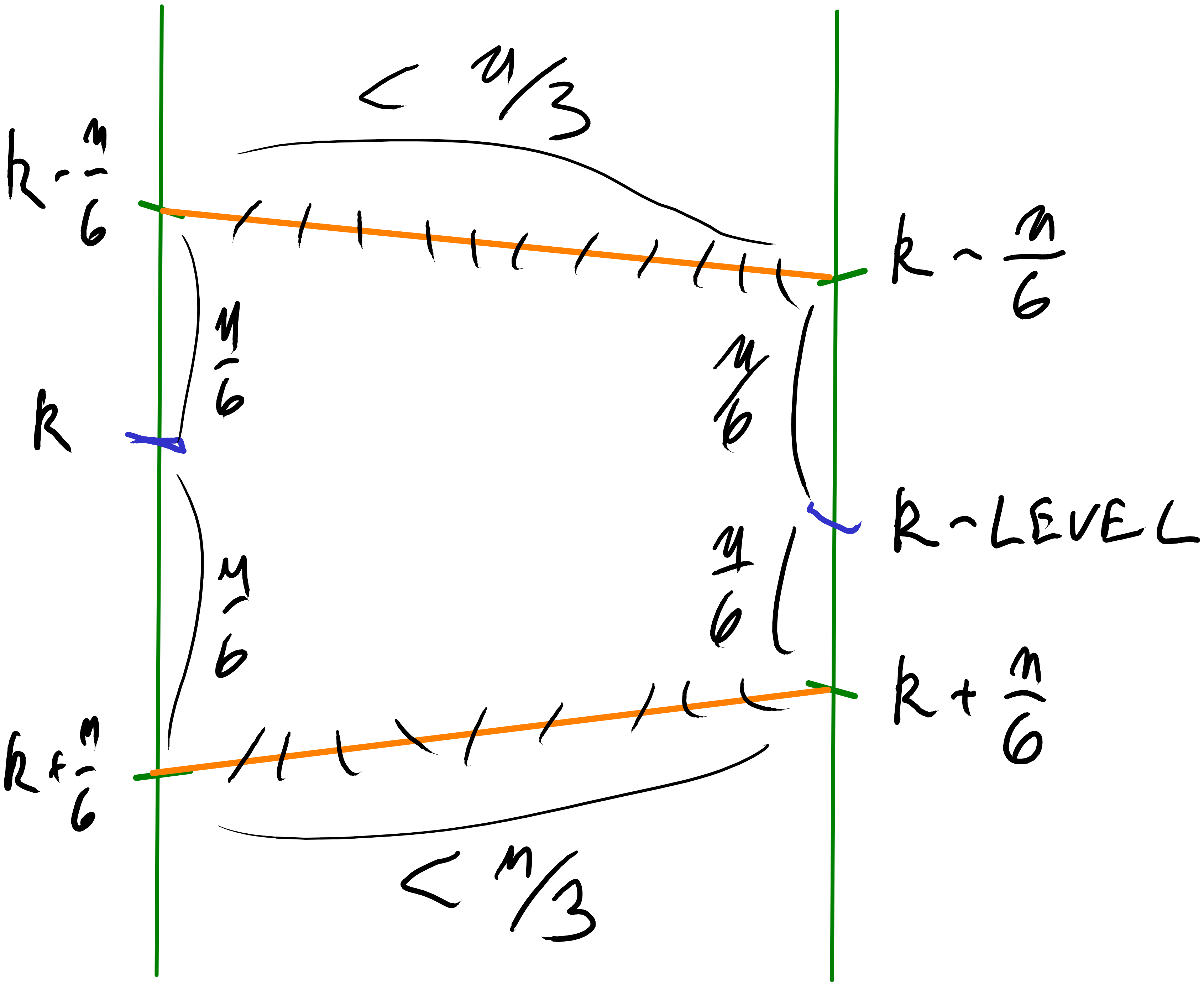


GOOD SLAB

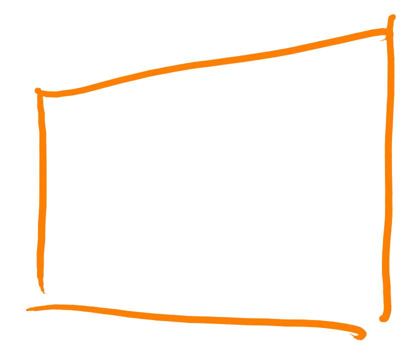


NEVER HAPPENS
$k - \frac{n}{6}$
$\geq n/6$
R
$\geq \frac{n}{6}$
$k - \frac{n}{6}$
$< \frac{n}{3}$
k
k-LEVEL
$k + \frac{n}{6}$
$k + \frac{n}{6}$

GOOD SLAB

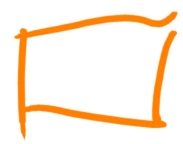


THROW AWAY LINES THAT DON'T TOUCH



$\Rightarrow \geq \frac{n}{3}$

Summary

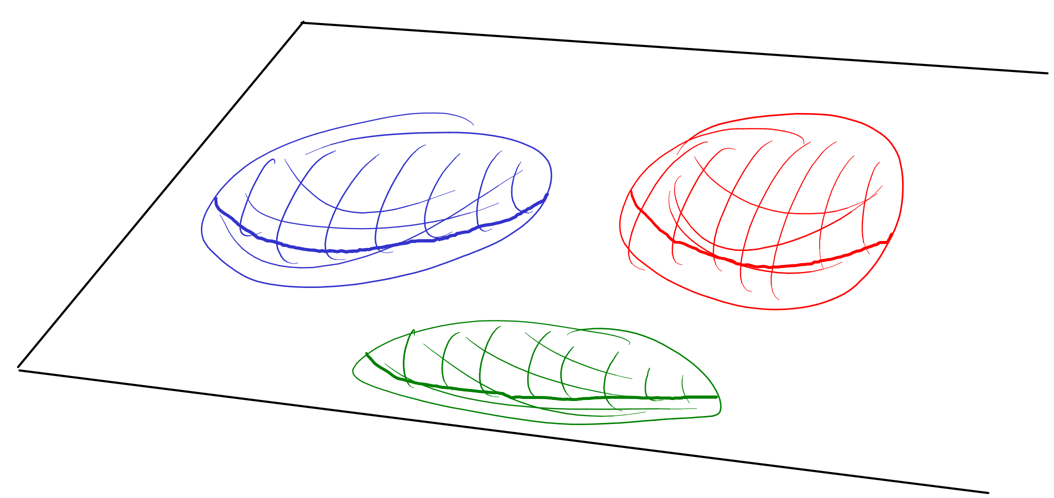
1. Sample vertices $O(n)$
2. Find odd slab $O(n)$
3. Throw away lines outside of  $O(n)$
4. Recurse

$$T(n) \leq O(n) + T\left(\frac{2n}{3}\right)$$

HS History (R^3)

- Posed by Steinhaus, problem 123 in "The Scottish Book":

  "Is it always possible to bisect three solids, arbitrarily located, with the aid of an appropriate plane?"



The Scottish Book

The enclosed collection of mathematical problems has its origin in a notebook which was started in Lwow, in Poland in 1935. If I remember correctly, it was S Banach who suggested keeping track of some of the problems occupying the group of mathematicians there The mathematical life was very intense in Lwow Some of us met practically every day, informally in small groups, at all times of the day to discuss problems of common interest, communicating to each other the latest work and results. Apart from the more official meetings of the local sections of the Mathematical Society (which took place Saturday evenings, almost every week!), there were frequent informal discussions mostly held in one of the coffee houses located near the University building - one of them a coffee house named "Roma" and the other "The Scottish Coffee House". This explains the name of the collection. A large notebook was purchased by Banach and deposited with the headwaiter of the Scottish Coffee House, who, upon demand, would bring it out of some secure hiding place, leave it at the table, and after the guests departed, return it to its secret location. [...]

S. Ulam, 1958

Scottish Coffee House

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- Attributed to Ulam by [Stone and Tukey 1942]

HS History ($\mathbb{R}^3$)

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- Attributed to Ulam by [Stone and Tukey 1942]
- Proved by Banach, published in a note in [Mathesis Polska 1938] (in polish, translated in 2004)

A Note on the Ham Sandwich Theorem
Hugo Steinhaus and others
From *Mathesis Polska* **XI**, (1938), pp. 26–28.

NOTES
From Topology

HS History (R^3)

- Posed by Steinhaus, problem 123 in "The Scottish Book"
- Proved by Banach, published in a note in [Mathesis Polska 1938] (in polish, translated in 2004)
- Generalized to R^d by [Stone and Tukey 1942]

HAM SANDWICH FOR MEASURES

GIVEN d NICE MEASURES $\mu_1, \mu_2, \ldots, \mu_d$ IN $\mathbb{R}^d$, THERE EXISTS A HYPERPLANE H SO THAT

$$\mu_i(H^+) = \tfrac{1}{2}\mu_i(\mathbb{R}^d) \qquad i = 1, 2, \ldots, d$$

Borsuk-Ulam Theorem

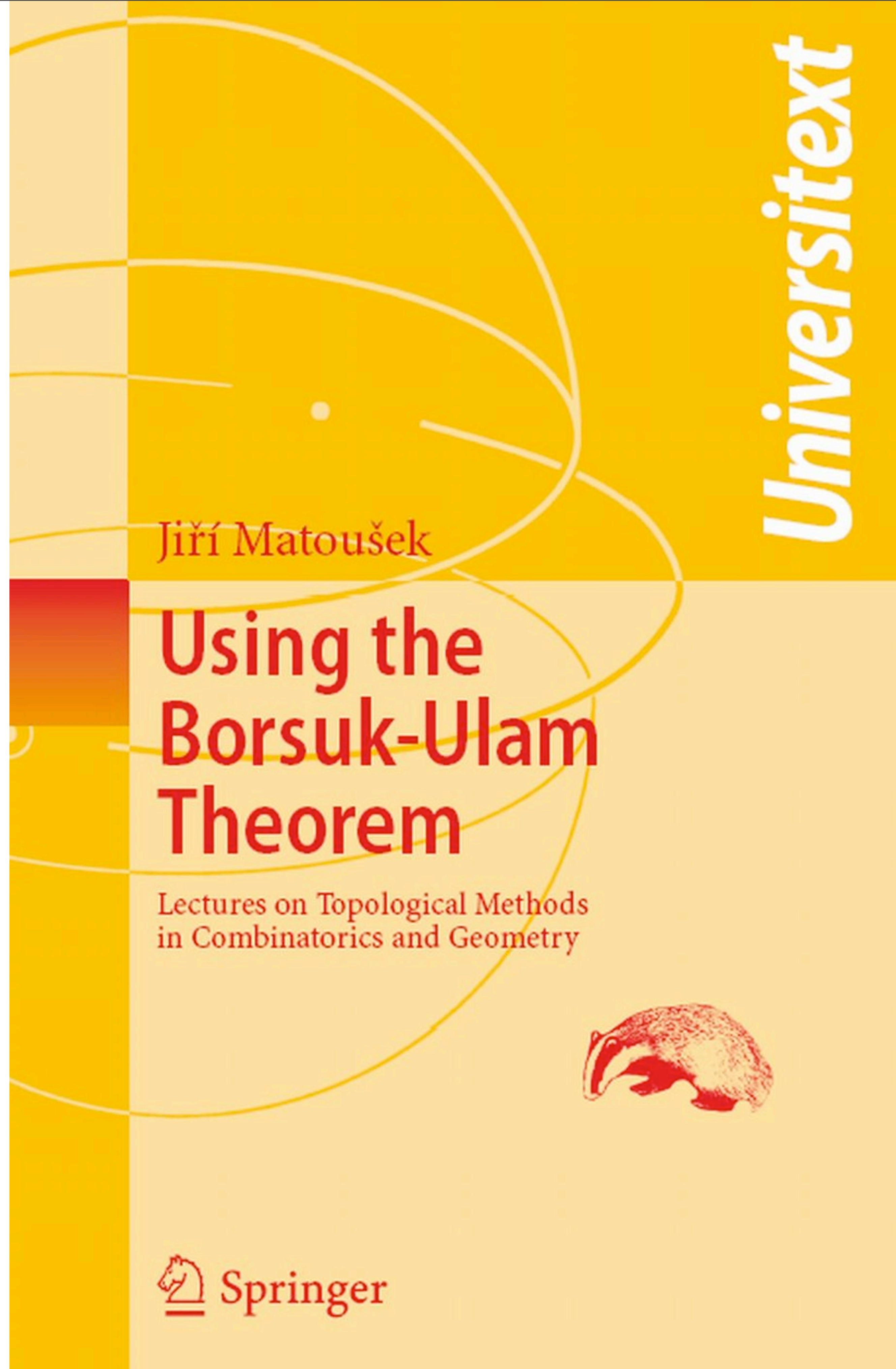
For every continuous mapping $f : S^n \to R^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$.

Conjectured by Ulam, Proof by Borsuk, 1933

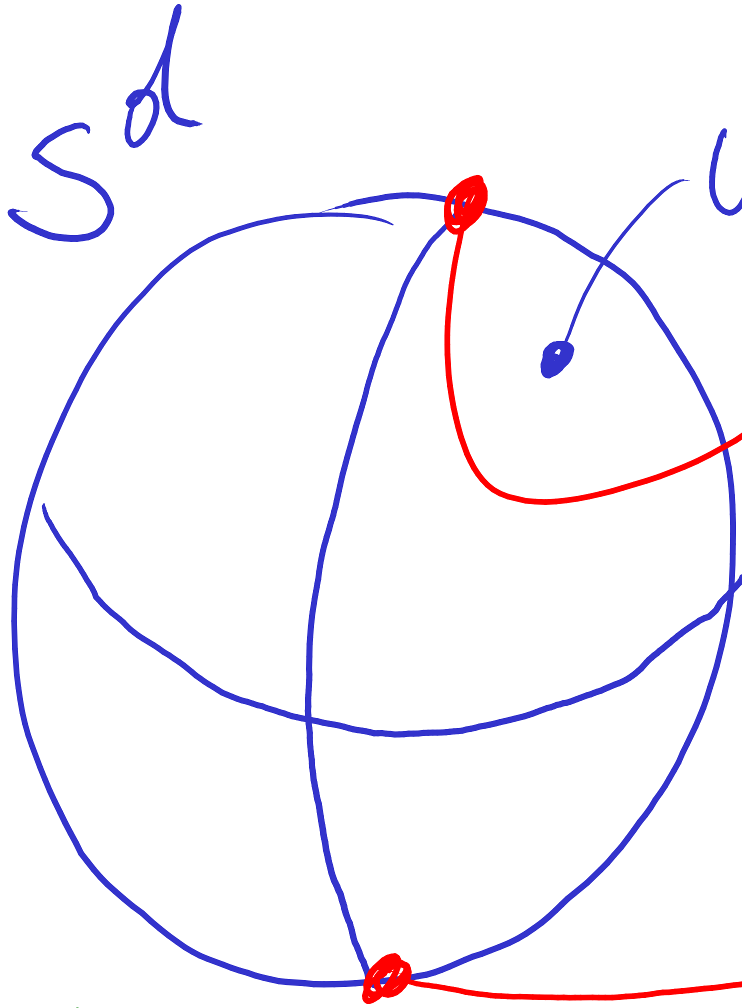
Borsuk-Ulam Theorem

For every continous mapping $f : S^n \to R^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$.

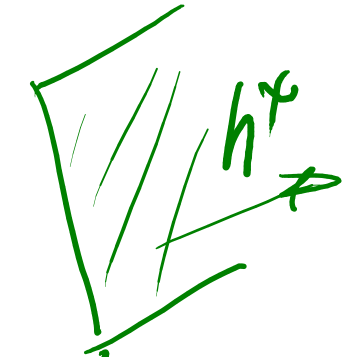
For every continous mapping $f : S^n \to R^n$, antipodal ($f(x)=-f(-x)$) there exists a point $x \in S^n$ such that $f(x) = 0$.


Universitext
Jiří Matoušek
Using the Borsuk-Ulam Theorem
Lectures on Topological Methods in Combinatorics and Geometry
Springer

PROOF OF HAM-SANDWICH THM



$v = (v_0, v_1, \dots, v_d) \rightarrow$



$$h^+(1,0,0,\dots,0) = \mathbb{R}^d$$

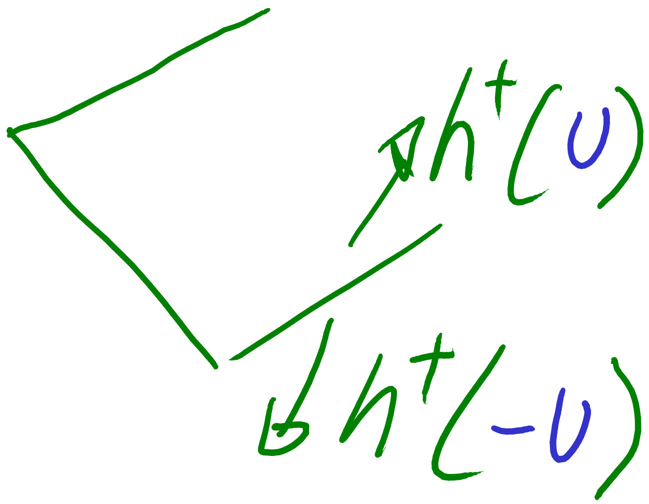
$$h^+(-1,0,0,\dots,0) = \emptyset$$

$$h^+(v) = \{(x_1,\dots,x_d) \in \mathbb{R}^d : v_1 x_1 + \dots + v_d x_d \le v_0\}$$

$$h^+(-v) = \{(x_1,\dots,x_d) \in \mathbb{R}^d : v_1 x_1 + \dots + v_d x_d \ge v_0\}$$

$$f_i(u) = \mu_i(h^+(u))$$

IF $f_i(u) = f_i(-u)$



$\Rightarrow h^+(u)$ BISECTS μ_i

$$f(u) = (f_1(u), f_2(u), \ldots, f_d(u))$$

$$f : S^d \longrightarrow \mathbb{R}^d$$

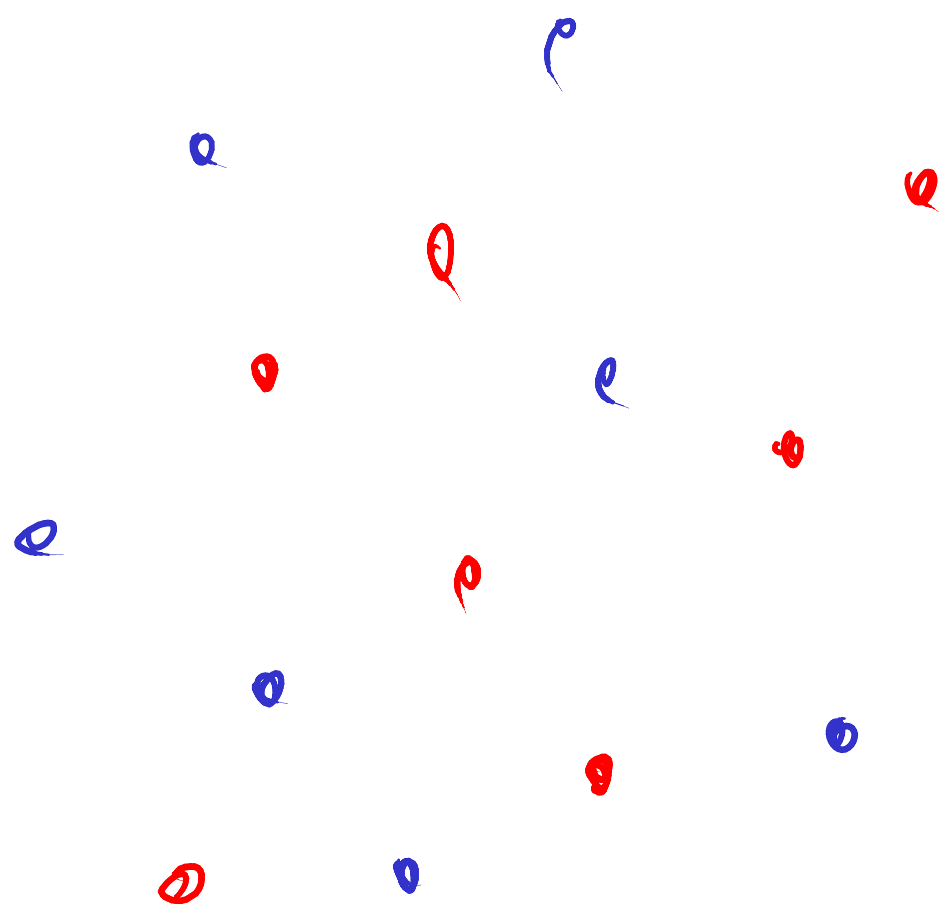
BORSOK-ULAM $\Rightarrow$

$\exists u : f(u) = f(-u) \equiv$ HAM-SANDWICH CUT

HAM SANDWICH FOR POINT SETS

$A_1, A_2, \ldots, A_d \subseteq \mathbb{R}^d$

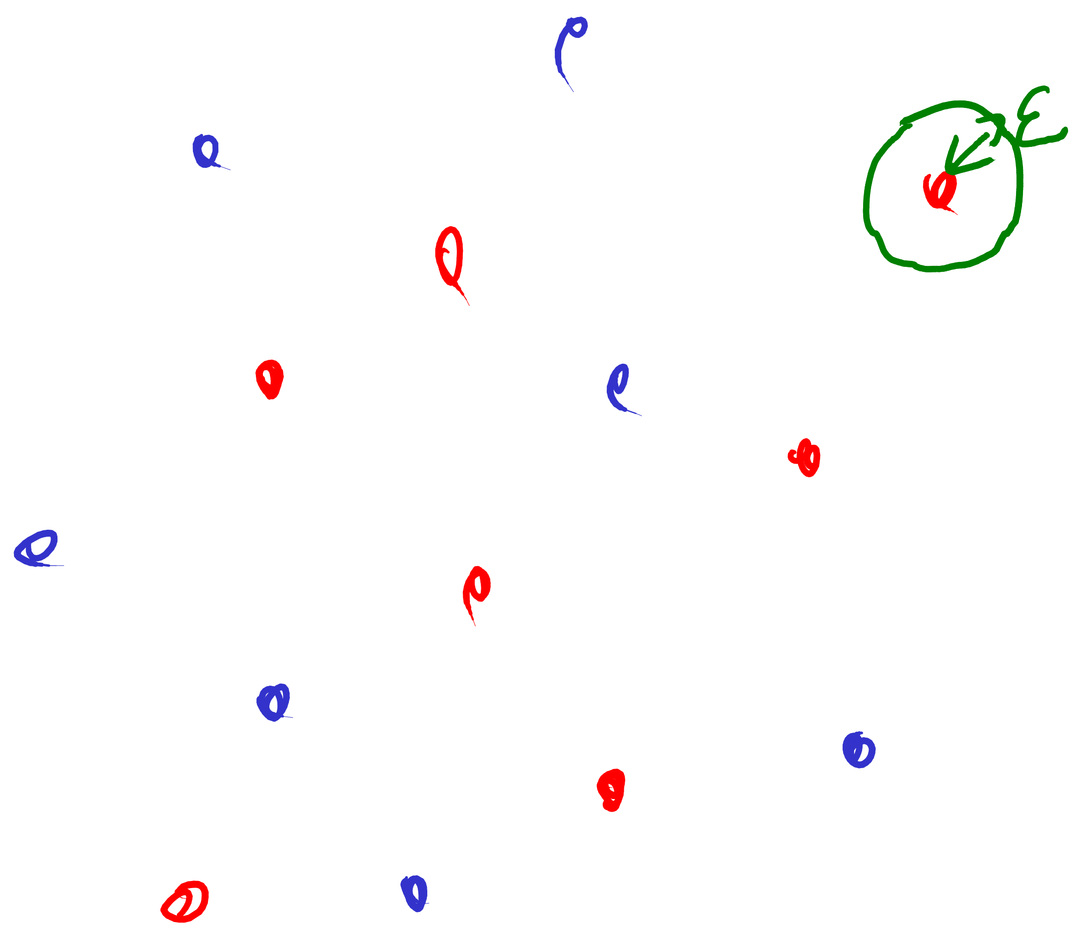
- BISECT $A_i \equiv \leq \left\lfloor \frac{|A_i|}{2} \right\rfloor$ (OPEN HALFSPACE)



- $|A_i|$ ODD
- GENERAL POSITION = NO $d+1$ POINTS ON A HYPERPLANE

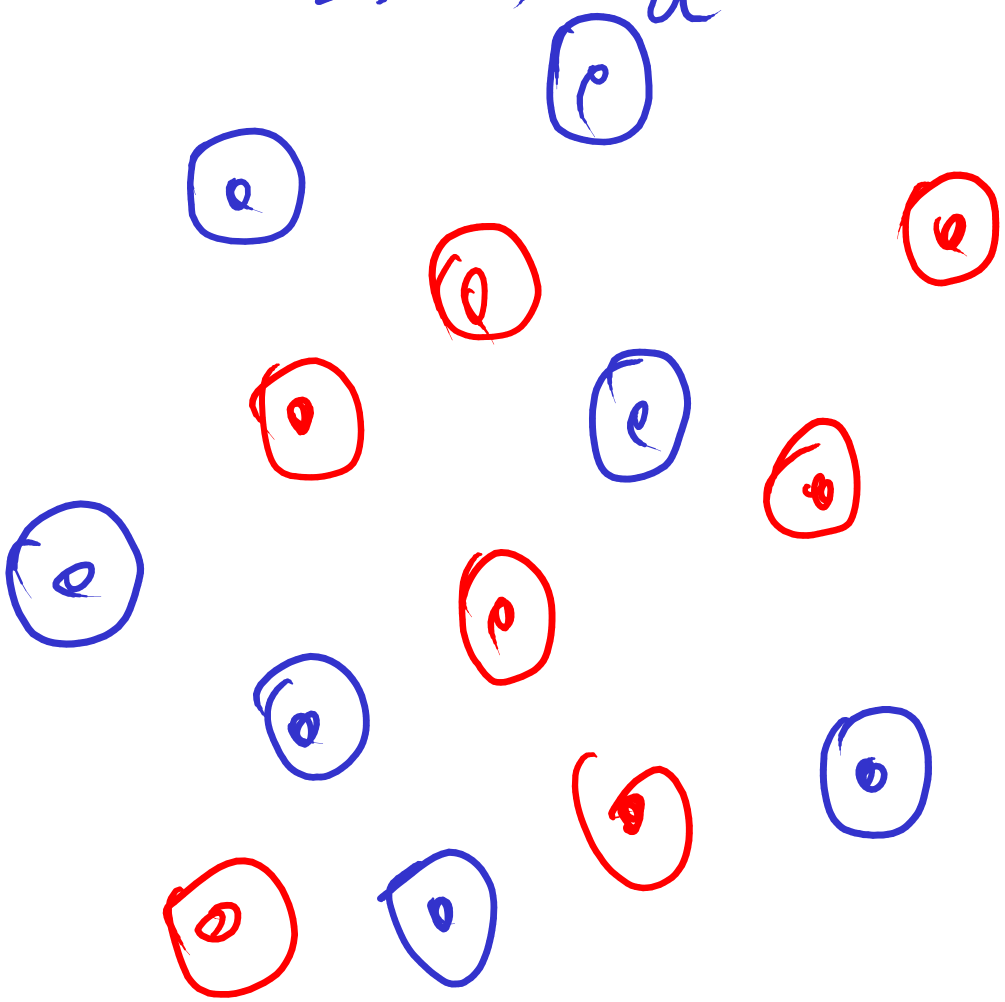
HAM SANDWICH FOR POINT SETS

$A_1, A_2, \ldots, A_d \subseteq \mathbb{R}^d$



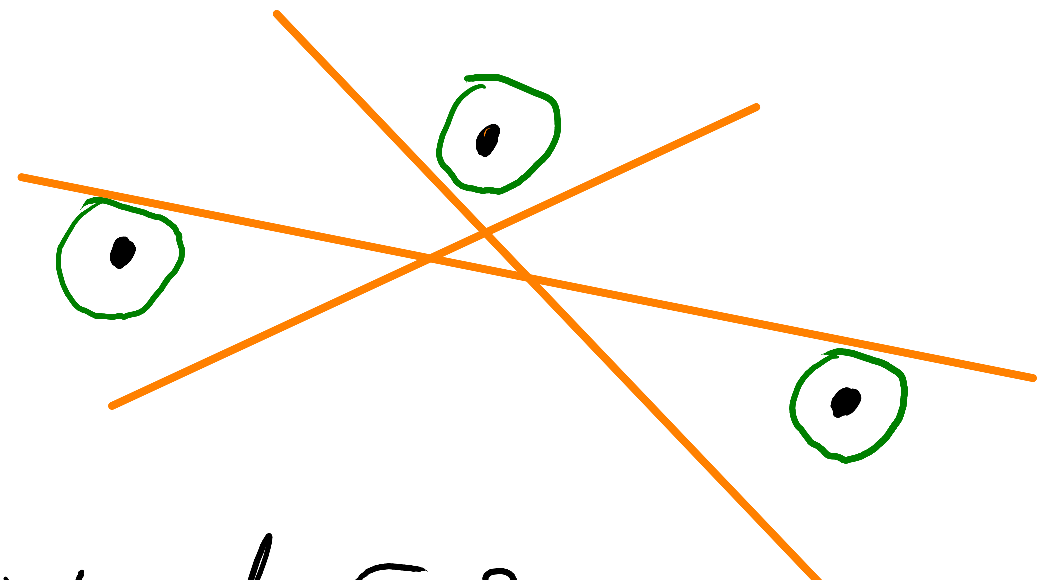
HAM SANDWICH FOR POINT SETS

$A_1, A_2, \ldots, A_d \subseteq \mathbb{R}^d$



ε SMALL

SO THAT

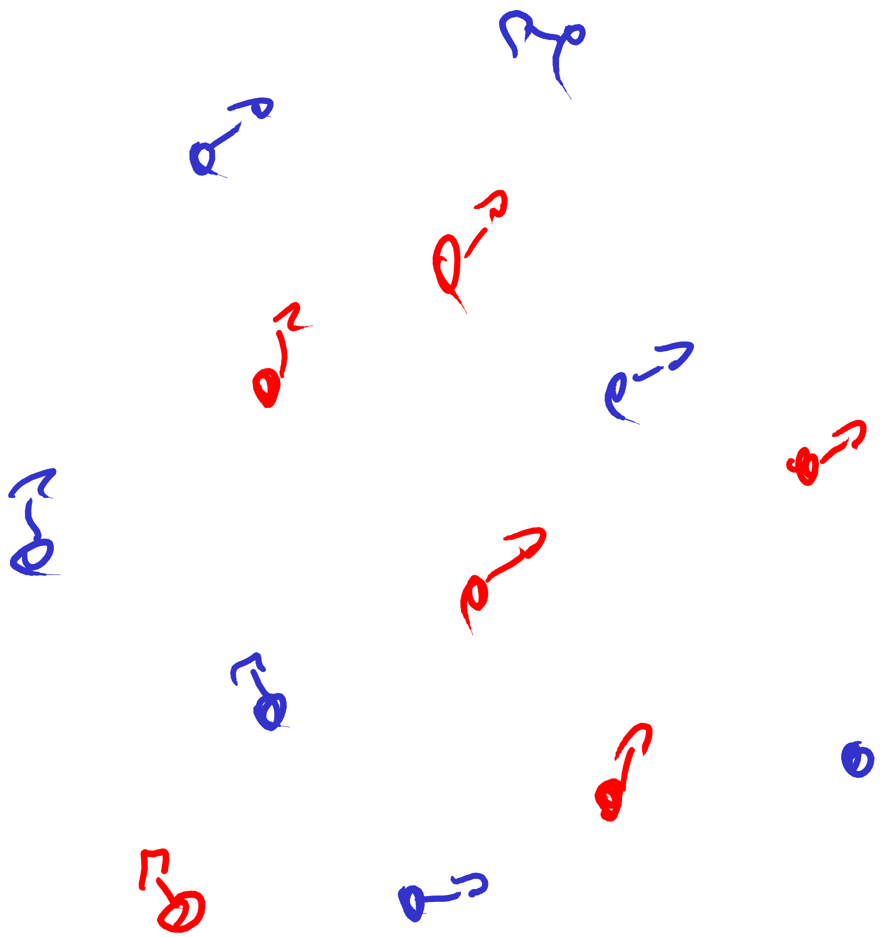


ALL d-TUPLES ARE WELL SEPARATED

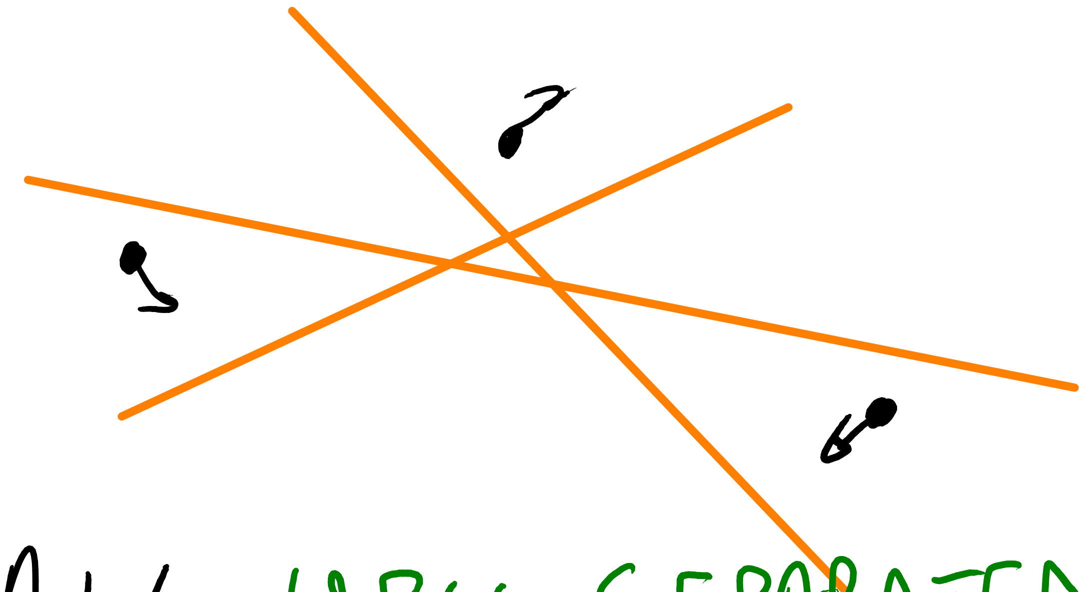
THEN TAKE LIMIT $\varepsilon \to 0$

DEGENERATE POINT SETS

$A_1, A_2, \ldots, A_d \subseteq \mathbb{R}^d$



PERTURB BY SMALL AMOUNT



ALL WELL SEPARATED d-TUPLES STAY WELL SEPARATED

THEN SNAP BACK TO ORIGINAL POSITION

R^3 and up

- [Lo Matousek Steiger 1994]

  Same as median level construction in R^{d-1}

  $O(n^{4/3} \log n)$ in R^3

  $O(n^{8/3+e})$ in R^4

  $O(n^{d-1-a(d)})$ in R^d

Well separated

- $S_1, S_2, \ldots, S_d$ are well separated iff any subset of the S_i can be separated from the others by a hyperplane
- $S_1, S_2, \ldots, S_d$ are well separated iff the affine hull containing one point in each set is a (d-1)-flat

Well separated

- $S_1, S_2, \ldots, S_d$ are well separated iff any subset of the S_i can be separated from the others by a hyperplane

- $S_1, S_2, \ldots, S_d$ are well separated then for any $(a_1, \ldots, a_d)$, $0 \leq a_d \leq |S_d|$, there is a hyperplane with a_i points of S_i for all i
  [Barany, Hubard, Jeronimo 2008]
  [Steiger&Zhao 2009]

Well separated

Well separated

- $O(n)$ in R^3 [Lo, Matousek, Steiger 1994]

Well separated

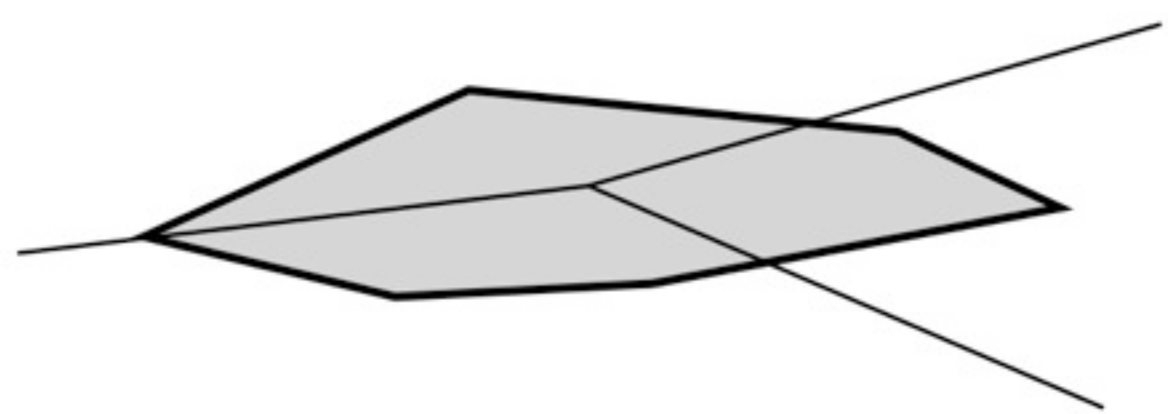
- $O(n)$ in R^3 [Lo, Matousek, Steiger 1994]
- $O(n \log^{d-3} n)$ in R^d [Steiger&Zhao 2009]

Partitions into convex sets

- Given qn red points qm blue points, are there q disjoint convex polygons that each contain n red and m blue points? [Kaneko & Kano 1999]

Cake cutting

- Partition the surface and the perimeter of a polygon into 3 equitable pieces. [Akiyama, Kaneko, Kano, Nakamura, Rivera-Campo, Tokunaga, Urrutia 2000]



Partitions by 3-fans

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- 3-fans for any 2 point sets [Bespamiatnikh, Kirkpatrick, Snoeyink 2000] [Ito, Uehara, Yokoyama 2000] [Sakai 2002].

Partitions by 3-fans

- 3-fans for any 2 point sets [Bespamiatnikh, Kirkpatrick, Snoeyink 2000] [Ito, Uehara, Yokoyama 2000] [Sakai 2002].
- constrained 3-fans [Bespamiatnikh, Kirkpatrick 2003]

Partitions by k-fans

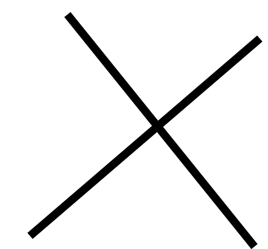
- [Barany & Matousek 2001]: $(a_1, a_2, \dots, a_k)$

	2 meas.	3 meas	4 meas
2-fan	always	(1/2,1/2) (2/3,1/3)	no
3-fan	(1/2,1/4,1/4)	no	no
4-fan	(2/5,1/5,1/5,1/5)	no	no
conv. 4-fan	no	no	no
5-fan	no	no	no

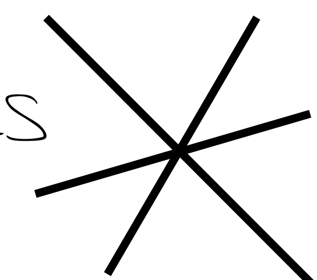
- [Barany & Matousek 2002]: 4-fan, 2 meas (1/4,1/4,1/4,1/4)
- [Bereg 2005]: 2-fans for 3 meas in $O(n \log^2 n)$.

More partitions

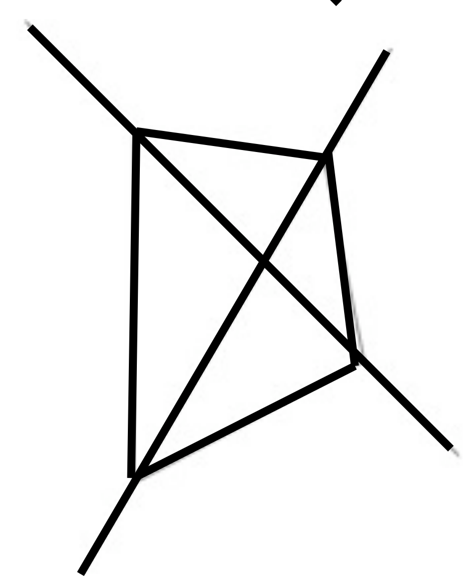
- Equitable 4-partition of n points by 2 orthogonal lines,



- Equitable 6-partition of n points by 3 lines through 1 point [Buck&Buck 1987]



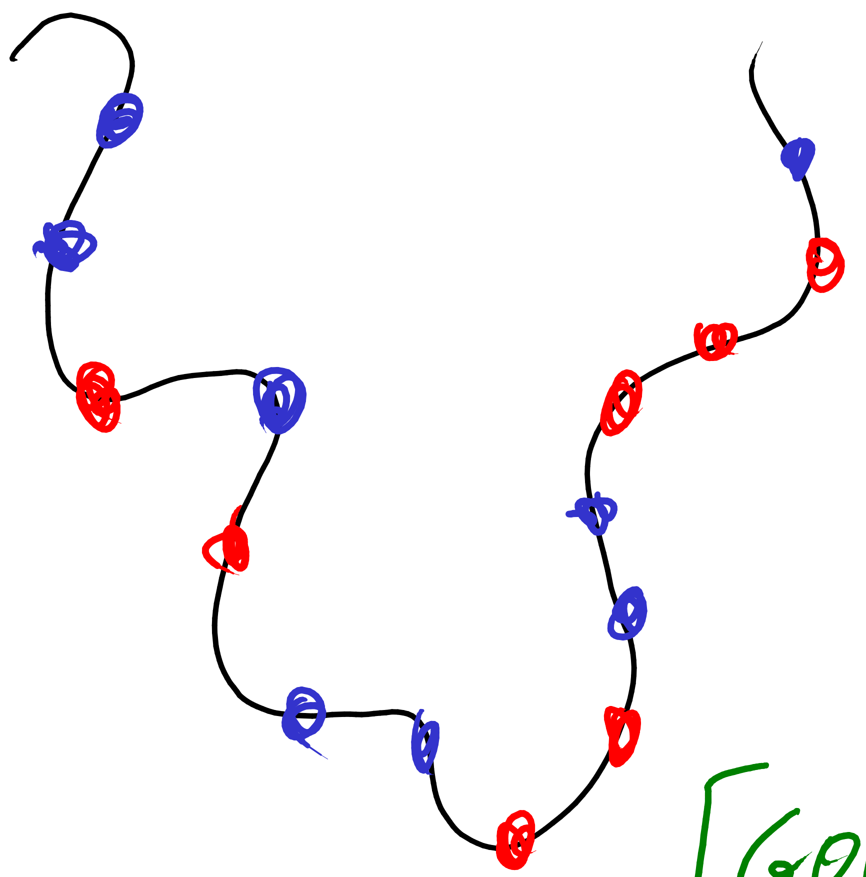
- Cobweb [Schulman 1992].
- $O(n \log n)$ algorithms [Roy & Steiger 2006]



Applications

- d-colored sets in R^d [Akiyama & Alon 1989]
- Necklace thieves

NECKLACE THIEVES



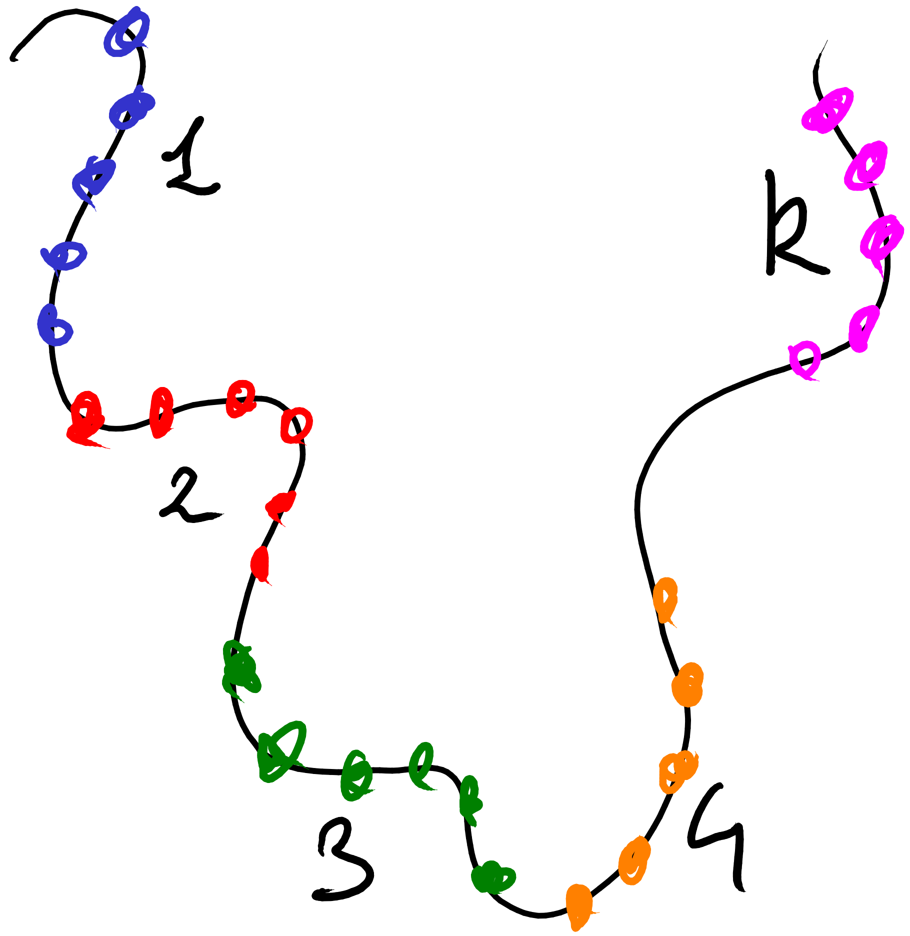
k KINDS OF BEADS

CUT INTO HOW MANY PIECES

SO BOTH THIEVES HAVE SAME # OF EACH KIND

[GOLDBERG & WEST 1985]

NECKLACE THIEVES



SOMETIMES NEED k CUTS

→ k+1 PIECES

MOMENT CURVE

$t \to (t, t^2, t^3, \dots, t^k)$

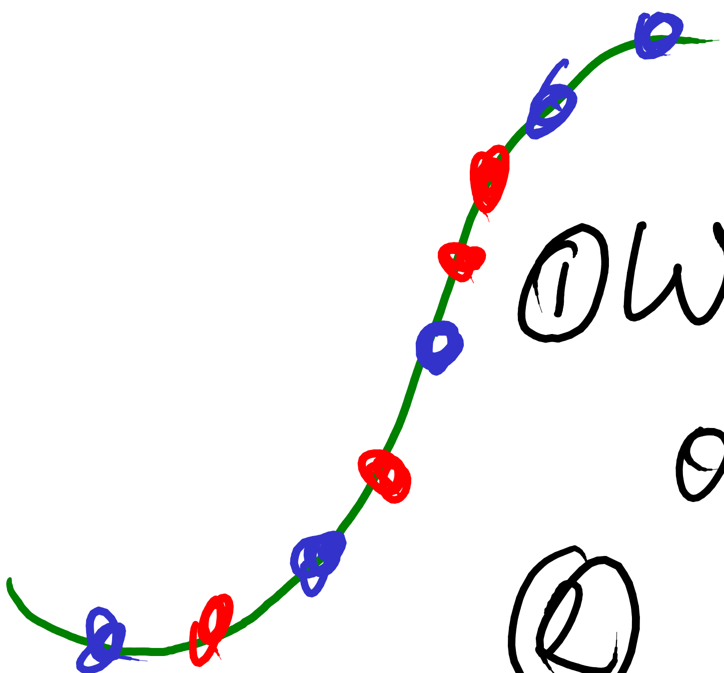
IN $\mathbb{R}^k$

ANY HYPERPLANE INTERSECTS IT

$\leq k$ TIMES

MOMENT CURVE IN $\mathbb{R}^k$

$t \to (t, t^2, t^3, \dots, t^k)$



① WRAP THE NECKLACE ON IT

② FIND HAM-SANDWICH CUT

[ALON]

Q: HOW TO DECIDE IF A HAM-SANDWICH CUT IS UNIQUE?

Q: HOW TO DECIDE IF
A HAM-SANDWICH CUT
IS UNIQUE ?

$O(n^{4/3} \log n)$

$\Omega(n \log n)$ [CHIEN & STEIGER '95]

WHS

BUT

WHAT IF I REALLY LIKE SESAME SEEDS?

BUT

WHAT IF I REALLY LIKE SESAME SEEDS?

→ PUT WEIGHTS ON THE POINTS

BUT

WHAT IF I REALLY LIKE SESAME SEEDS?

→ PUT WEIGHTS ON THE POINTS

WHAT IF I REALLY DON'T LIKE OLIVES

BUT

WHAT IF I REALLY LIKE SESAME SEEDS?

→ PUT WEIGHTS ON THE POINTS

WHAT IF I REALLY DON'T LIKE OLIVES

→ THE WEIGHTS CAN BE NEGATIVE.

RESULTS

ALGO: THE WEIGHTED HAM-SANDWICH CUT OF n POINTS IN $\mathbb{R}^2$ CAN BE COMPUTED IN $O(n \log n)$

RESULTS

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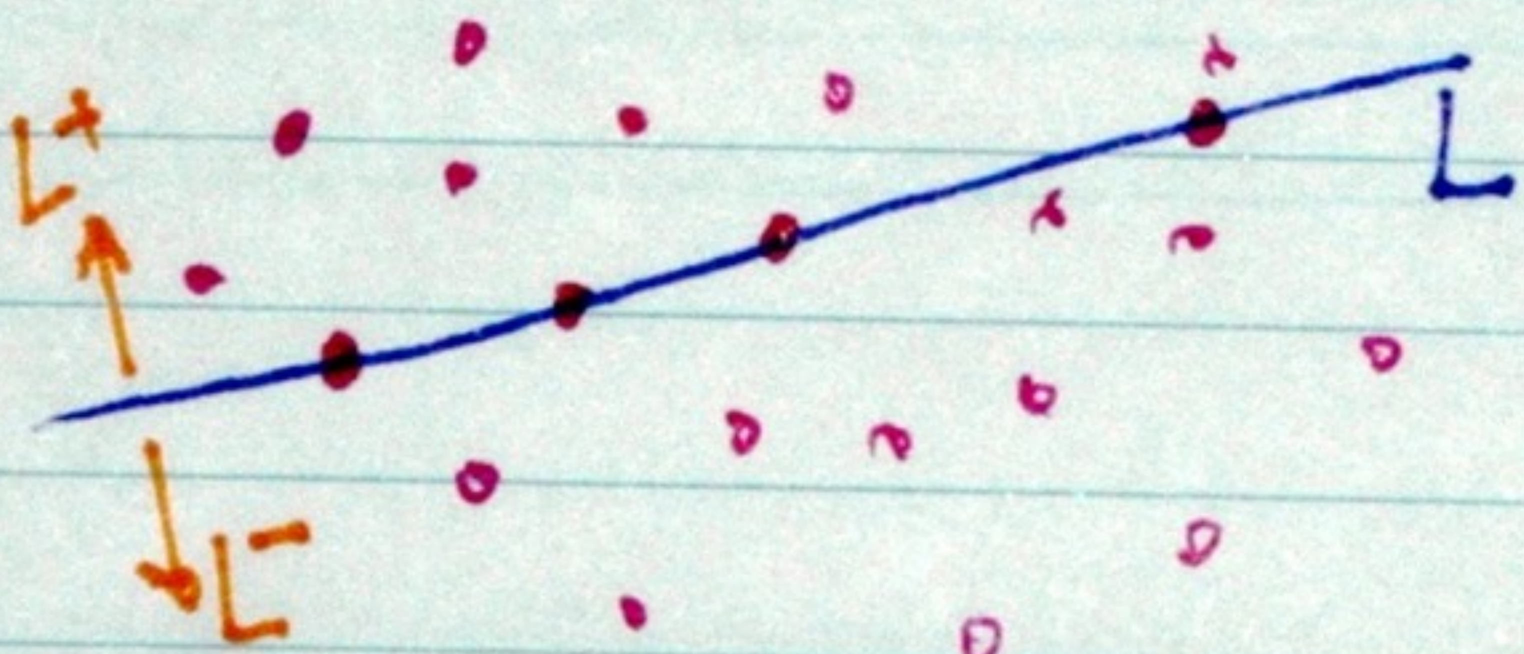
THM: DECIDING IF THE WEIGHTED HAM-SANDWICH CUT IS UNIQUE IS 3SUM-HARD

DEFINITION:

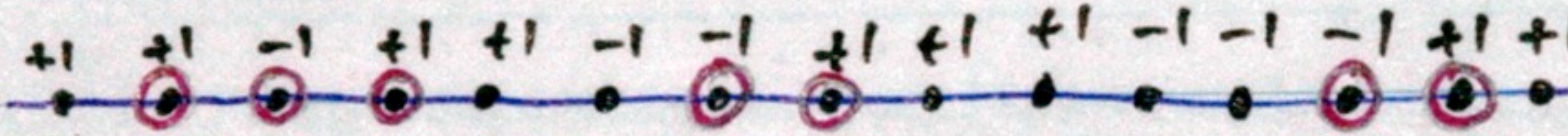
A LINE L BISECTS A WEIGHTED SET OF POINTS S IF

$$|w(L^+ \cap S) - w(L^- \cap S)| \leq |w(L \cap S)|$$



1-D

. THE WEIGHTED MEDIAN IS NOT UNIQUE

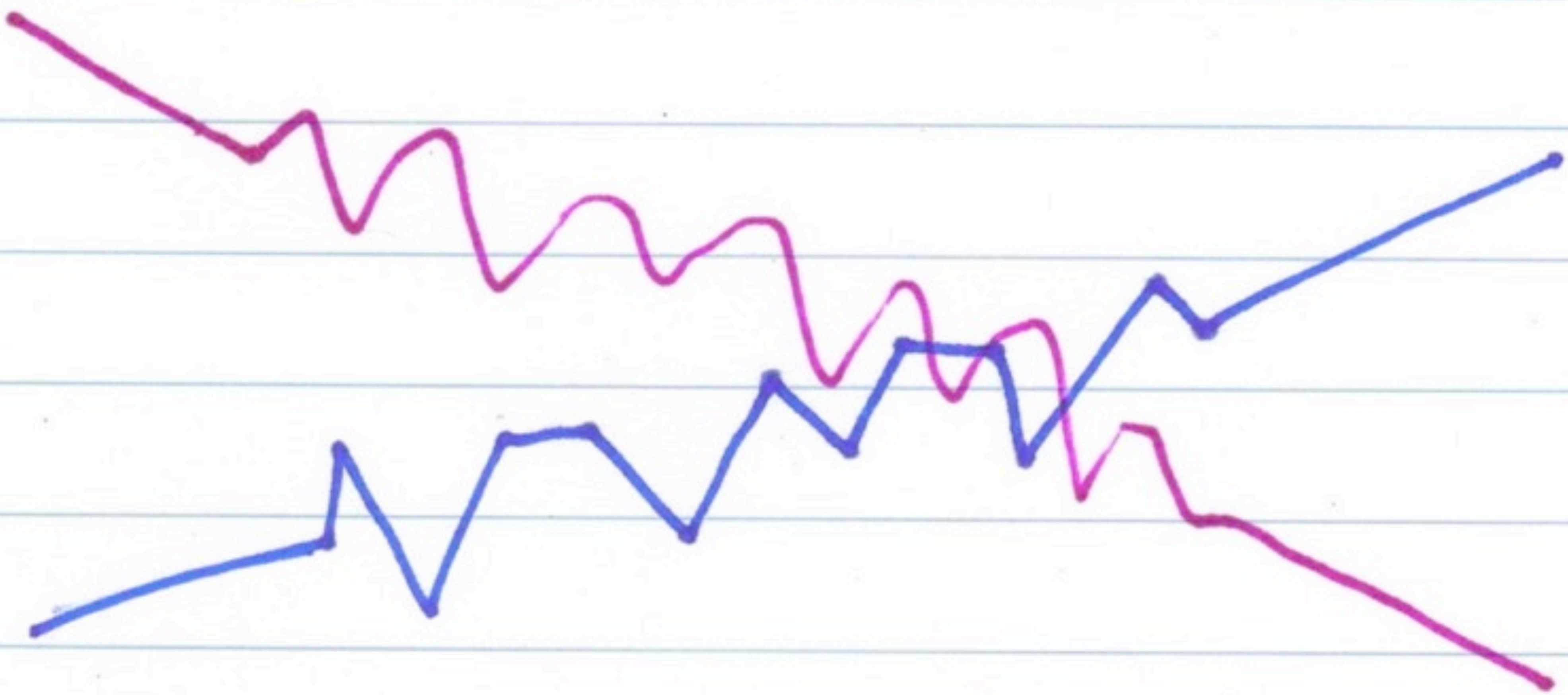


2-D: MEDIAN LEVELS

2-D: MEDIAN LEVELS

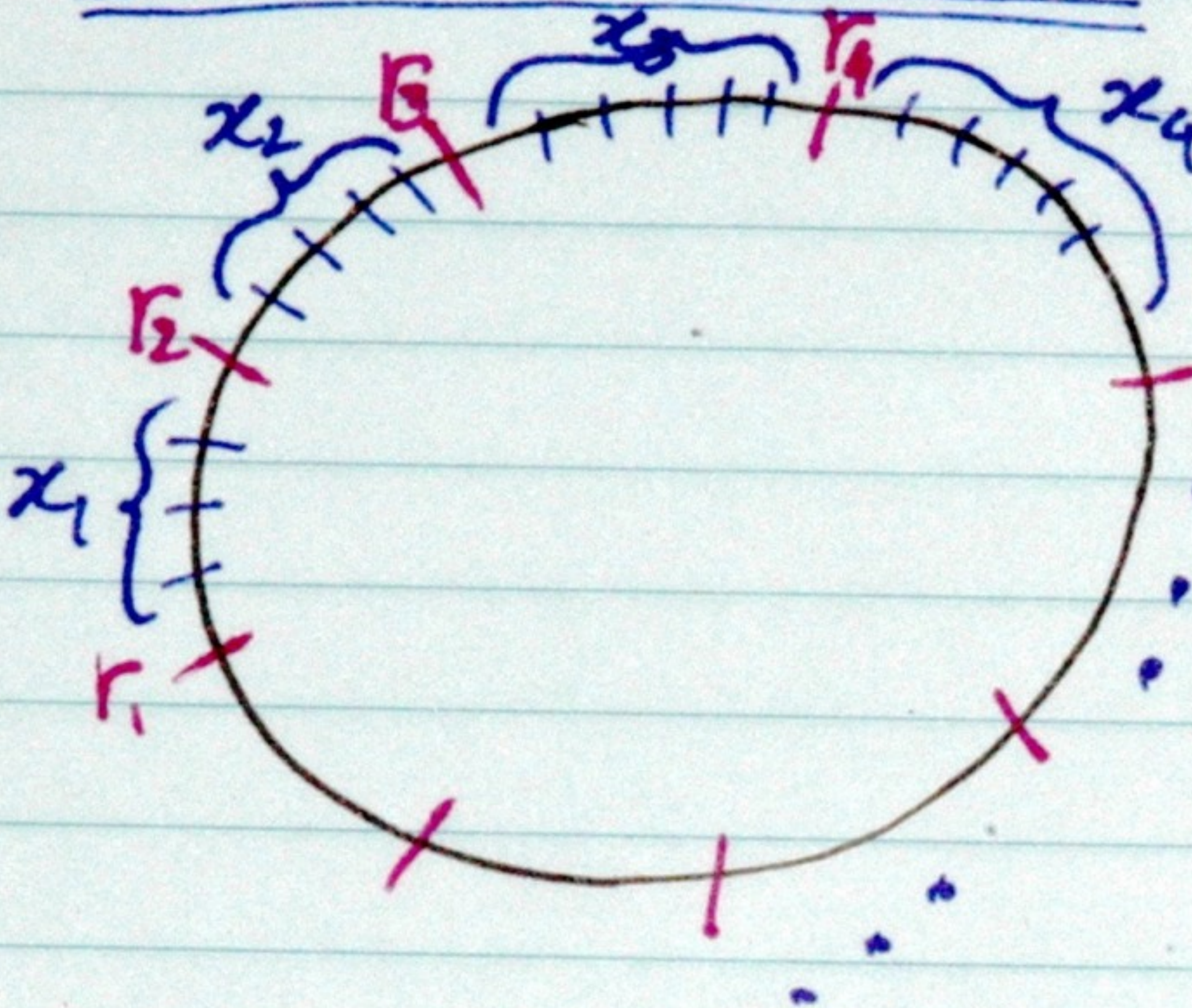
2-D: MEDIAN LEVELS

RED & BLUE MEDIAN LEVELS

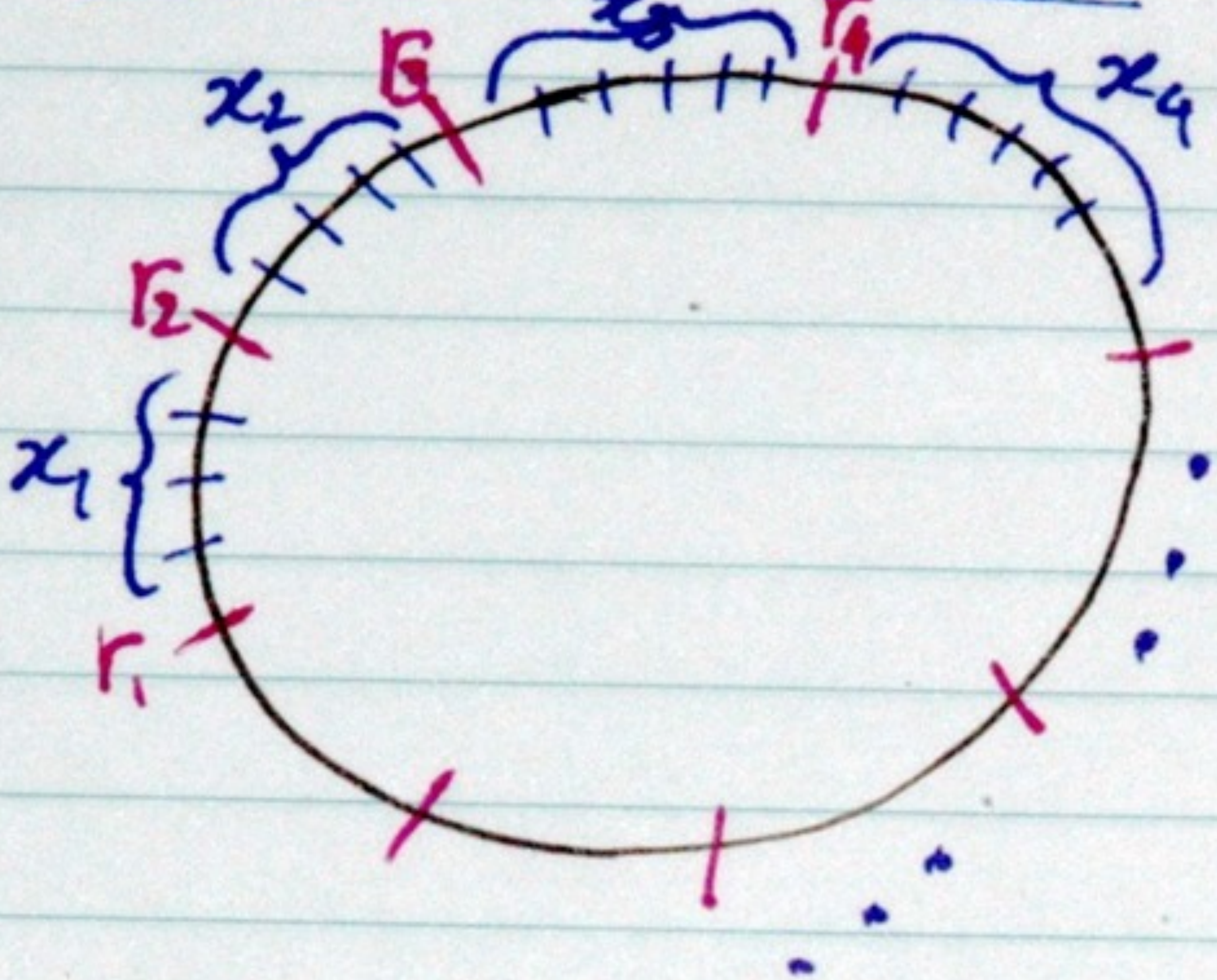


INTERSECTION ⤴
= HAM-SANDWICH CUT

OF INTERSECTIONS



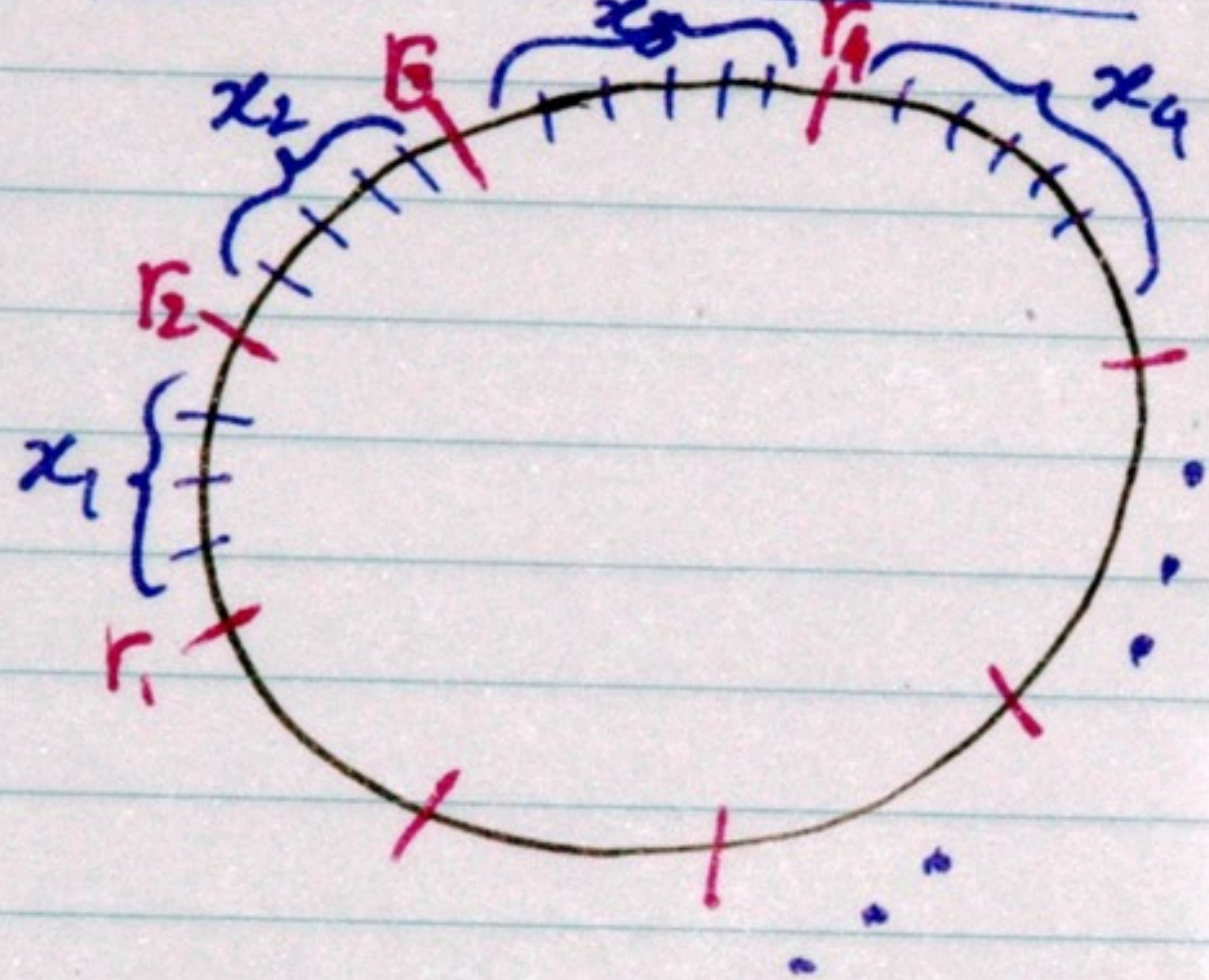
OF INTERSECTIONS



LEMMA: THE PARITY OF THE NUMBER OF INTERSECTIONS = THE PARITY OF

$x_2 + x_4 + x_6 + x_8 + \dots$

OF INTERSECTIONS

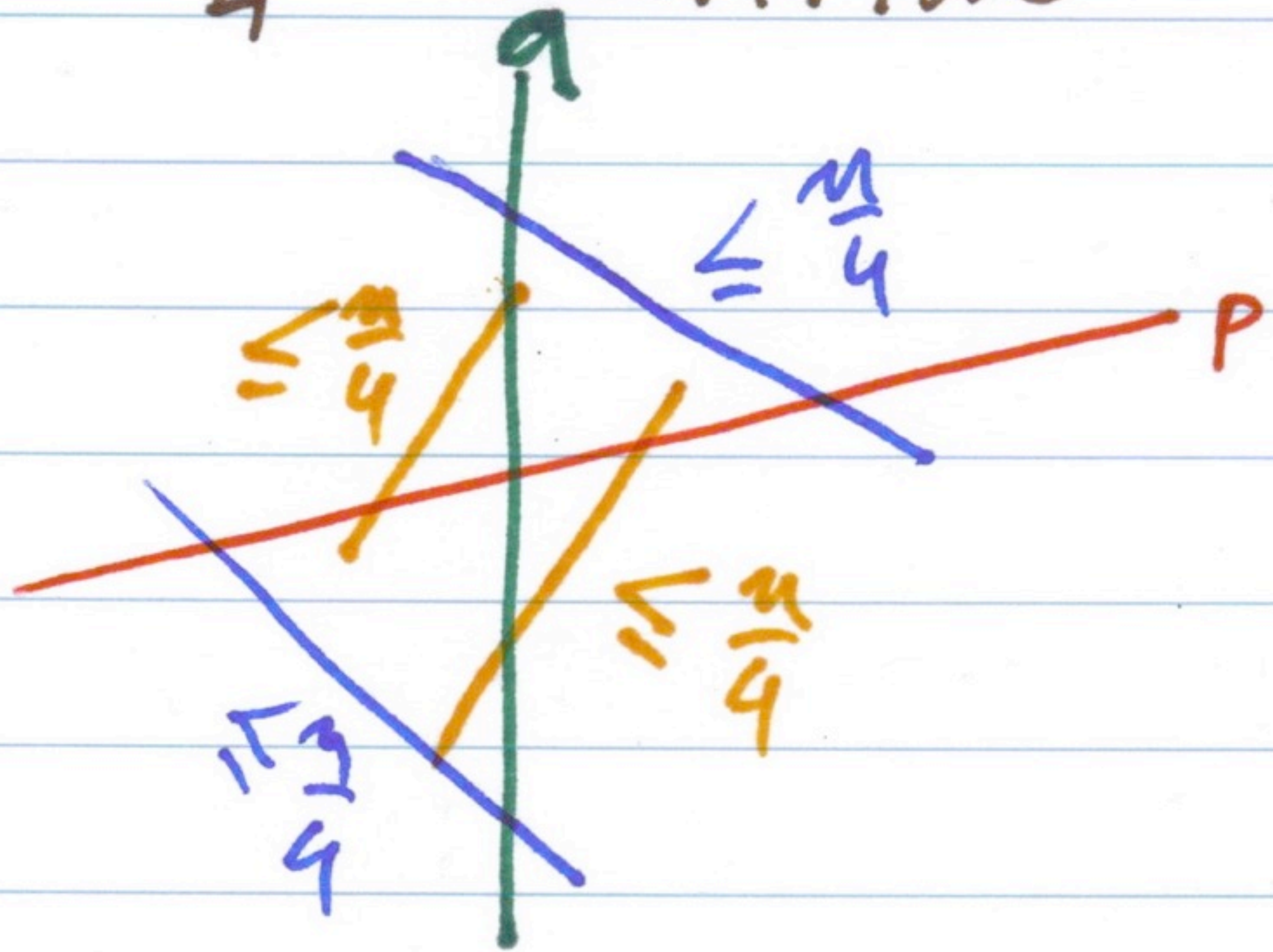


LEMMA: THE PARITY OF THE NUMBER OF INTERSECTIONS = THE PARITY OF

$$x_2 + x_4 + x_6 + x_8 + \dots$$

$\Rightarrow$ $O(n \log n)$ ALGO!

4 - PARTITION



PRUNING

1. COMPUTE A 4-PARTITION OF THE LINES

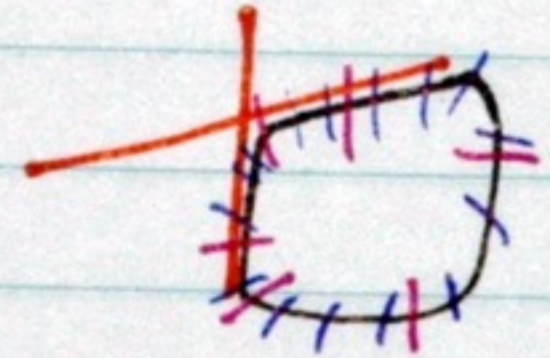
$O(n)$

PRUNING

1. COMPUTE A 4-PARTITION OF THE LINES

$O(n)$

2. DECIDE WHICH QUARTER CONTAINS A H-S CUT

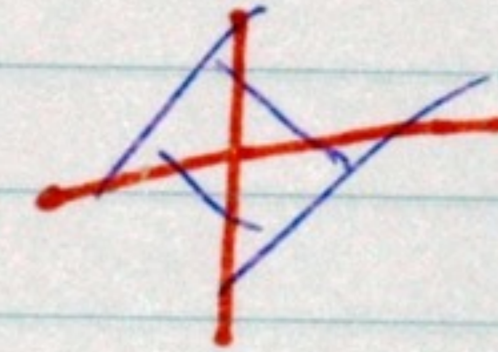


$O(n \log n)$

3. REMOVE 1/4 OF THE LINES NOT INTERSECTING THAT QUARTER

PRUNING

1. COMPUTE A 4-PARTITION OF THE LINES



$O(n)$

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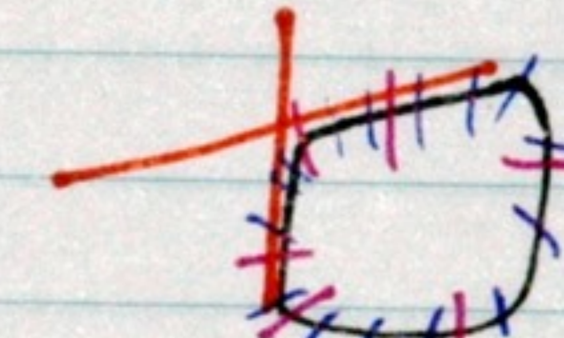
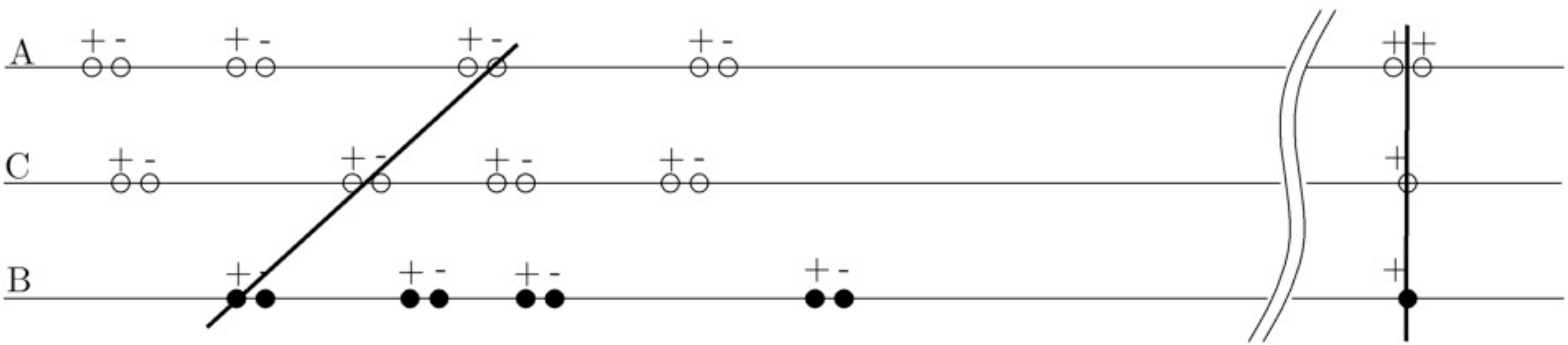


$O(n \log n)$

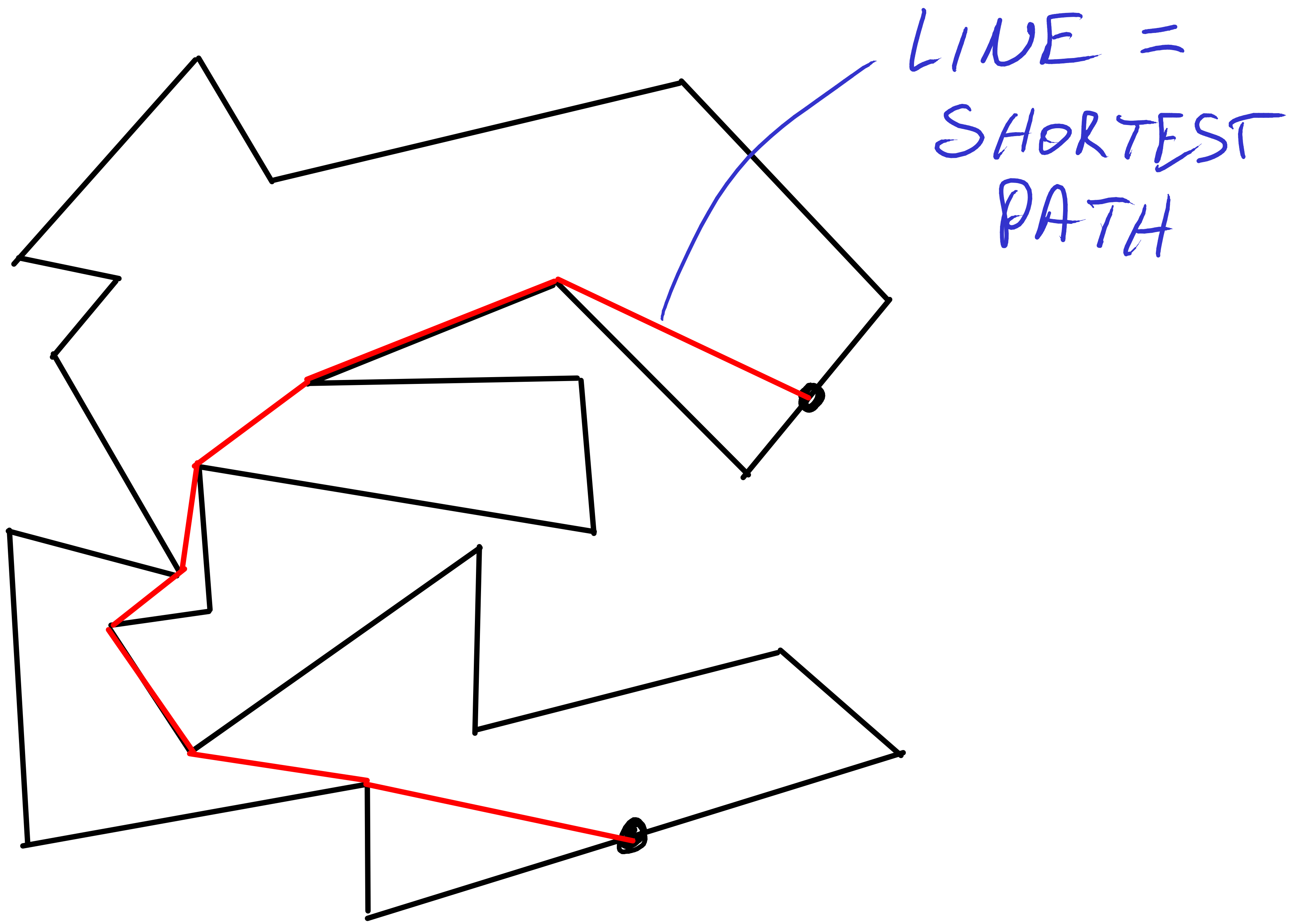
3. REMOVE 1/4 OF THE LINES NOT INTERSECTING THAT QUARTER

$\Rightarrow O(n \log n)$ ALGO!

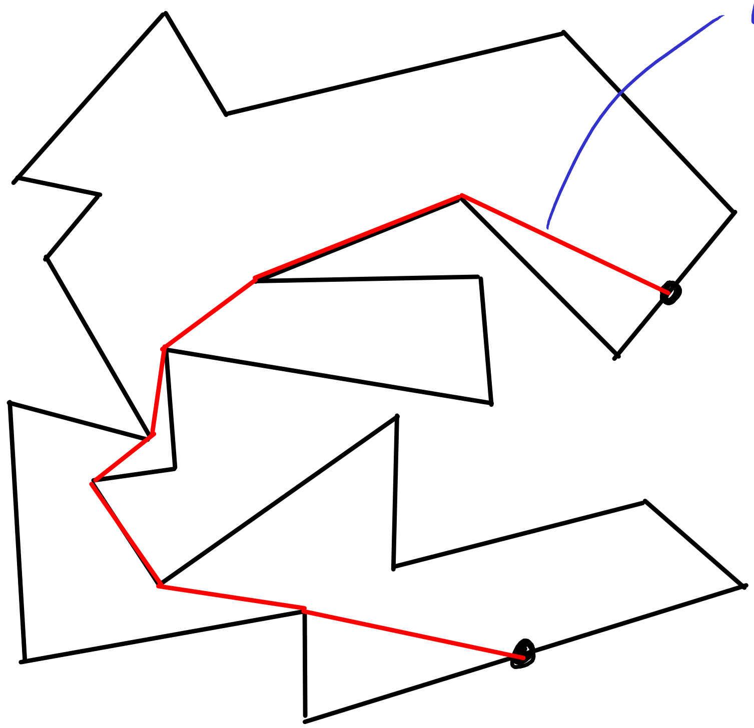
THM: DECIDING IF THE WEIGHTED HAM-SANDWICH CUT IS UNIQUE IS 3SUM-HARD


A
C
B

Geodesic



Geodesic Ham sandwich



Theorem:

Given blue and red points inside a polygon, there is a geodesic shortest path that bisects both simultaneously.

Can be found in $O(n \log k)$

n = # of points + vertices, k = # of reflex vertices

[Bose, Demaine, Erickson, Hurtado, Iacono, Langerman, Meijer, Morin, Overmars, Whitesides 2003]

Partitioning with hyperplanes

- Can we partition R^d into 2^d regions with $n/2^d$ points with d hyperplanes? [Grunbaum 1960s]
- Motivation...

Partitioning with hyperplanes

Partitioning with hyperplanes

- R^1 -> Easy

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- R^2: Yes (ham-sandwich cut) Algorithmic problem posed by [Willard 1982], solved $O(n)$ by [Megiddo 1985].

Partitioning with hyperplanes

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- R^2: Yes (ham-sandwich cut) Algorithmic problem posed by [Willard 1982], solved $O(n)$ by [Megiddo 1985].
- R^3: Yes [Yao, Dobkin, Edelsbrunner, Paterson 1989]. $O(n^6 \log n)$.

Partitioning with hyperplanes

- R^4 -> OPEN!!!
- R^5: NO!!!!

MOMENT CURVE

$t \to (t, t^2, t^3, \ldots, t^k)$

IN $\mathbb{R}^k$

ANY HYPERPLANE INTERSECTS IT $\leq k$ TIMES

So k hyperplanes intersect it $\leq k^2$ times

-> at most k^2+1 regions have points

[Avis 1984].

Part. w/ hyperplanes

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- What is the smallest dimension $d(j,k)$ such that j distributions can be equipartitioned by k hyperplanes?

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Part. w/ hyperplanes

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- $d(k,1) = k$ (Ham-sandwich thm)
- $d(1,2) = 2$, $d(1,3) = 3$, $d(1,5) > 5$
- $d(2,2) = 3$ [Edelsbrunner 1986].
- $j2^{k-1} \geq d(j,k) \geq j(2^k-1)/k$ [Ramos 1996].

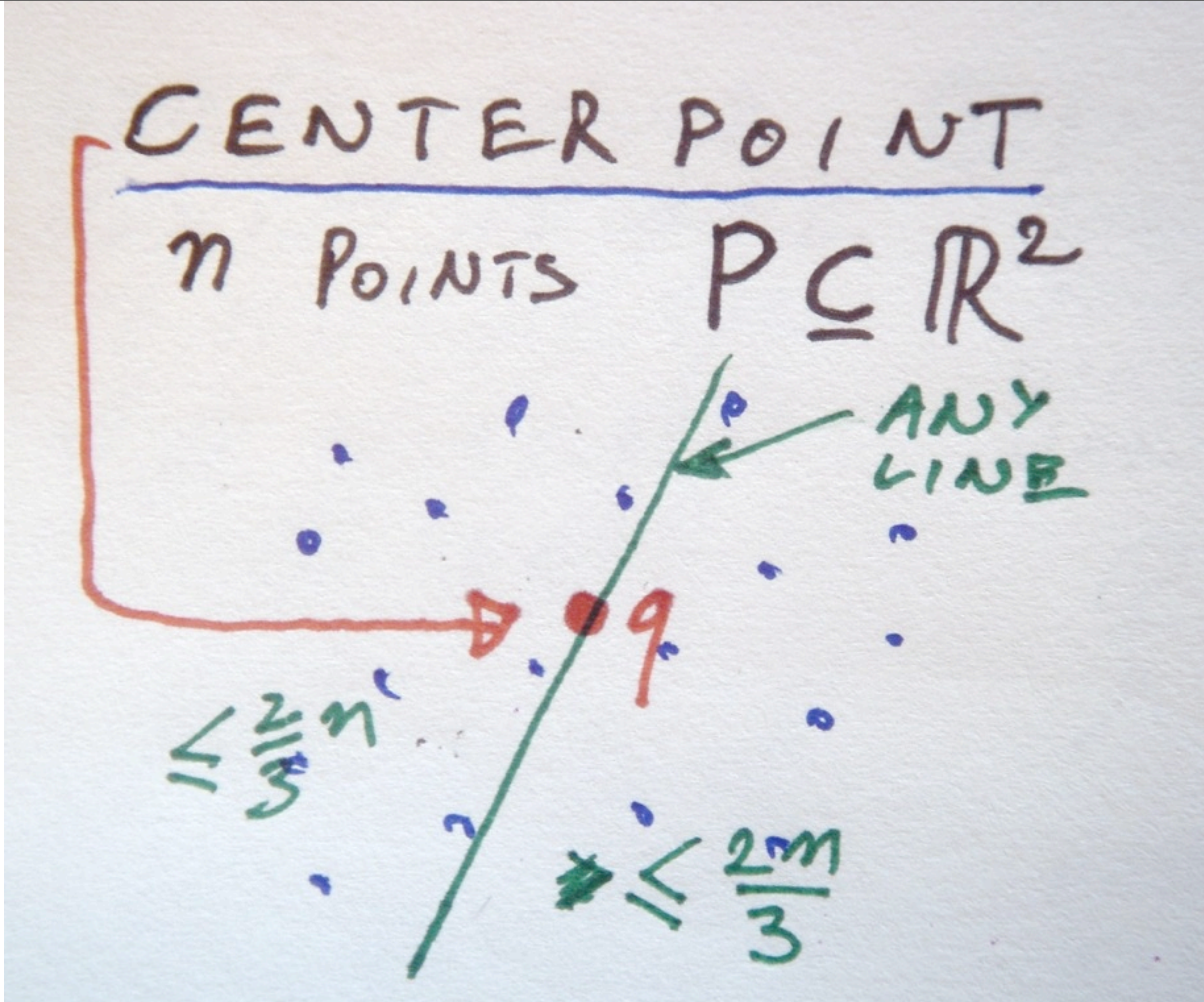
Partitioning with points

CENTER POINT

n POINTS $P \subseteq \mathbb{R}^2$

CENTER POINT

n POINTS $P \subseteq \mathbb{R}^2$


CENTER POINT
n POINTS P ⊆ ℝ²
ANY LINE
q
≤ 2/3 n
≤ 2n/3

Central Transversal Thm

- For any $k+1$ mass distributions in R^d there exists a k-flat s.t. any hyperplane containing f has > $1/(d-k\_1)$ of the i^{th} mass on each side.

[Dol'nikov 1992]

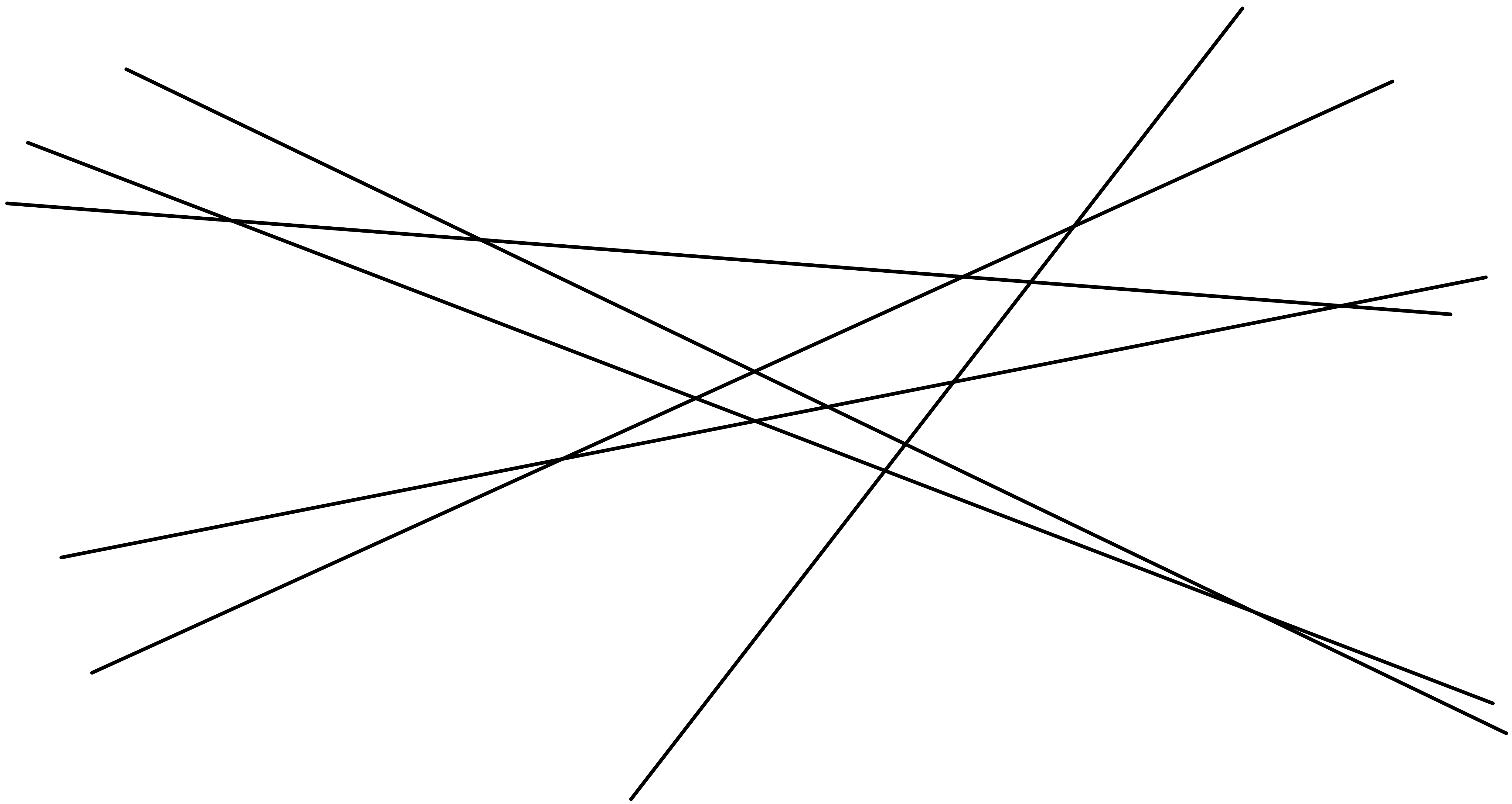
[Zivaljevic&Vrecica 1990]

Arrangements

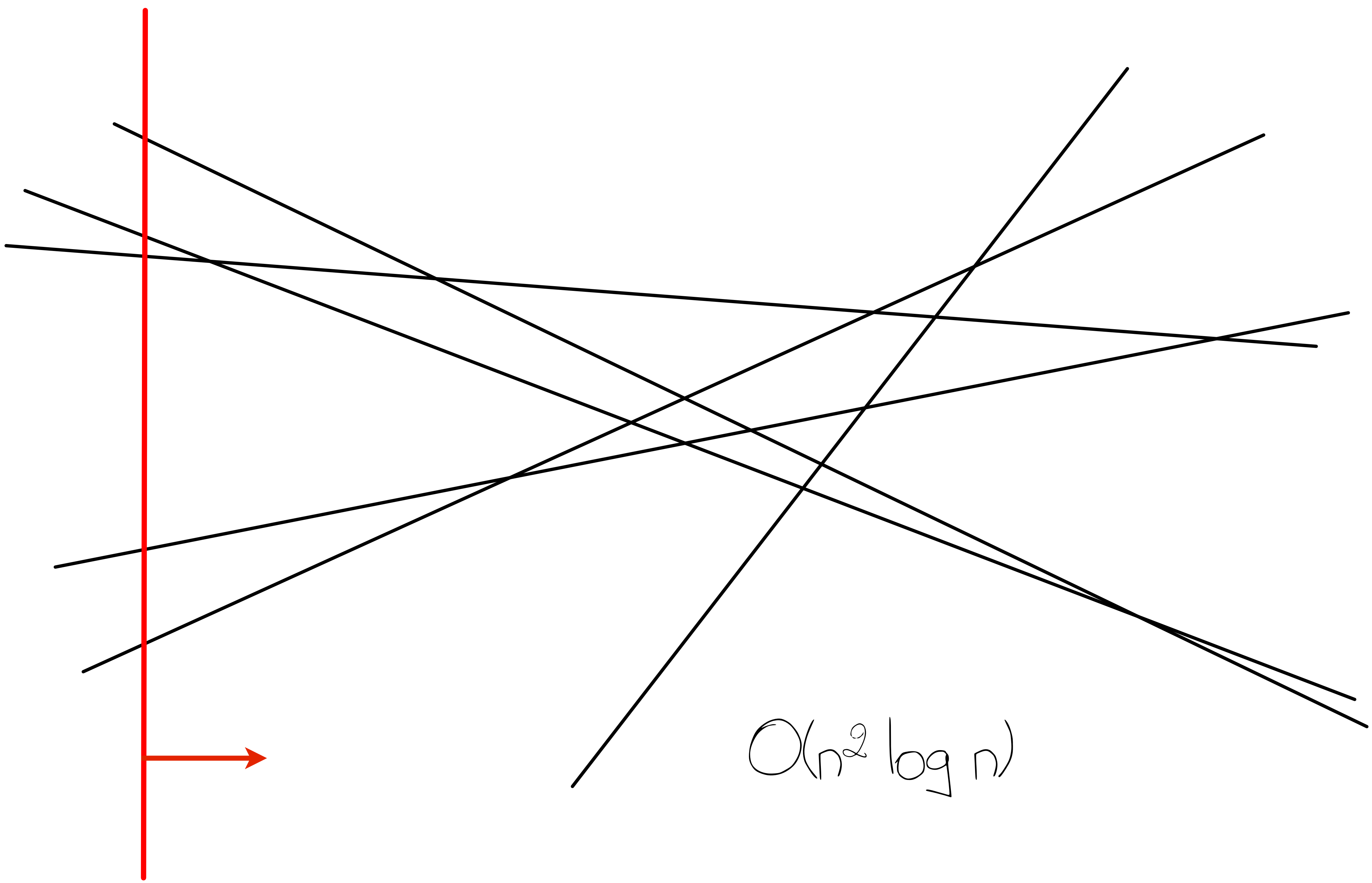
Arrangements

- S = Set of curves (2D) or surfaces (3D) (Here: S = lines, planes or hyperplanes)
- Arrangement A(S) = Decomposition of space R^d into connected cells of R^d - S

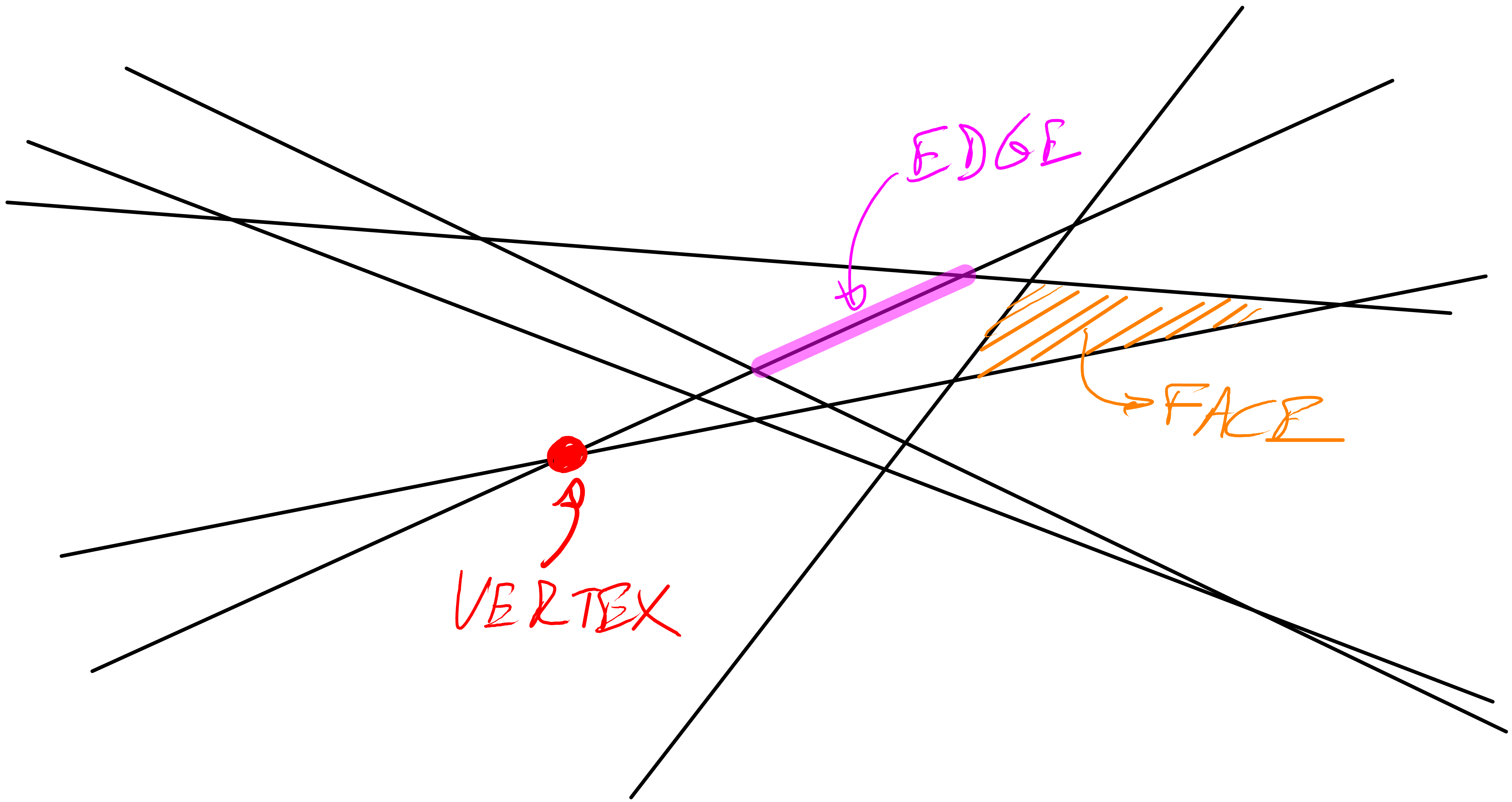
Line arrangement



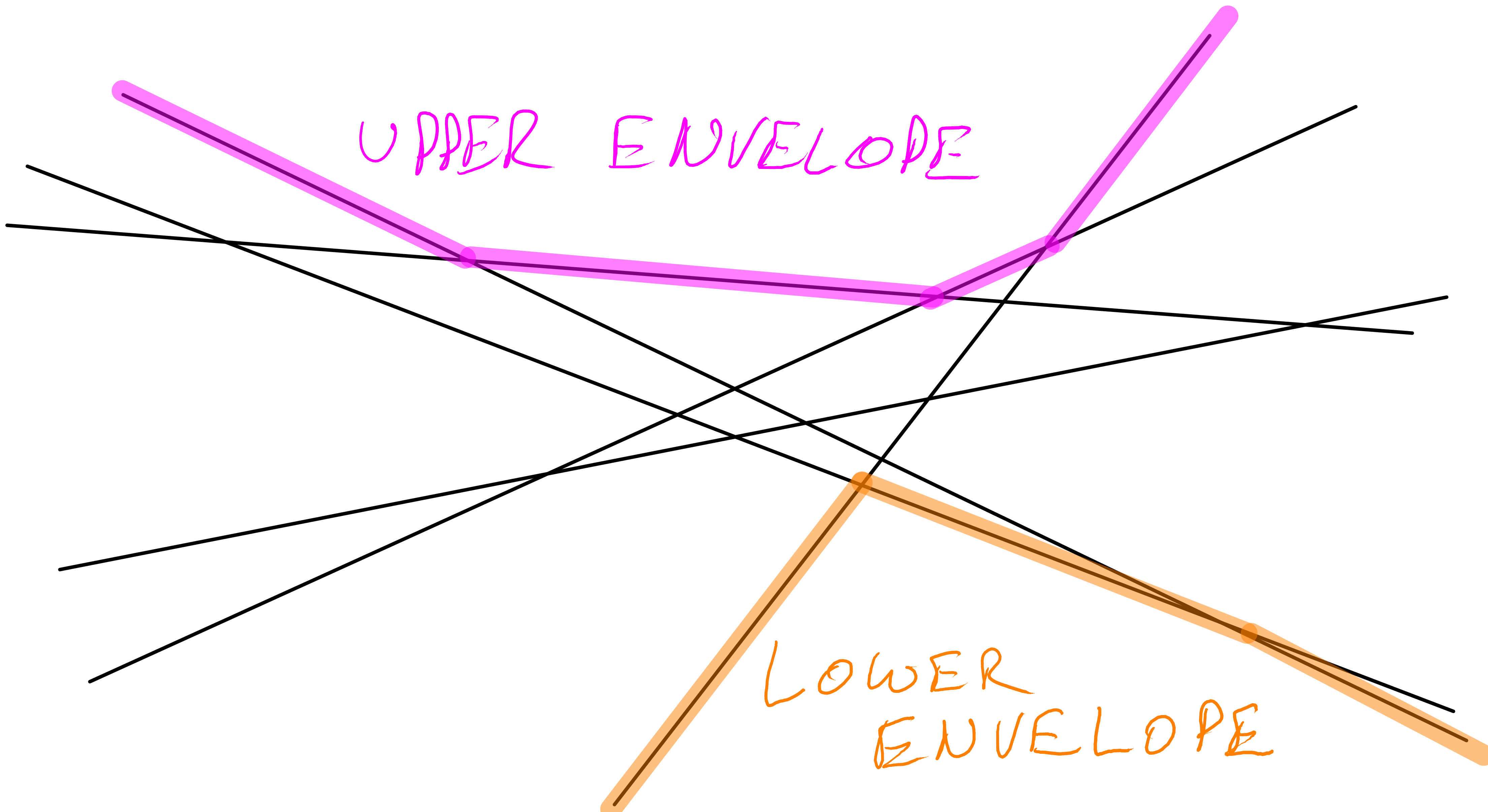
Algorithm: Linesweep



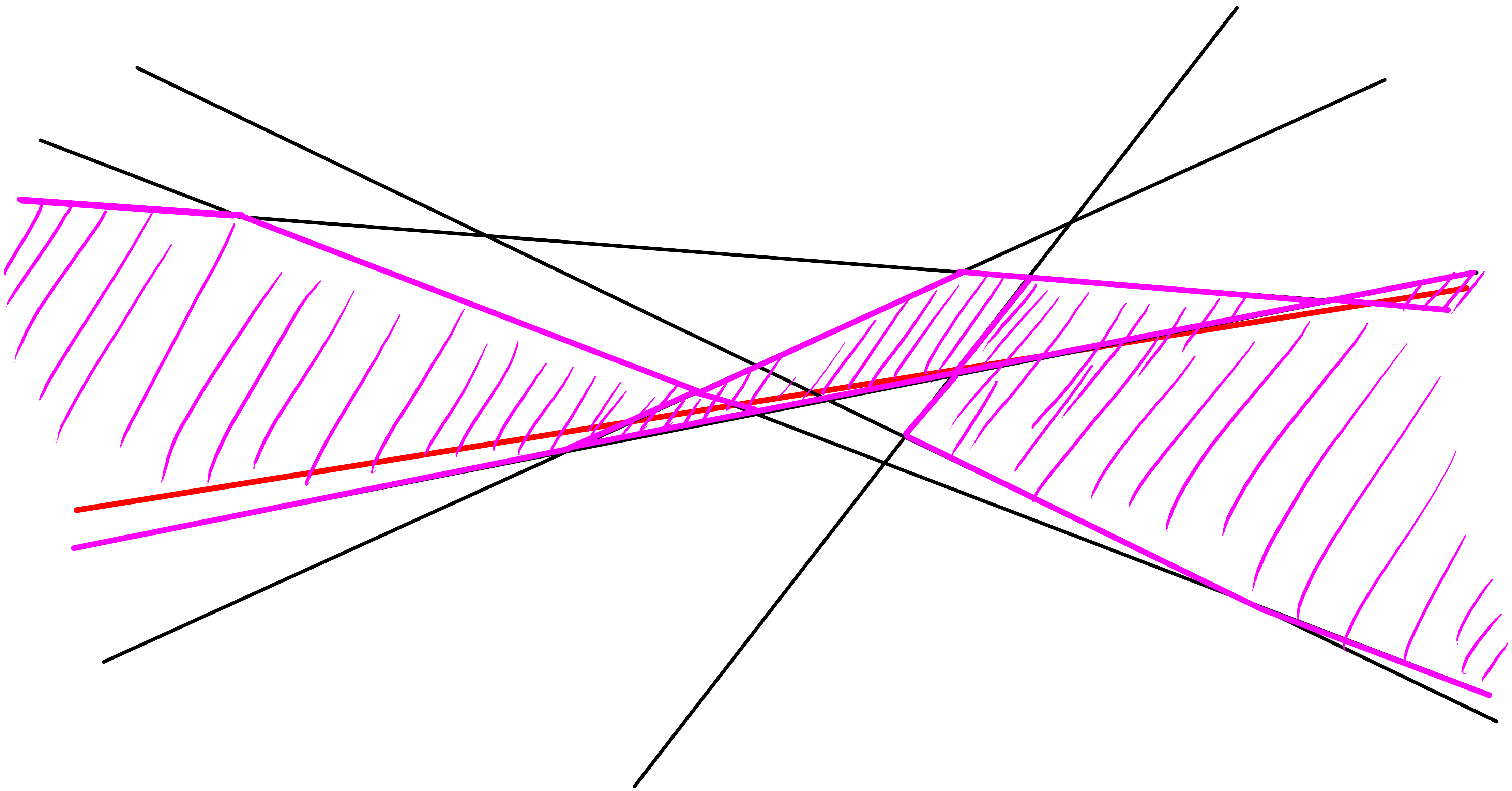
Vertices, Edges, Faces



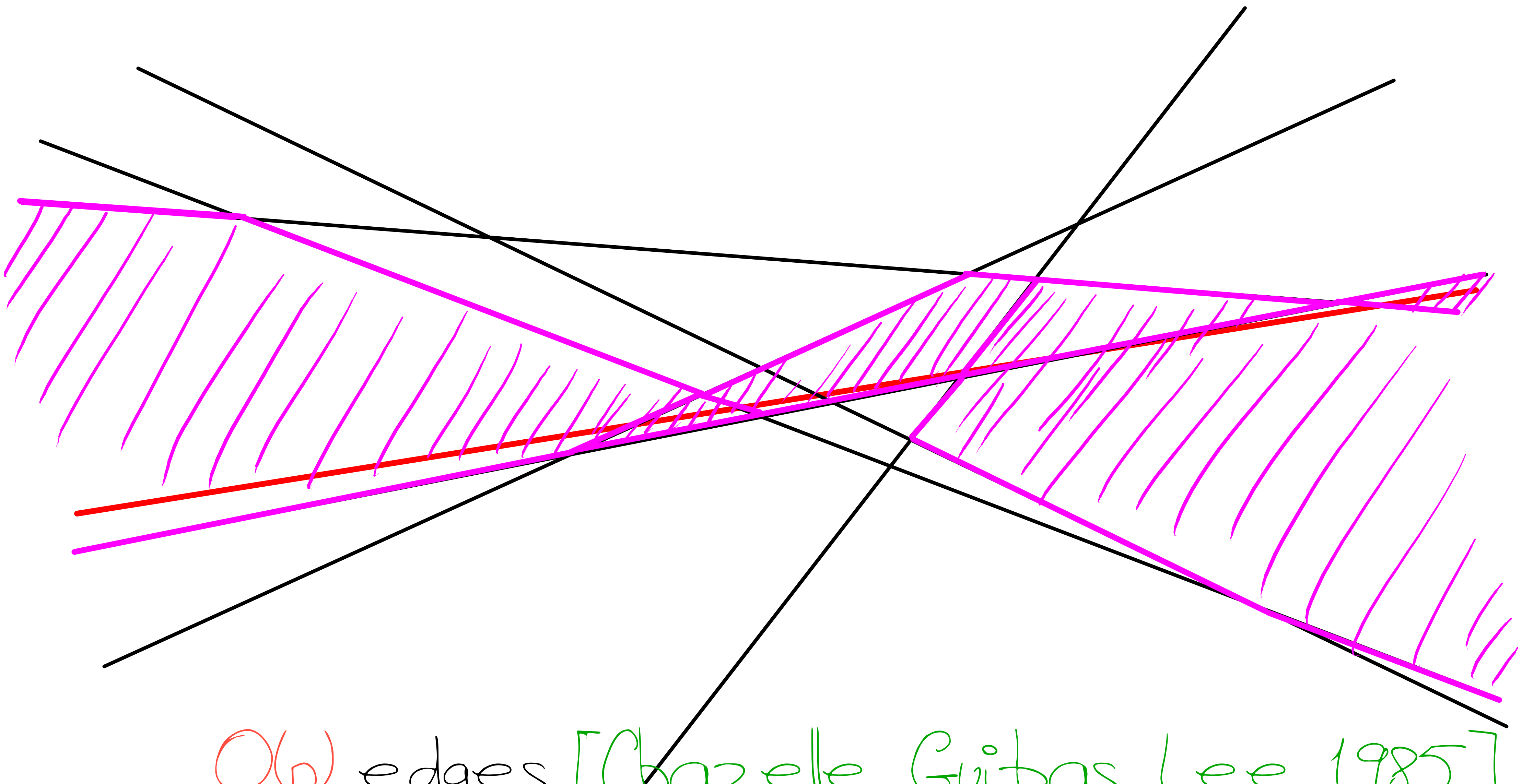
Envelopes



Zone

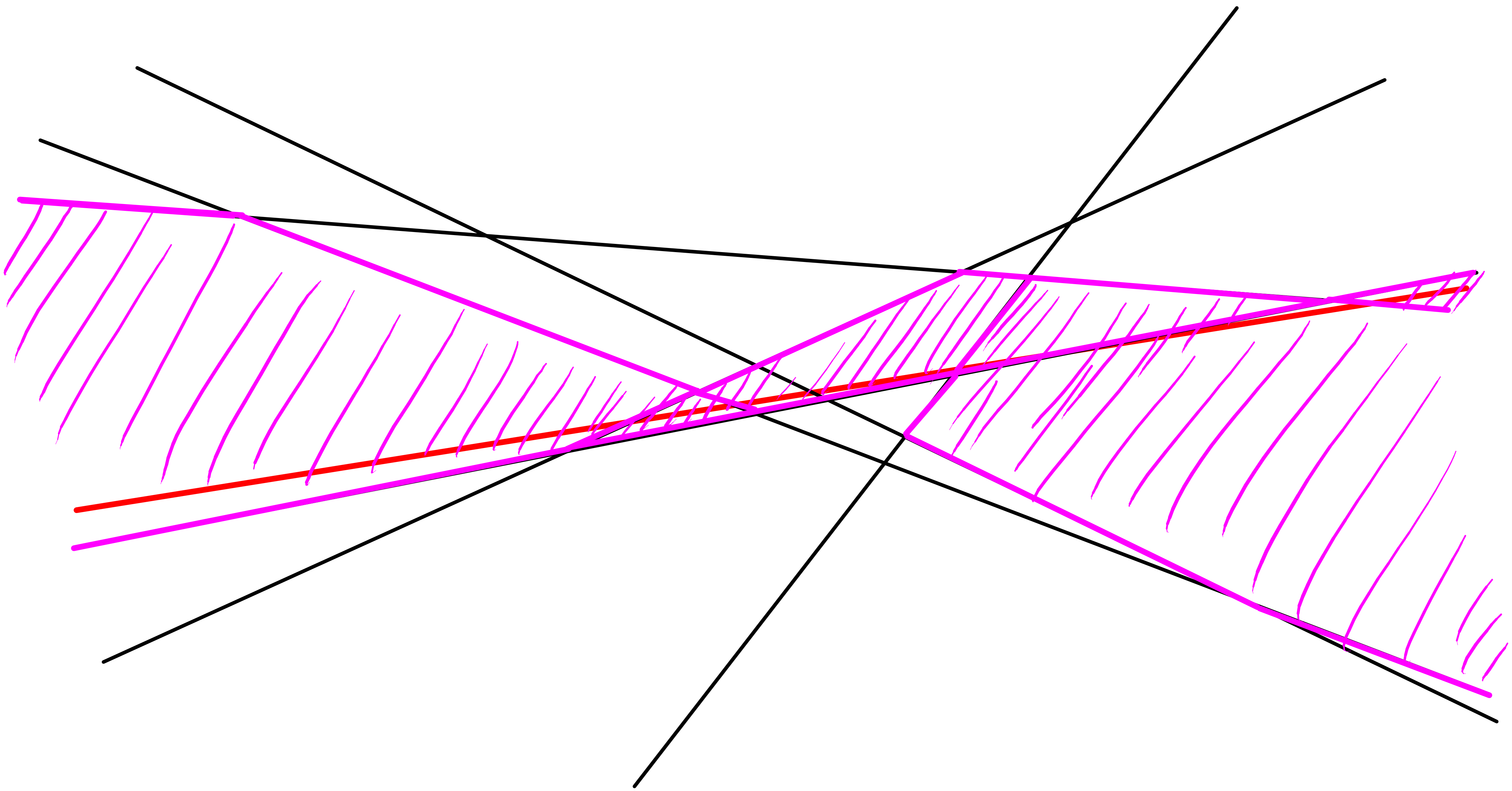


Zone

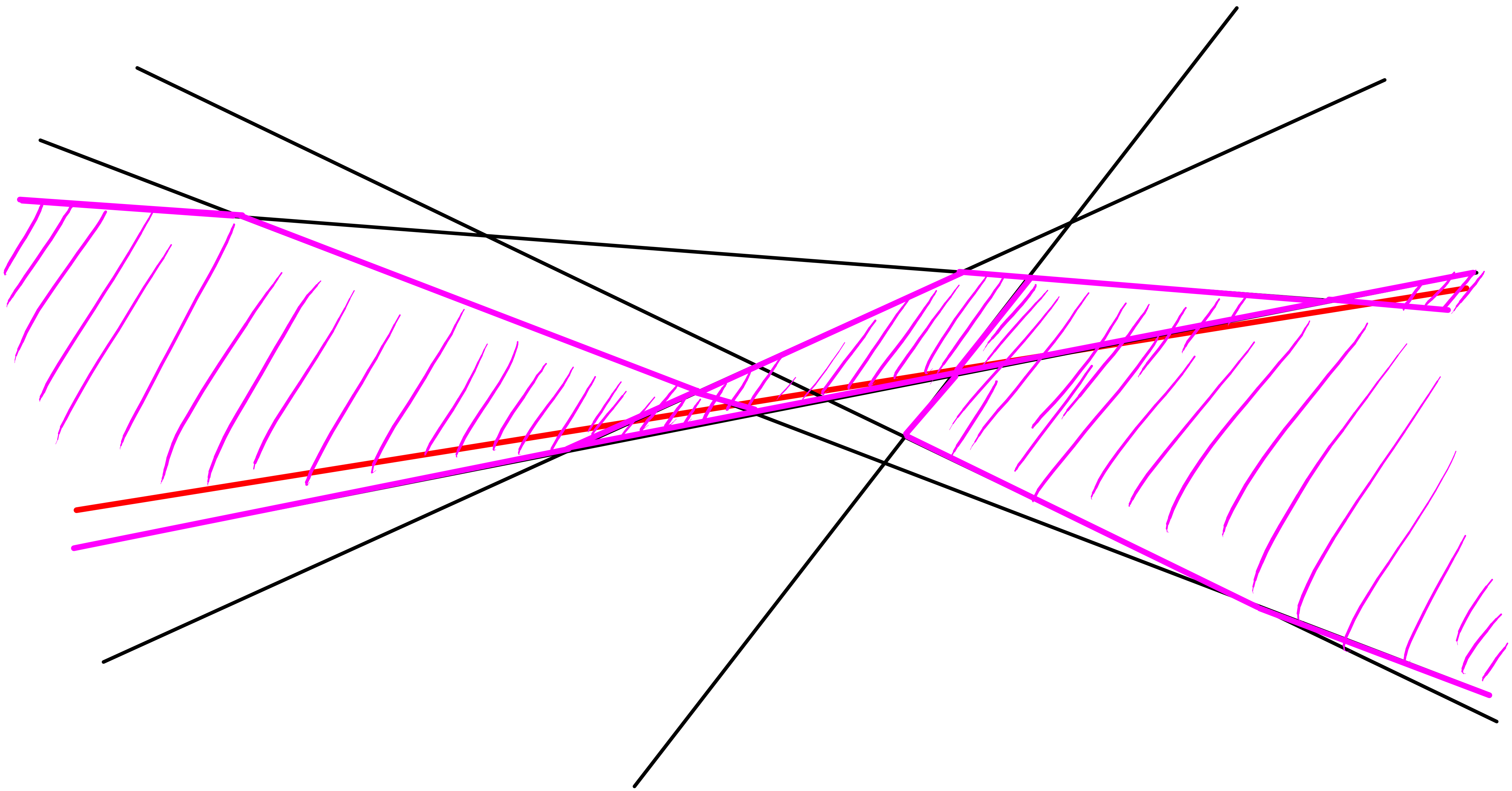


O(n) edges [Chazelle, Guibas, Lee 1985]
[Edelsbrunner, O'Rourke, Seidel 1986]

Zone: Algorithm?



Arrangement: algorithm



How big is the median level (= (n/2)-level)?

HOW BIG IS THE MEDIAN LEVEL?

$O(n^{3/2})$ [ERDÖS, LOVASZ, SIMMONS, STRAUS '73]

$\Omega(n \log n)$

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$n \cdot 2^{\Omega(\sqrt{\log n})}$ [TOTH '00]

Monotone paths

- How long can they be?

Monotone paths

- How long can they be?

$\Omega(n^{3/2})$ [Sharir <1987]

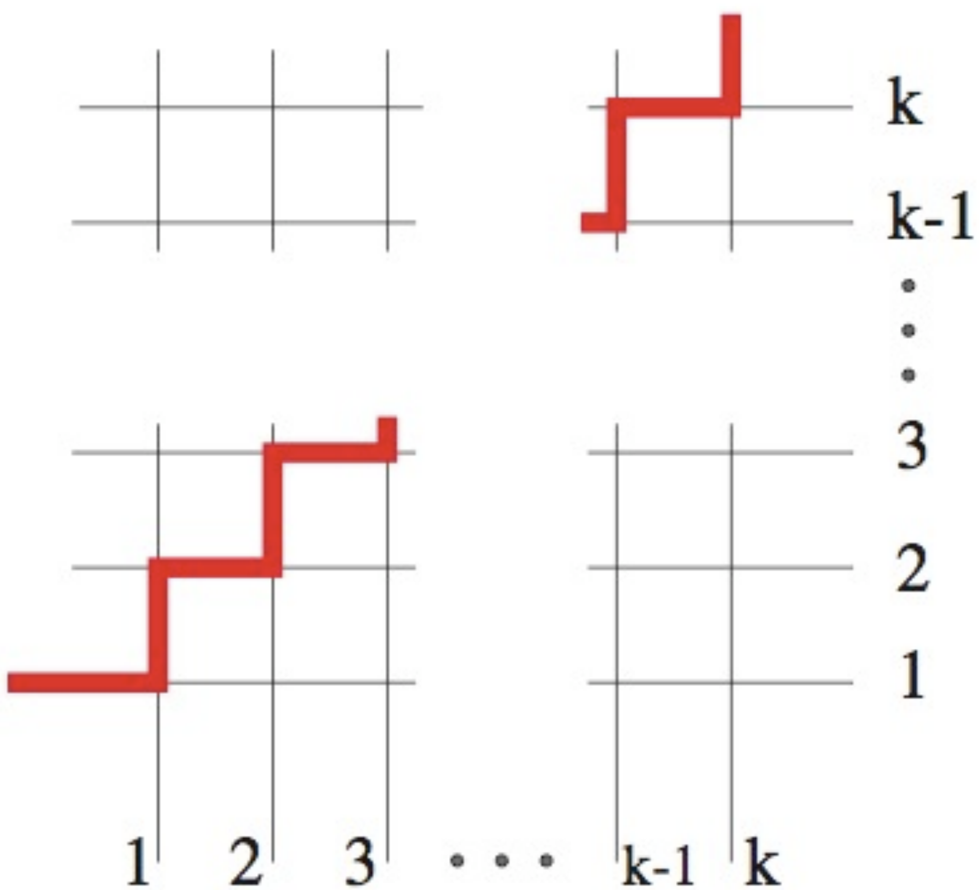
$\Omega(n^{5/3})$ [Matousek 1991]

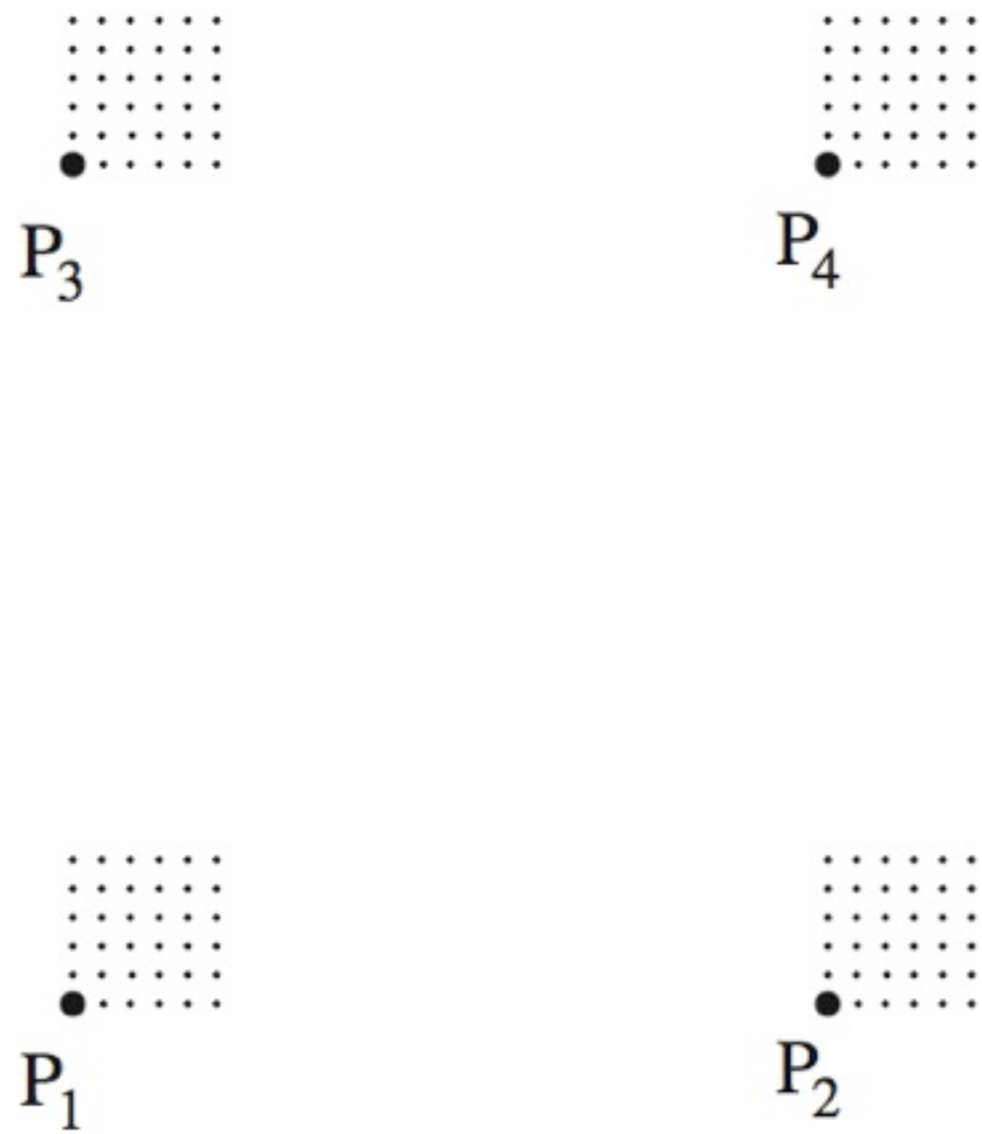
$\Omega(n^{7/4})$ [Radoicic and Toth 2001]

Monotone paths

- How long can they be? $\Omega(n^{2-(d/\sqrt{\log n})})$

[Balogh, Smyth, Steiger, Szegedy 2004]

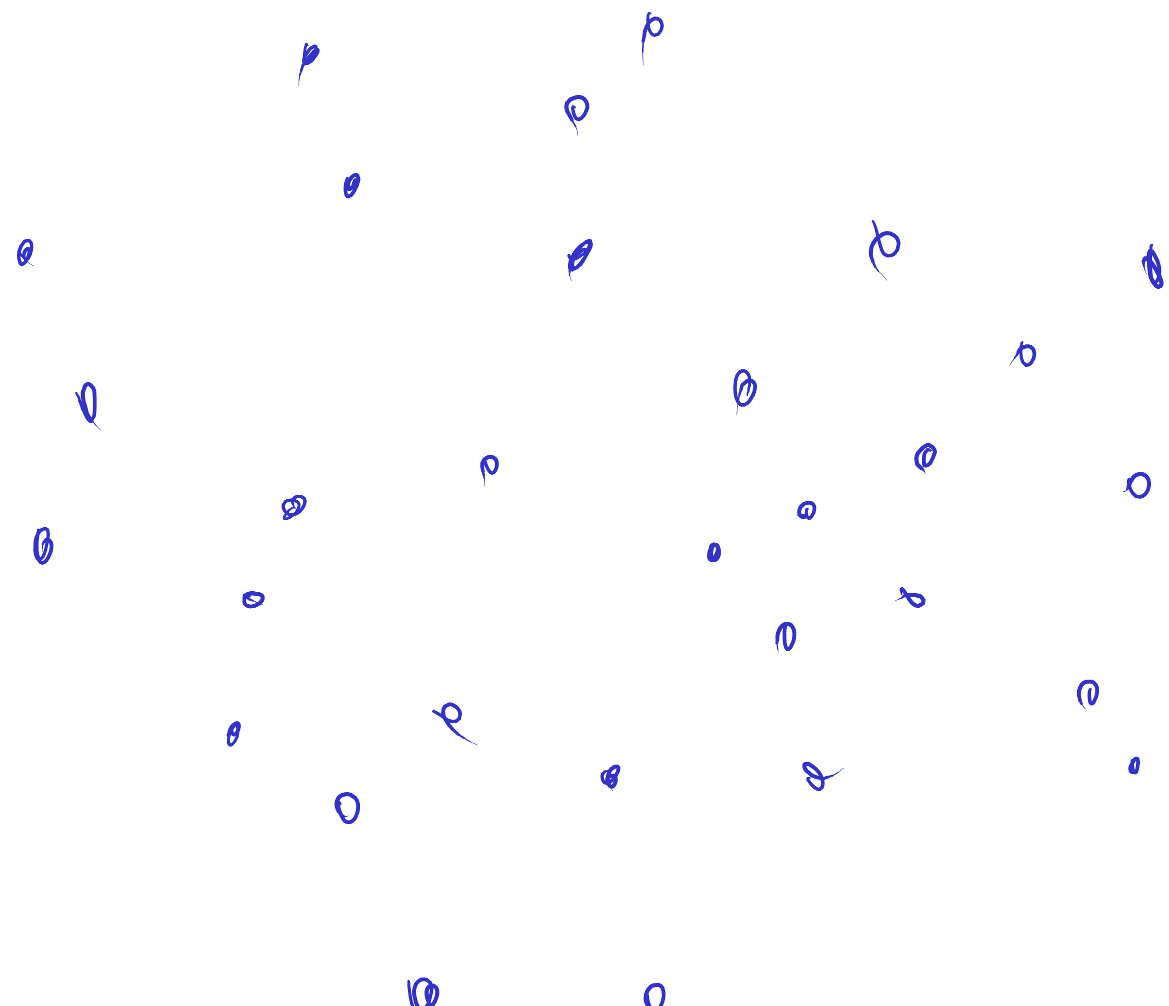




Slope selection

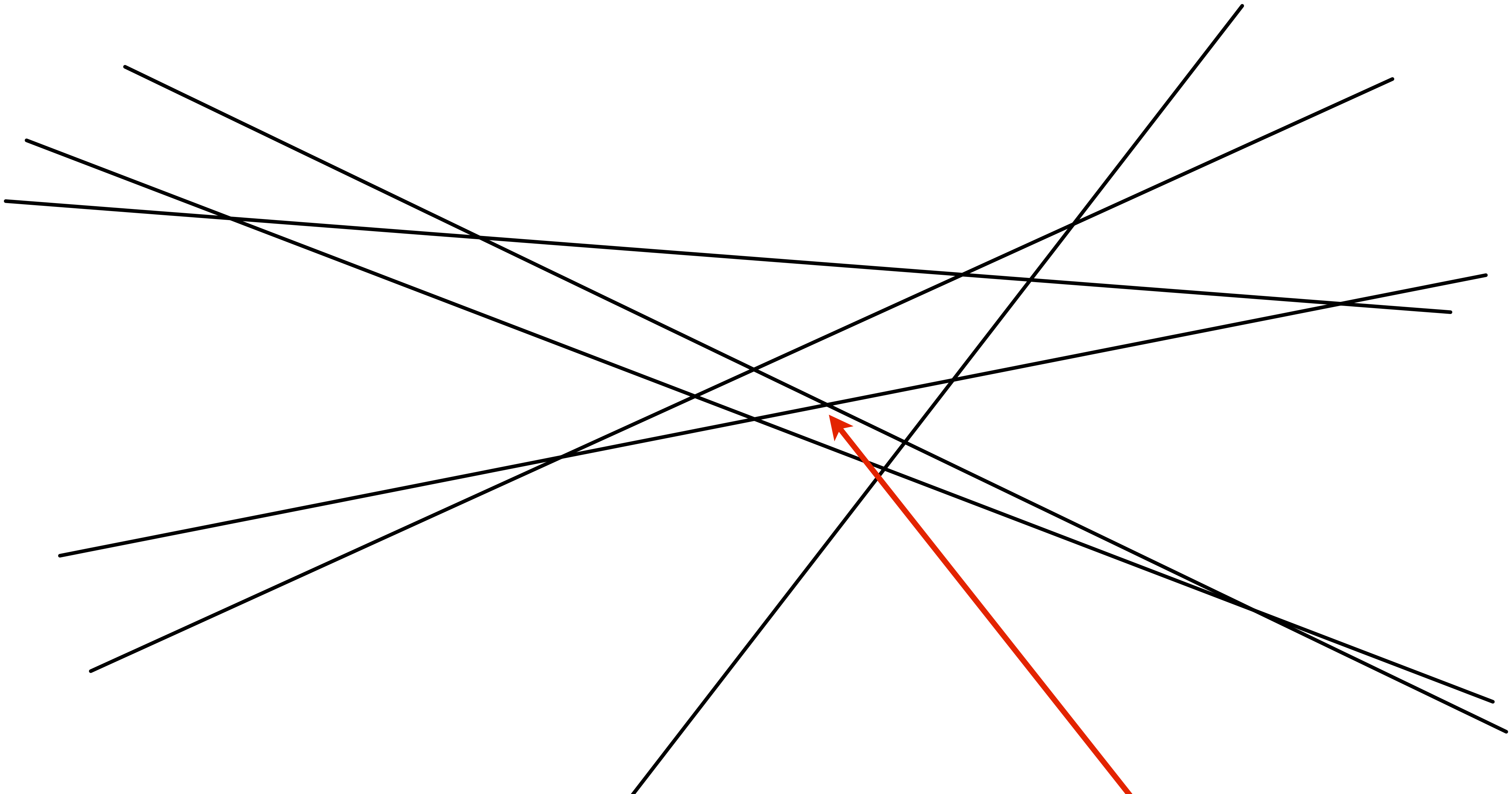
n points in the plane -> $O(n^2)$ different slopes

Select the k^{th} smallest.



Thiel-Sen regression estimator

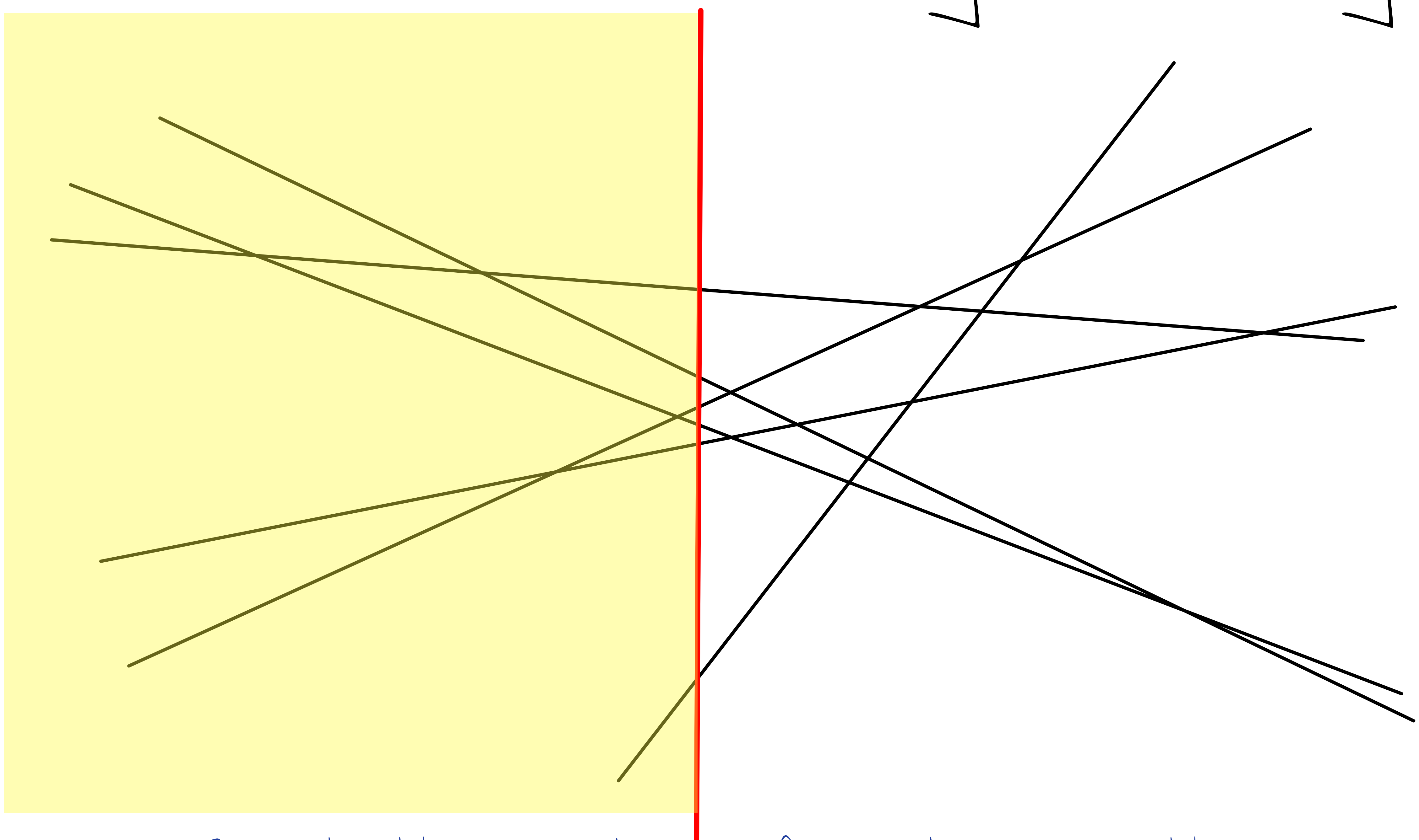
Vertex Selection



n lines in the plane -> $O(n^2)$ different vertices

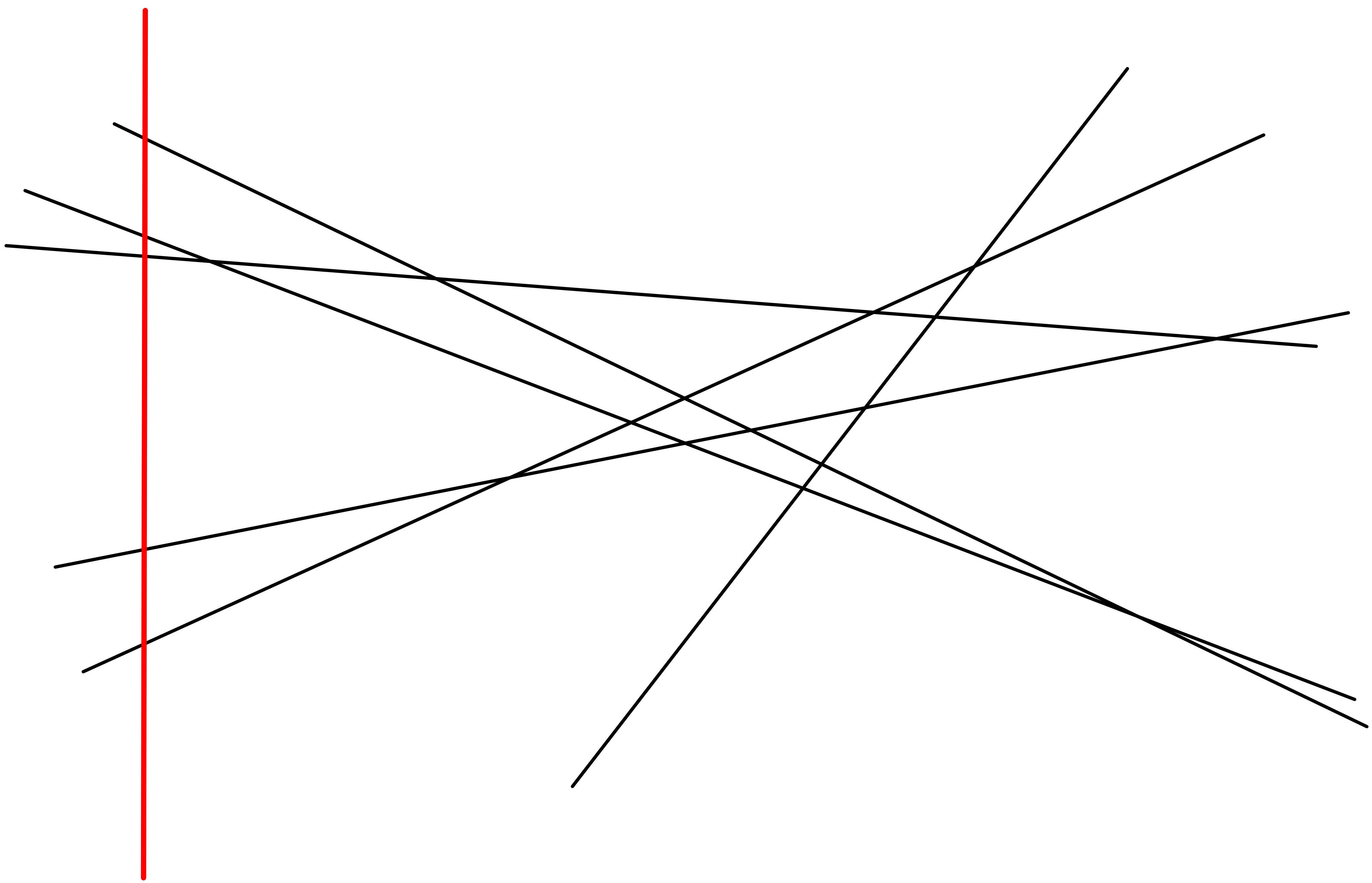
Select the one with the k^{th} smallest x-coordinate

Vertex counting/ranking

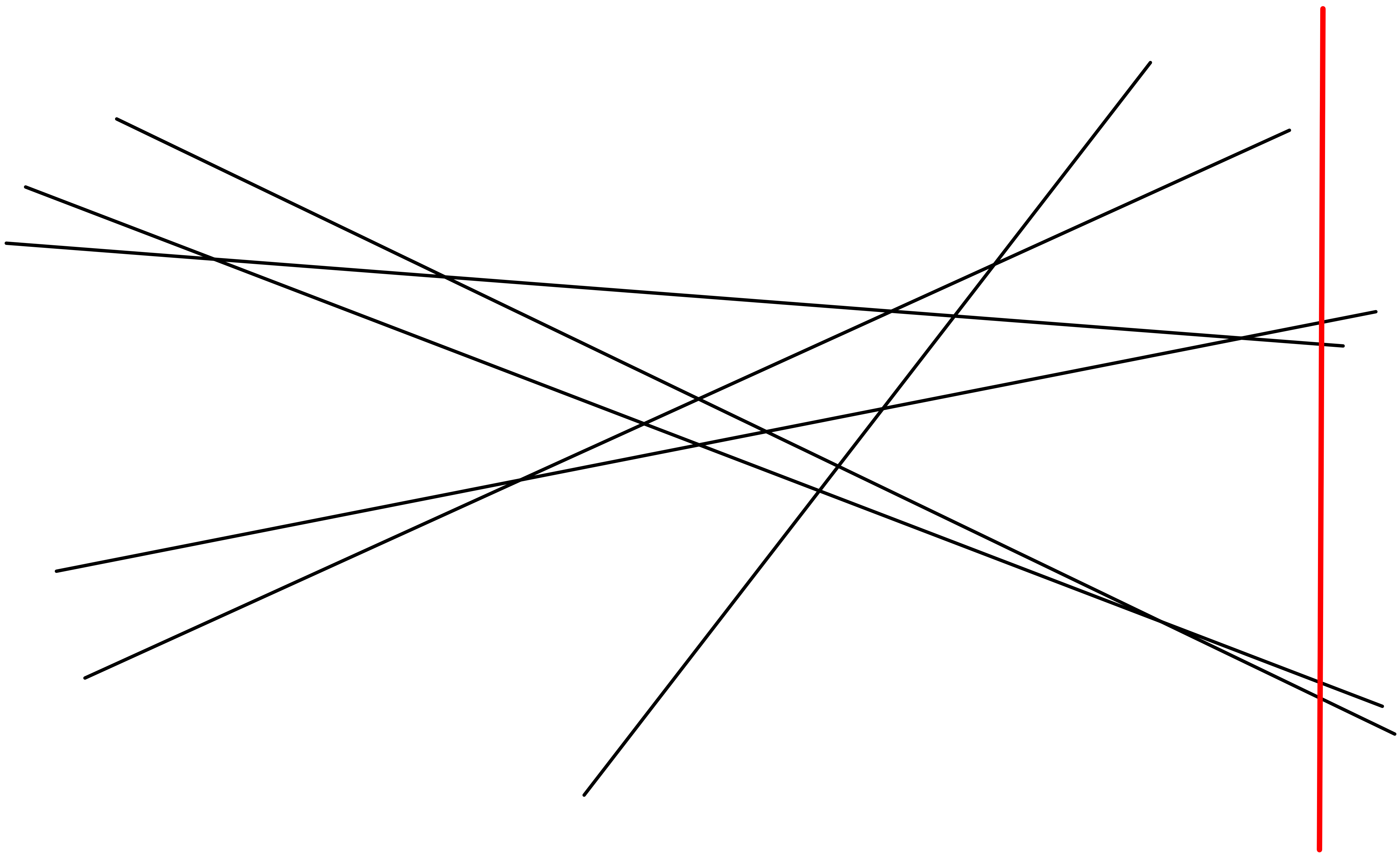


Count the number of vertices with x-coordinate < x_0

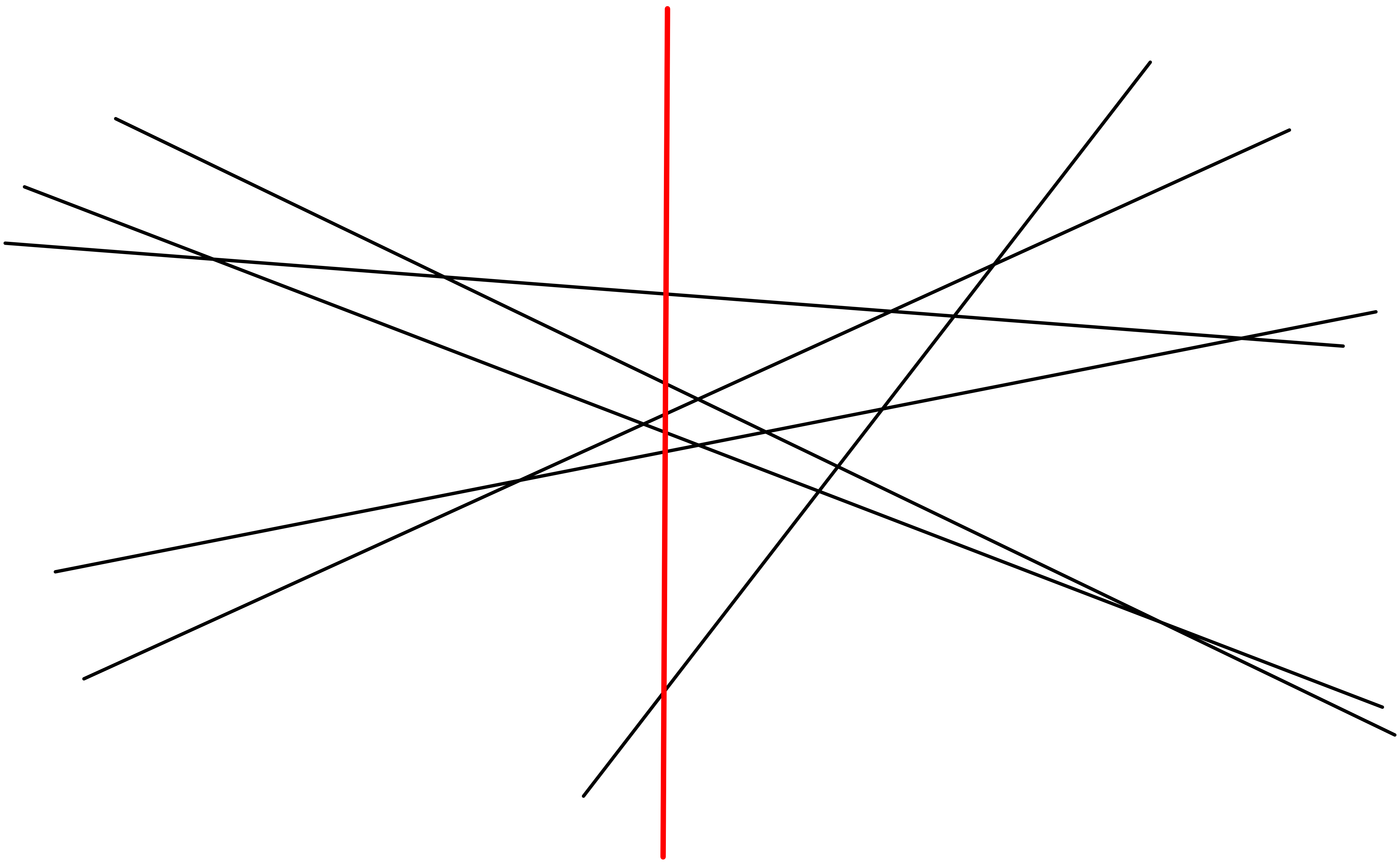
Vertex counting/ranking



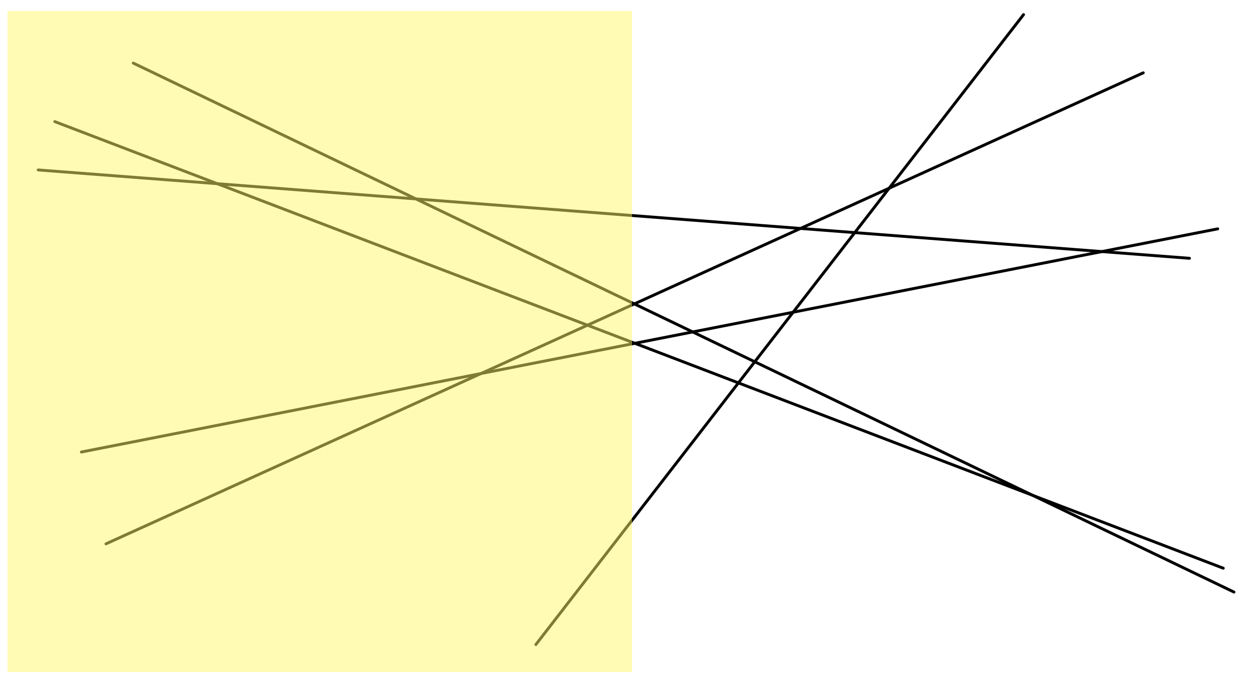
Vertex counting/ranking



Vertex counting/ranking



Binary search



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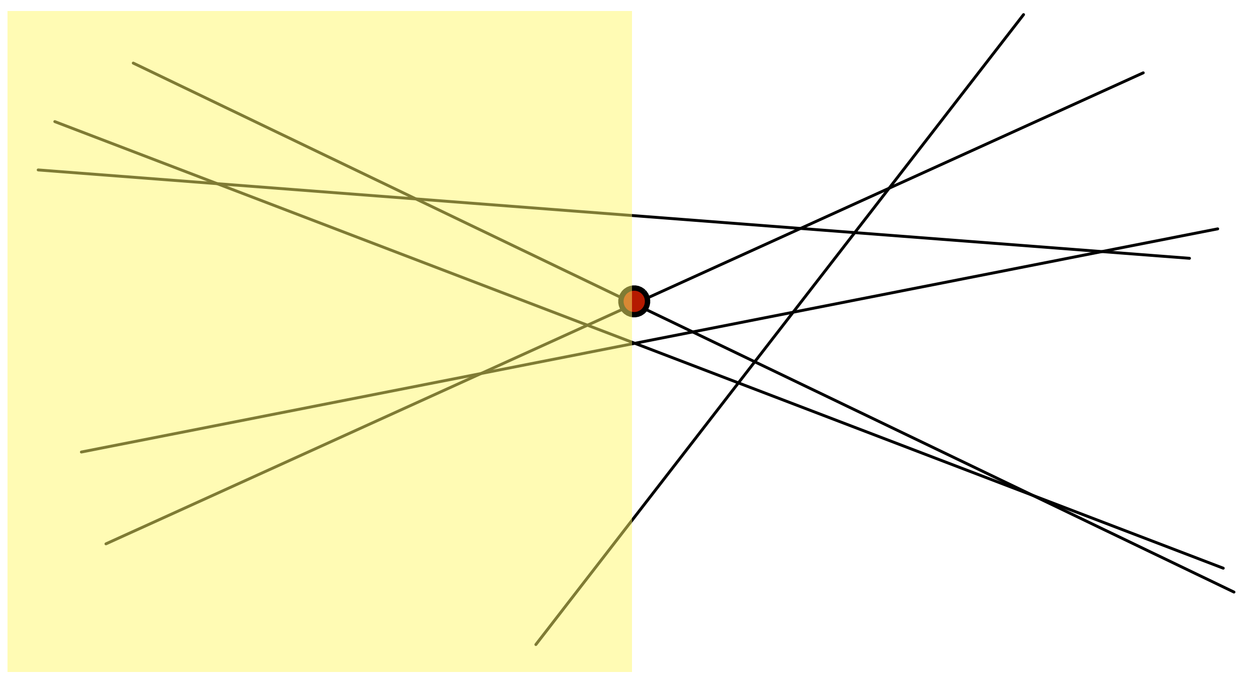
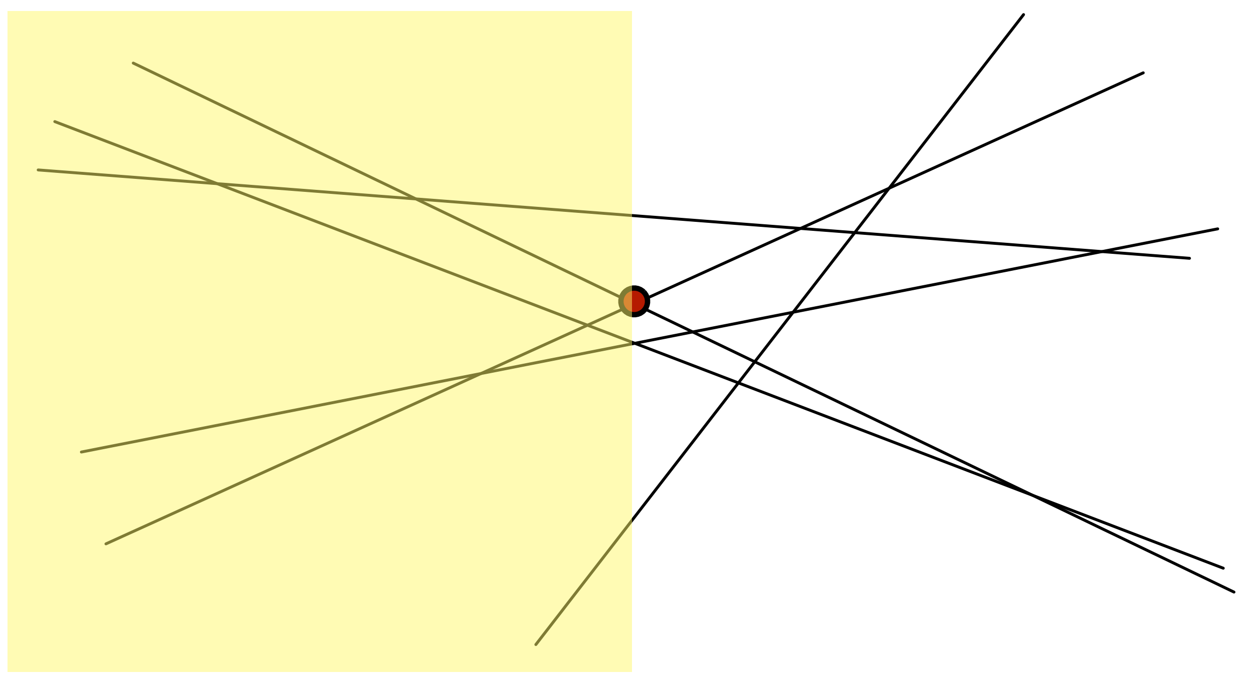


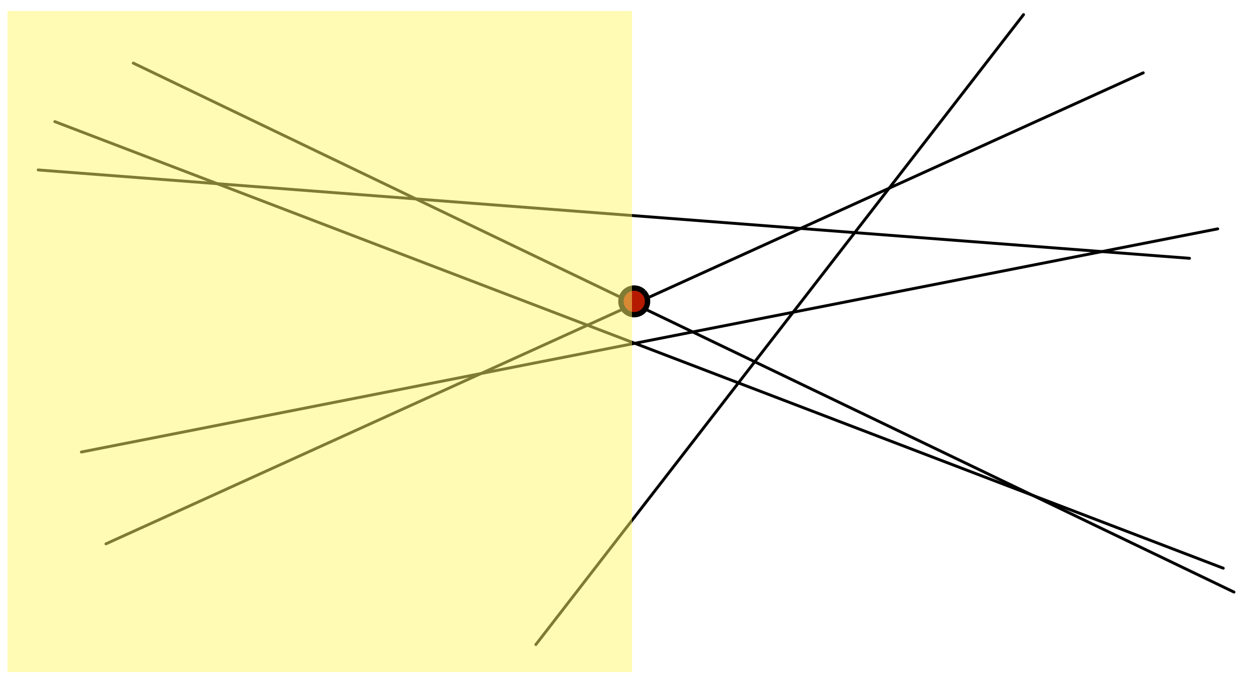
Binary search

- Pick a vertex at random



Binary search

- Pick a vertex at random
- Rank the vertex (r)



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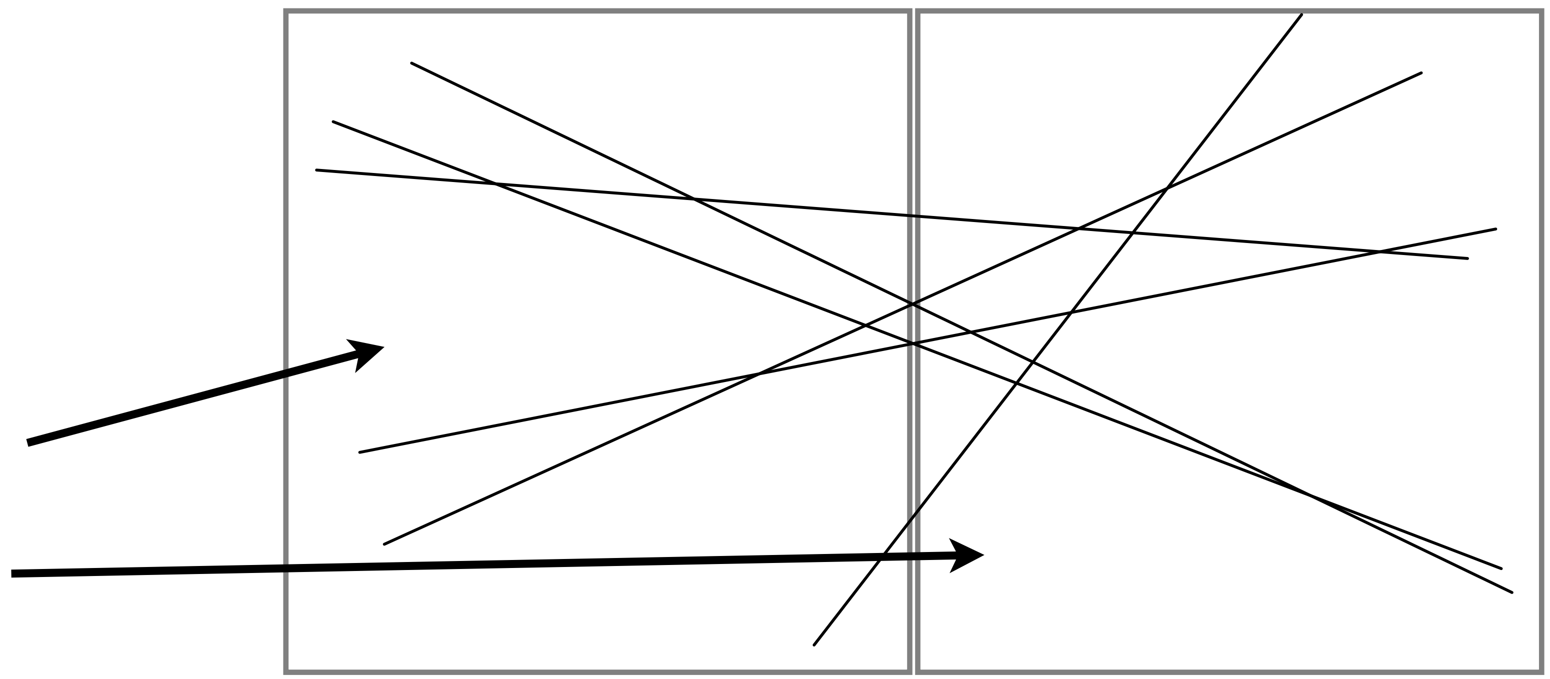
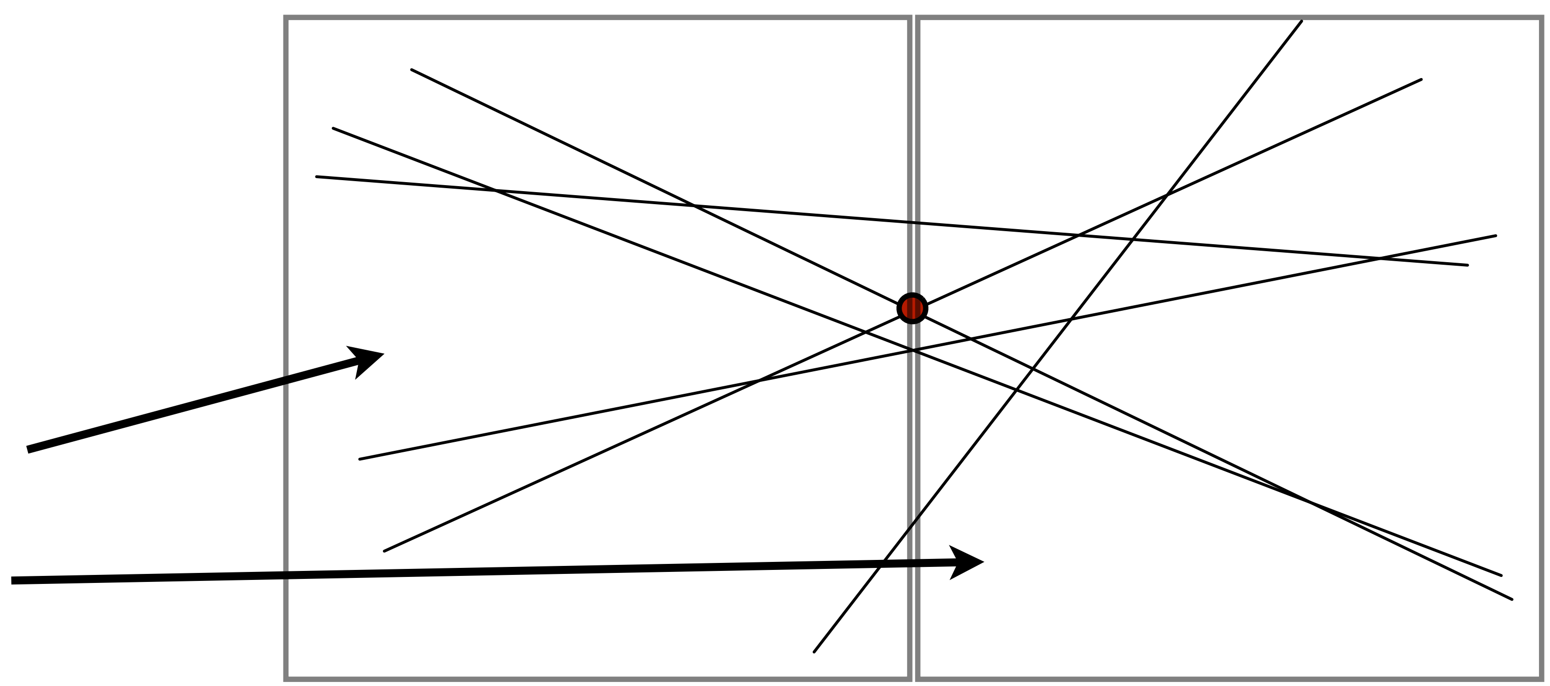


Binary search



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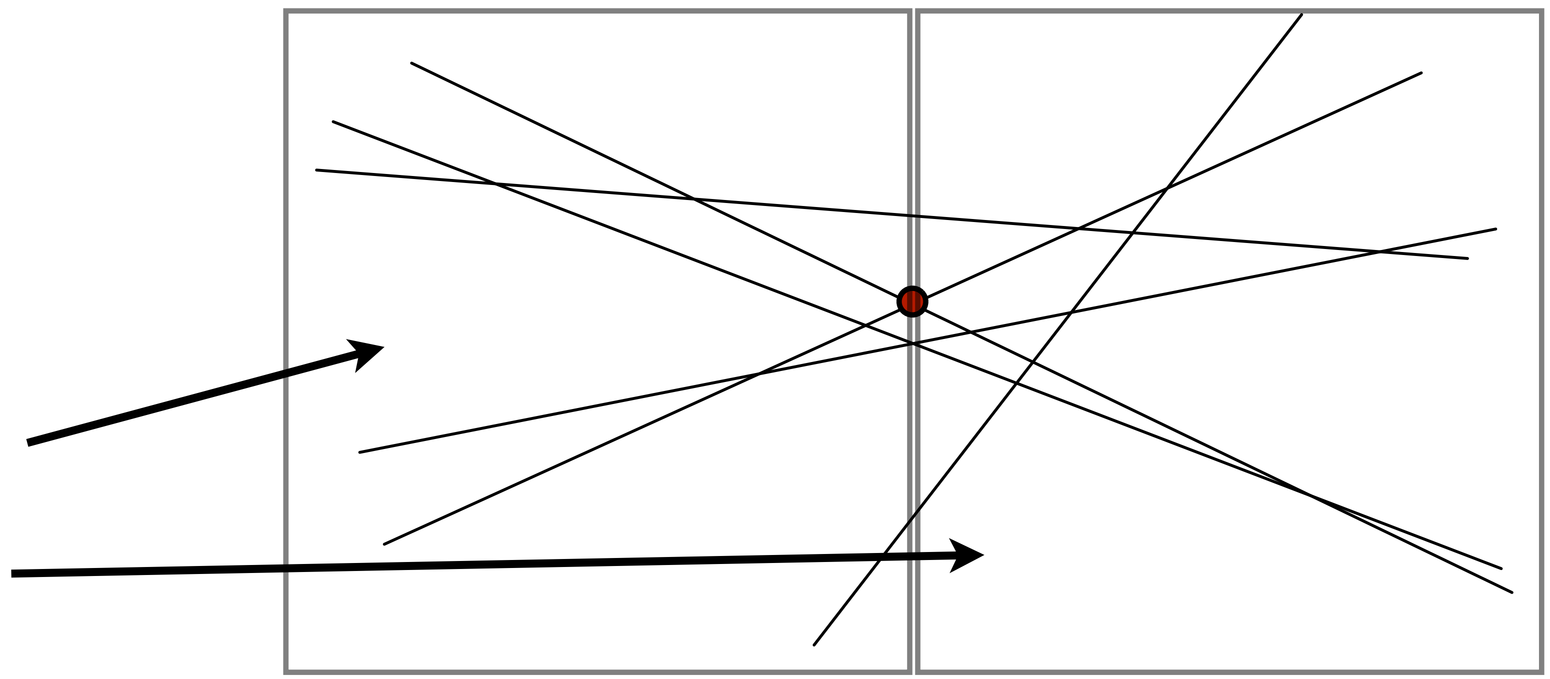

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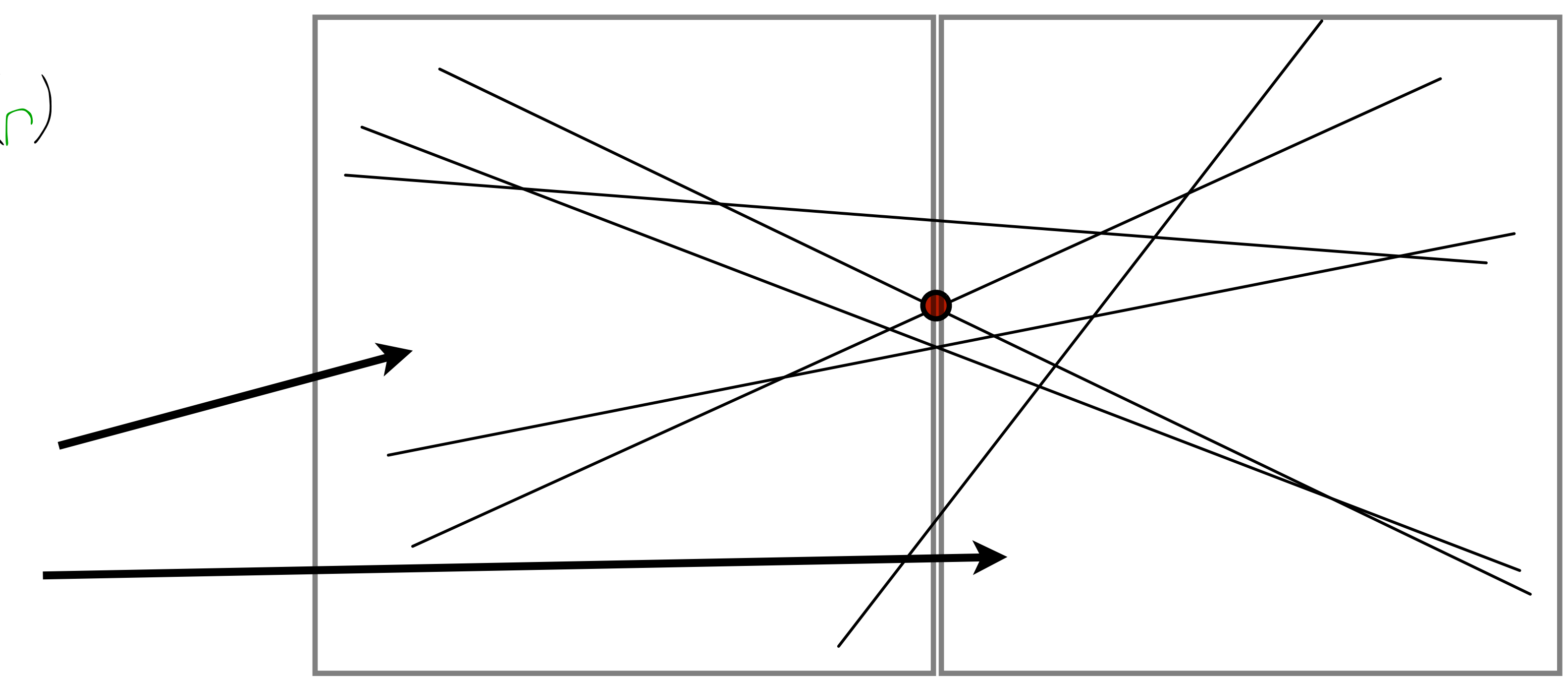
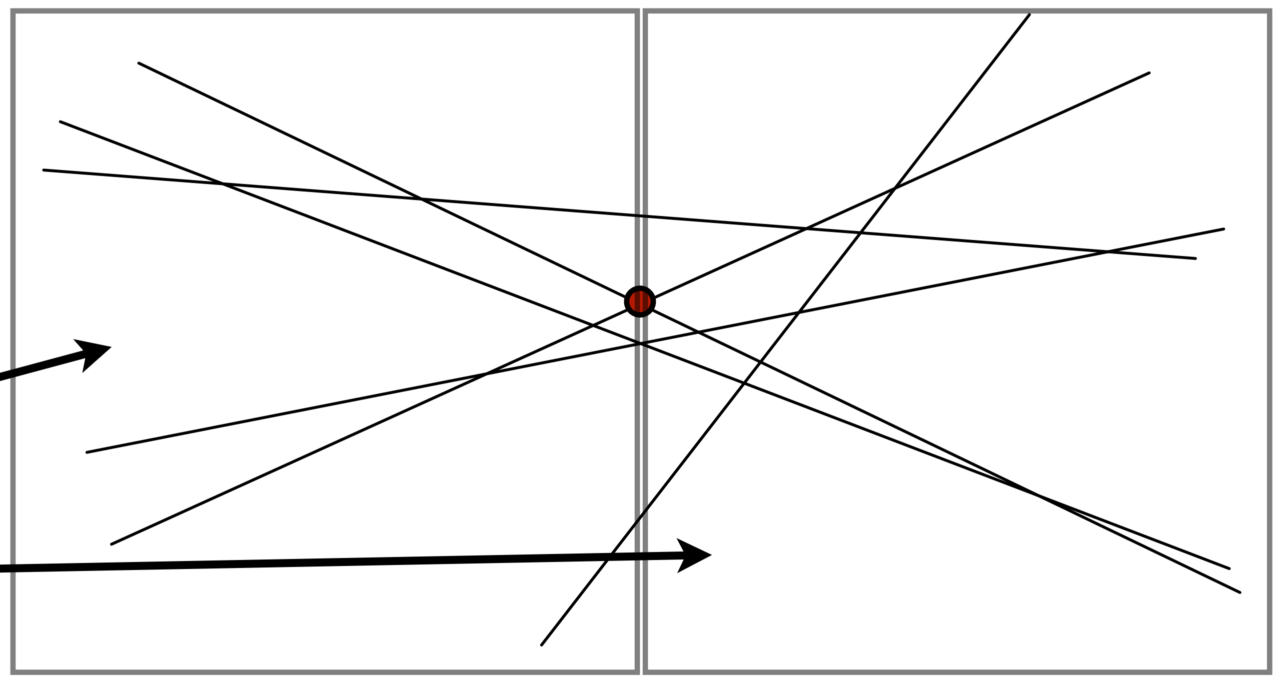


Binary search

- Pick a vertex at random
- Rank the vertex (r)
- If r=k, done!
  if r>k, recurse left,
  if r<k recurse right



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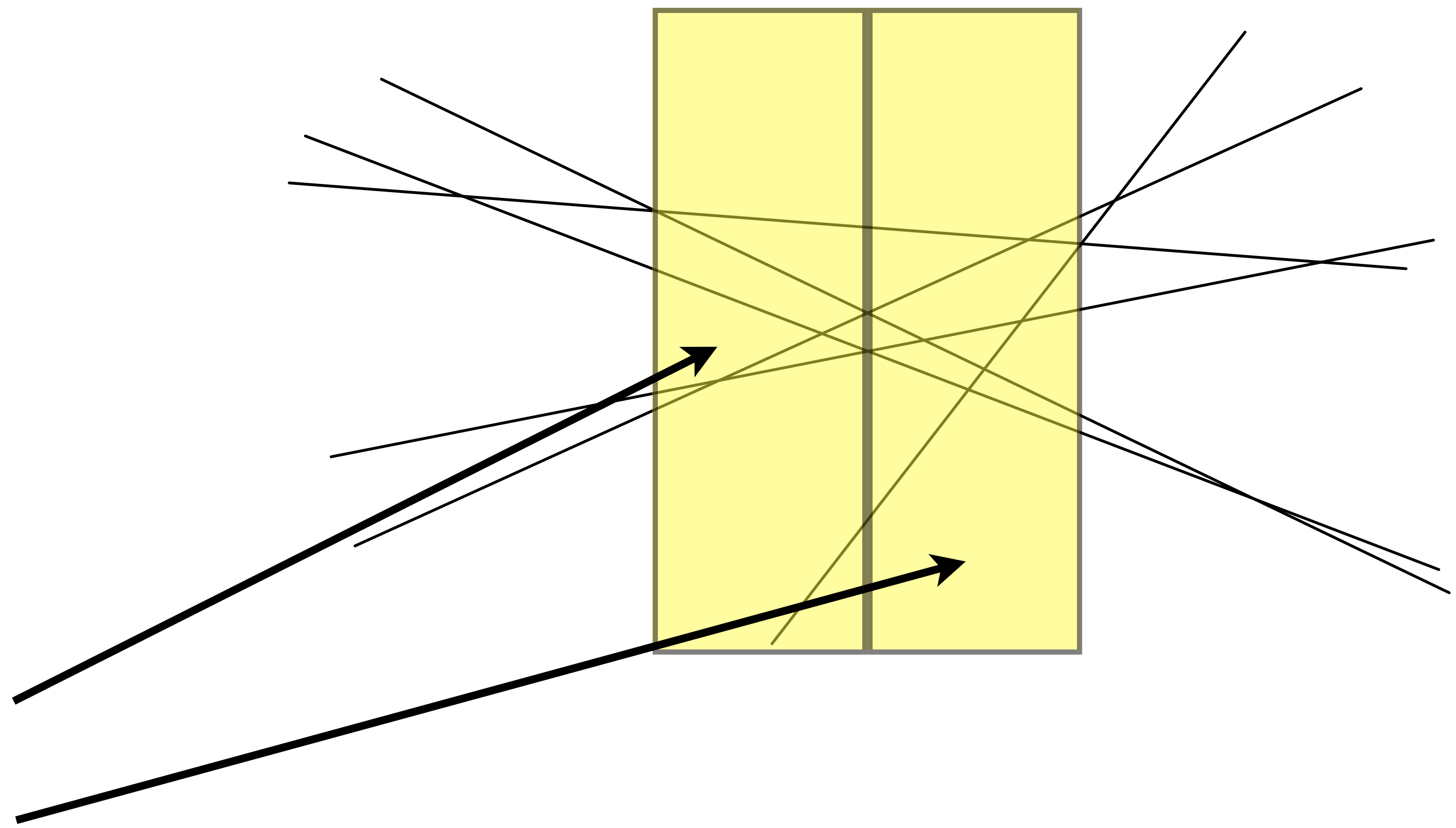
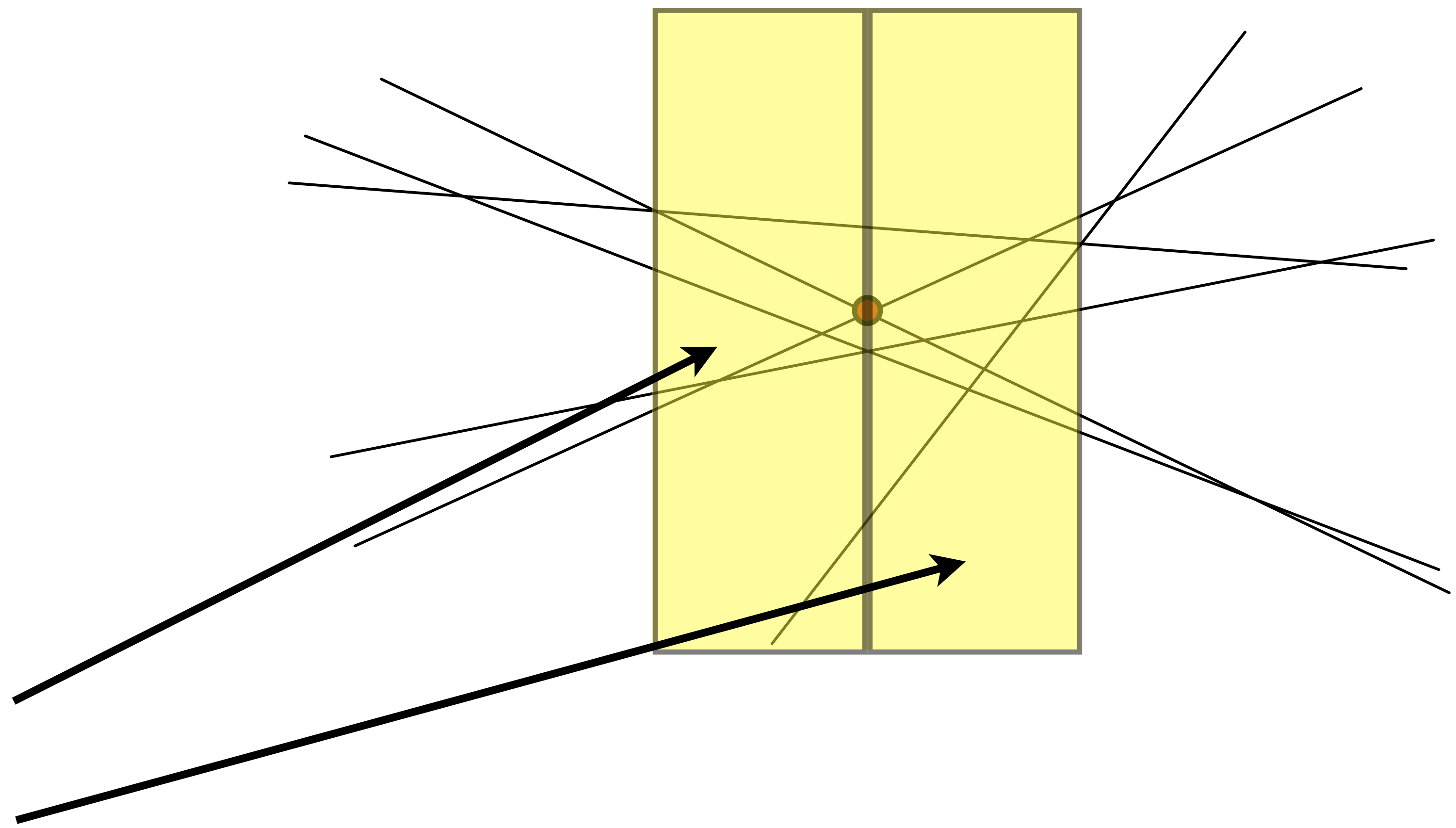


Expected $O(\log n)$ recursion steps

-> $O(n \log^2 n)$ expected time

Binary search

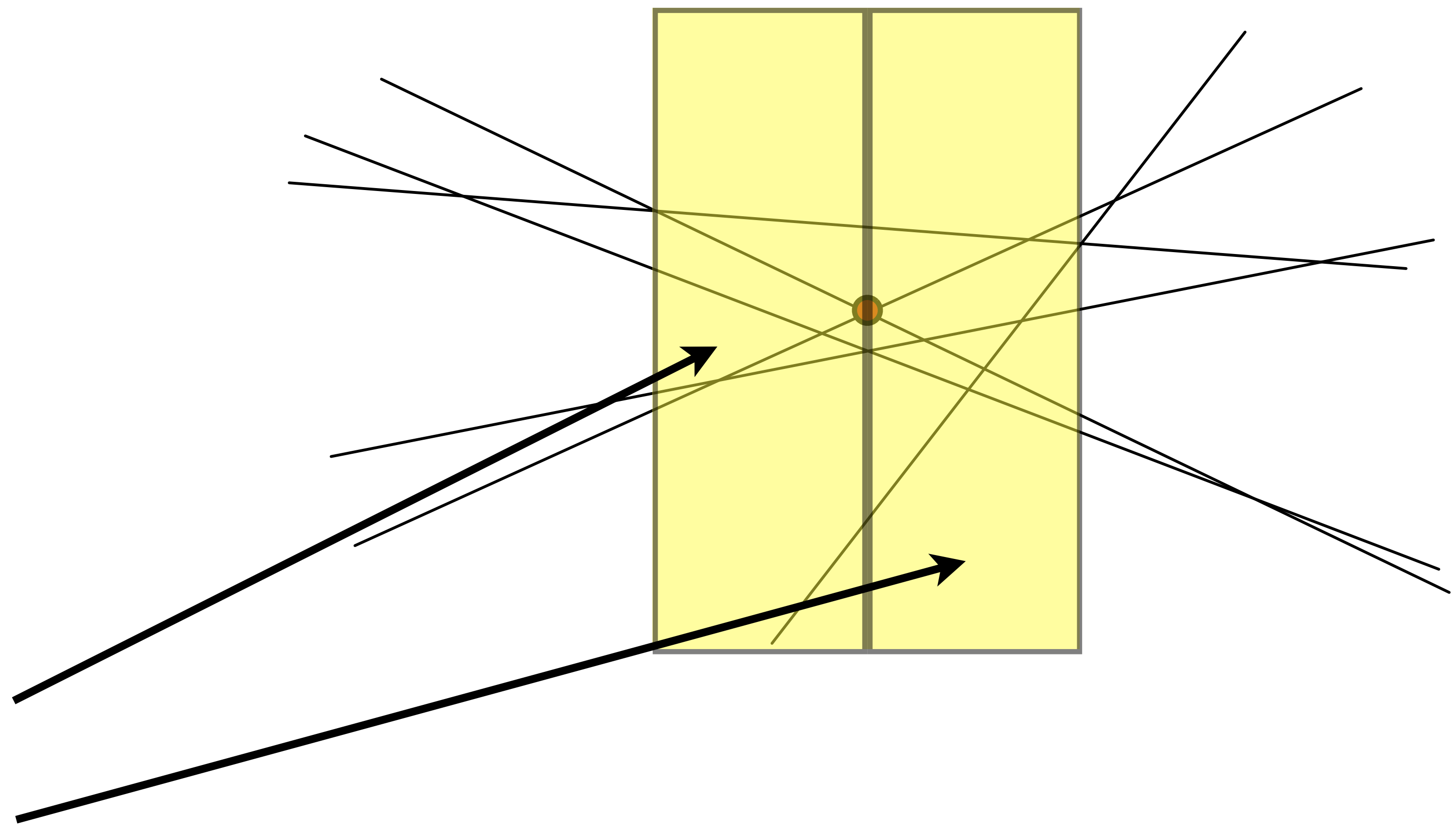


Expected O(log n) recursion steps

-> O(n log² n) expected time

Binary search

- Pick a vertex at random inside the active slab
  O(n log n)

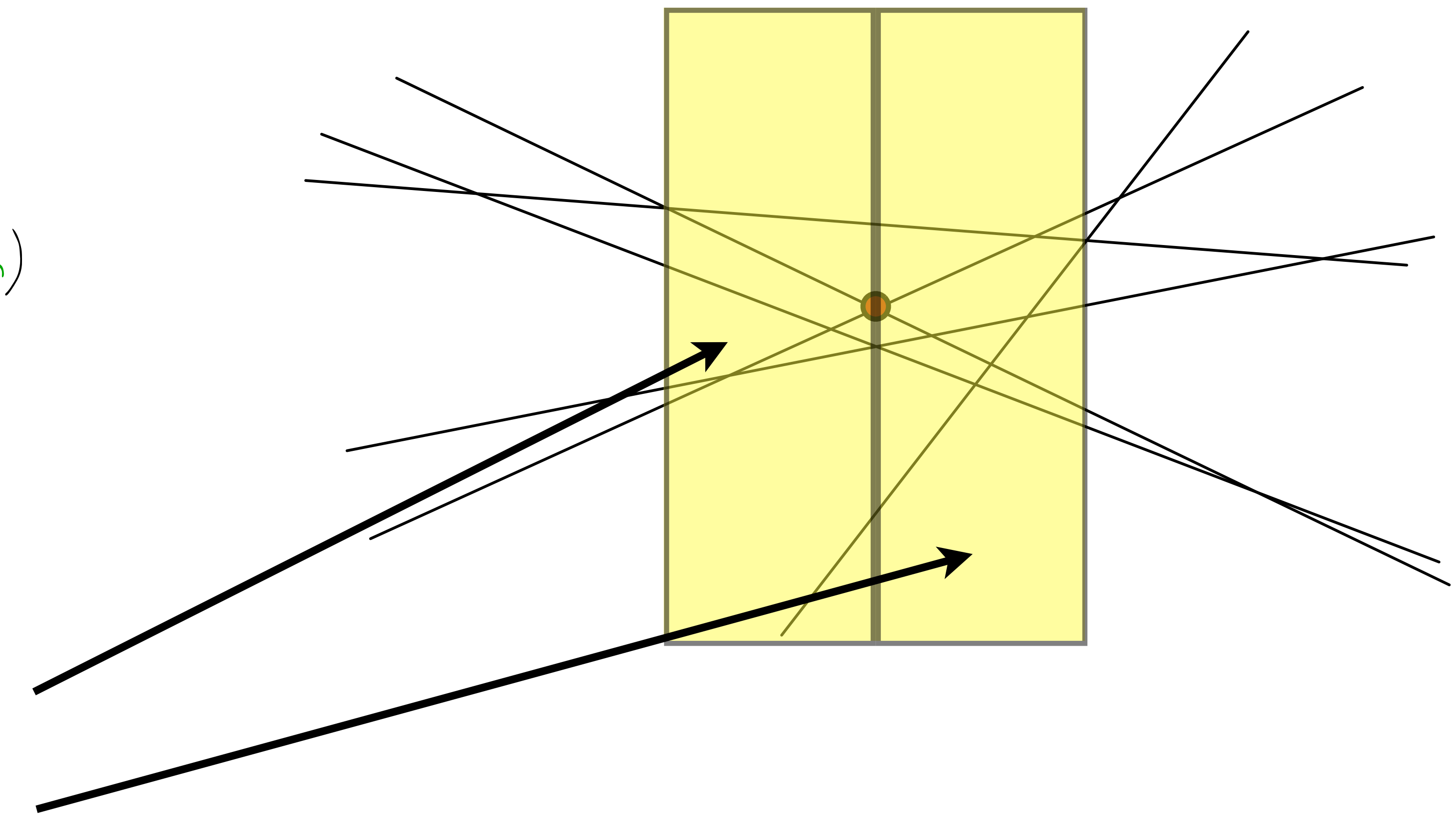


Expected O(log n) recursion steps

-> $O(n \log^2 n)$ expected time

Binary search

- Pick a vertex at random inside the active slab
  O(n log n)
- Rank the vertex (r)
  O(n log n)



Expected O(log n) recursion steps
-> O(n log² n) expected time

Binary search

- Pick a vertex at random inside the active slab
  O(n log n)
- Rank the vertex (r)
  O(n log n)
- If r=k, done!
  if r>k, recurse left,
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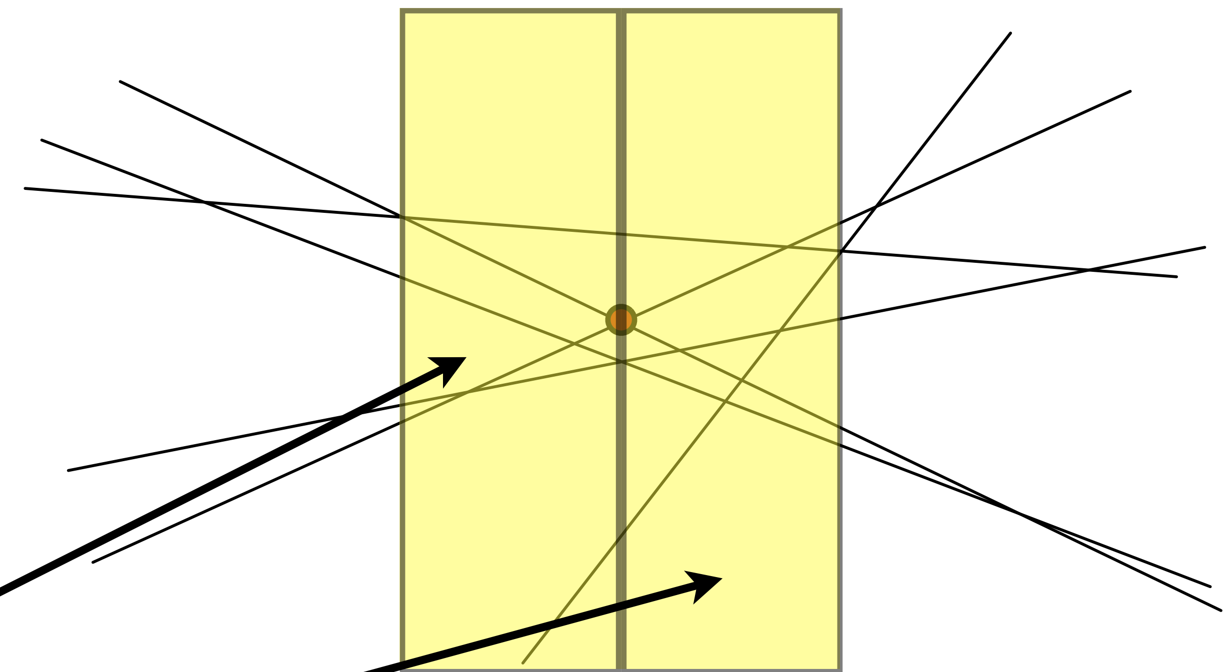


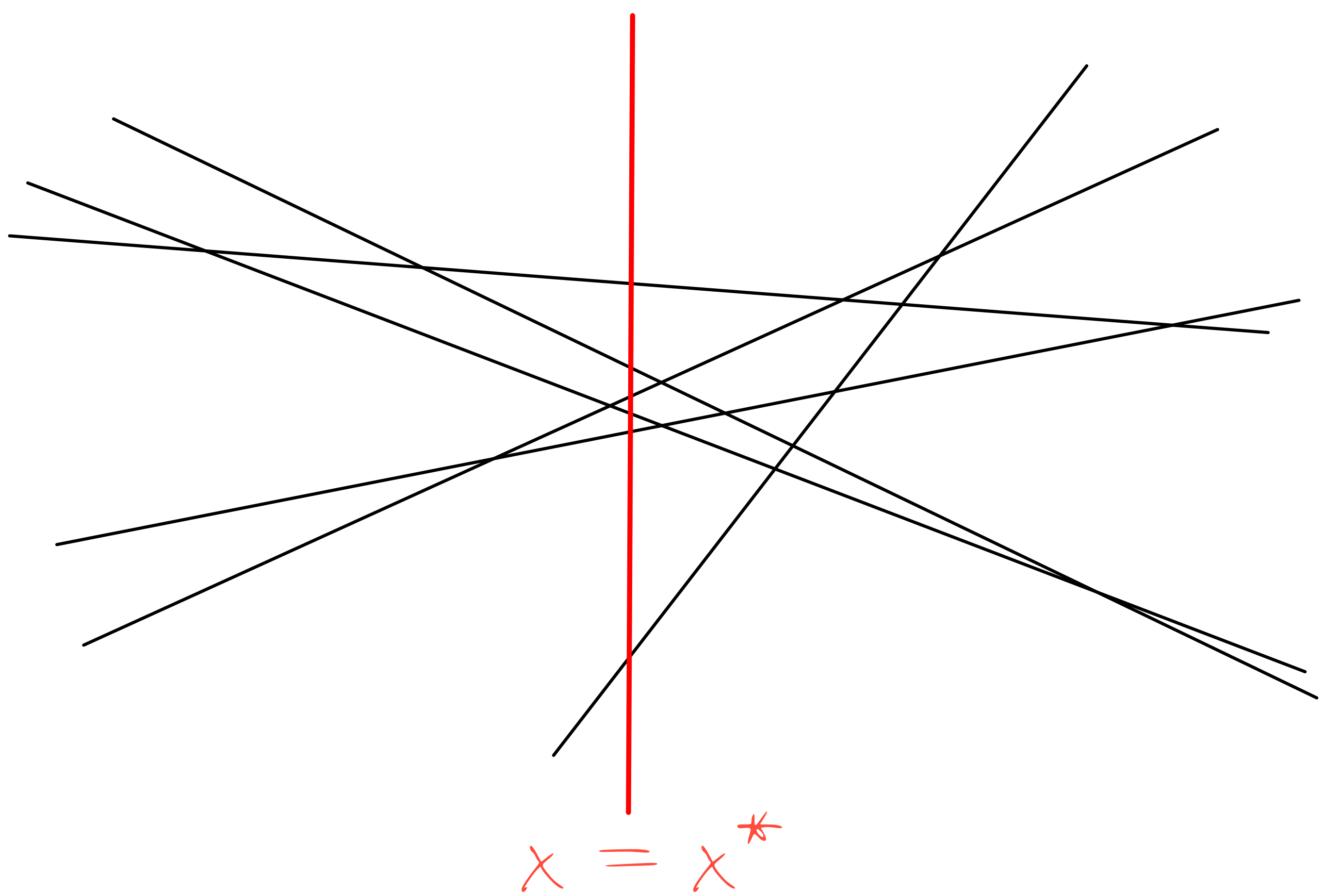
Expected O(log n) recursion steps

-> $O(n \log^2 n)$ expected time

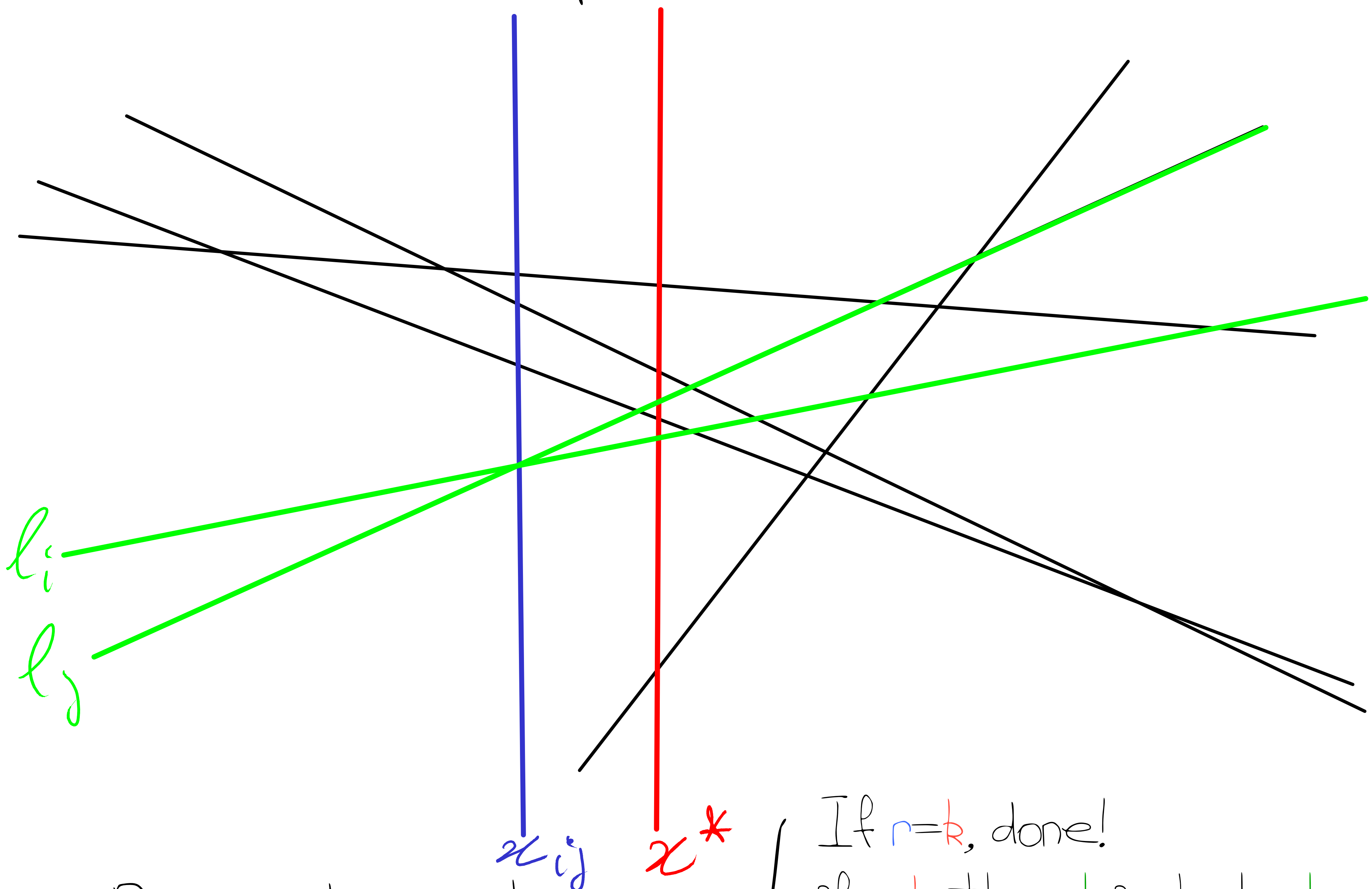
Deterministic?

Use Megiddo's parametric search to transform ranking to selection:

run the ranking algorithm at $x = x^*$



Comparison



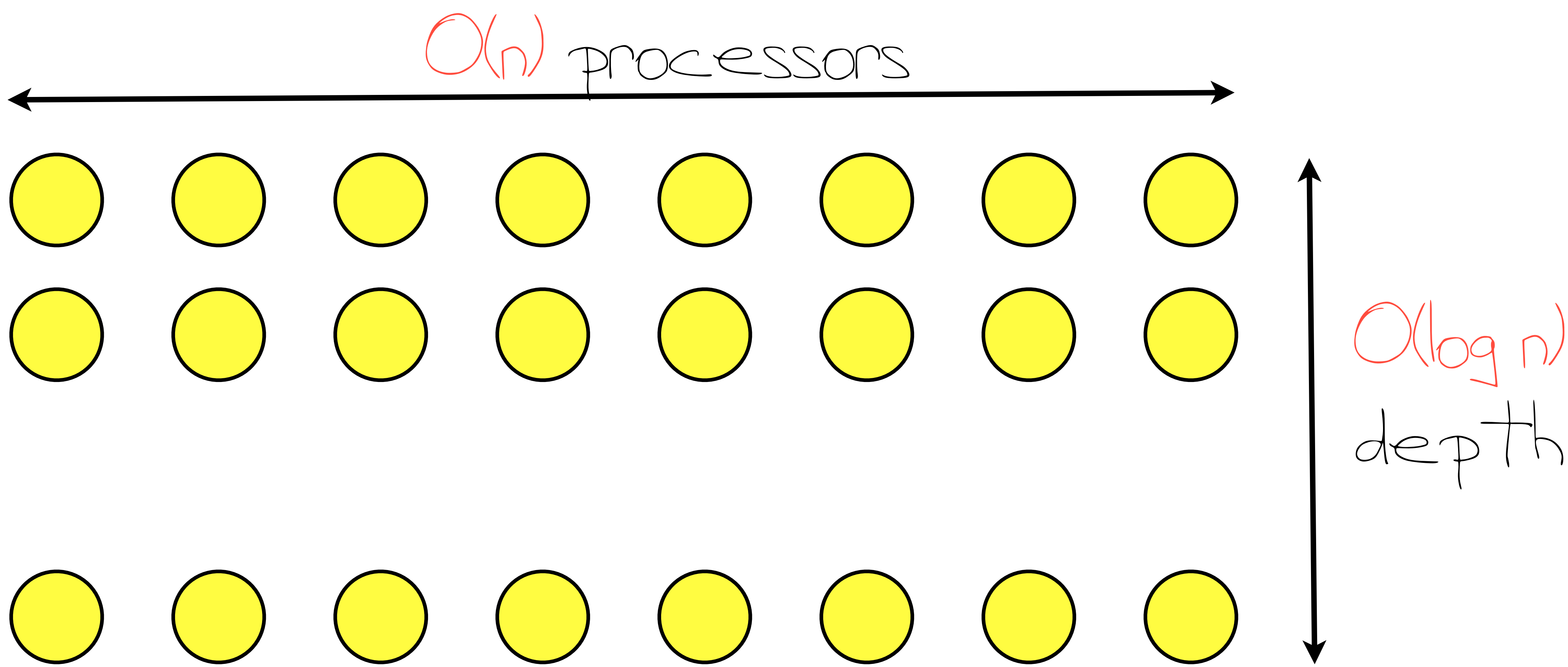
Run ranking at $x = x_{ij}$.

- If $r = k$, done!
- if $r < k$ then l_i is below l_j,
- if $r > k$ then l_i is above l_j.

Running ranking at $x = x^*$

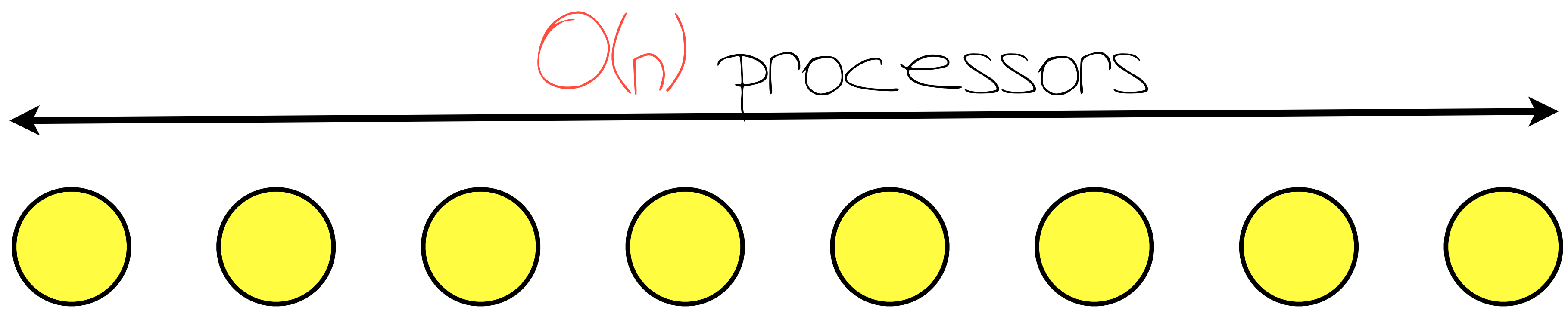
- Each comparison = ranking $O(n \log n)$
- Ranking at $x = x^*$ uses $O(n \log n)$ comparisons so $O(n^2 \log^2 n)$
- Too much! But...

Rank/Sort using parallel sorting network



Each processor does one comparison = is x_{ij} left of x^*?

Look at the first level



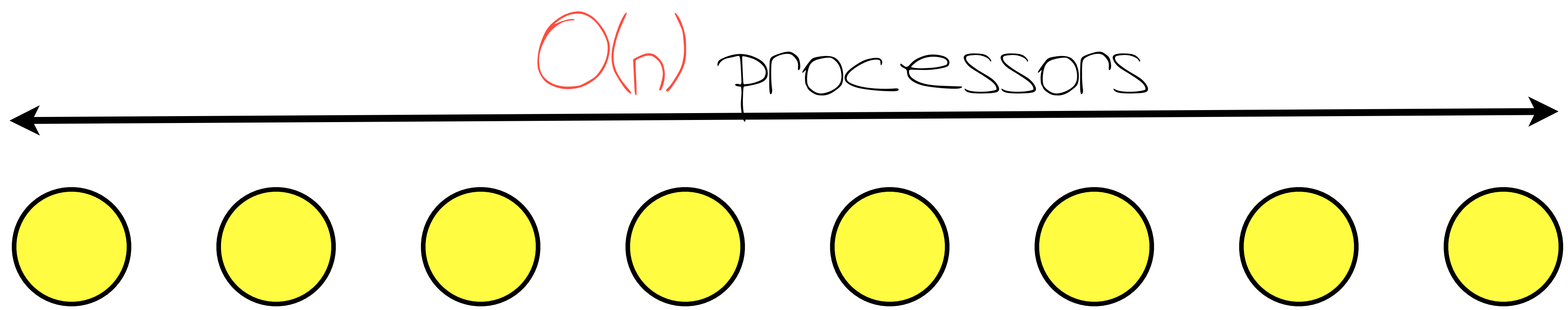
Each processor does one comparison =

is x_{ij} left of x^*?

Sort the x_{ij}:

$x_0 < x_1 < x_2 < x_3 < x_4 < x_5 < \dots < x_n$

Look at the first level



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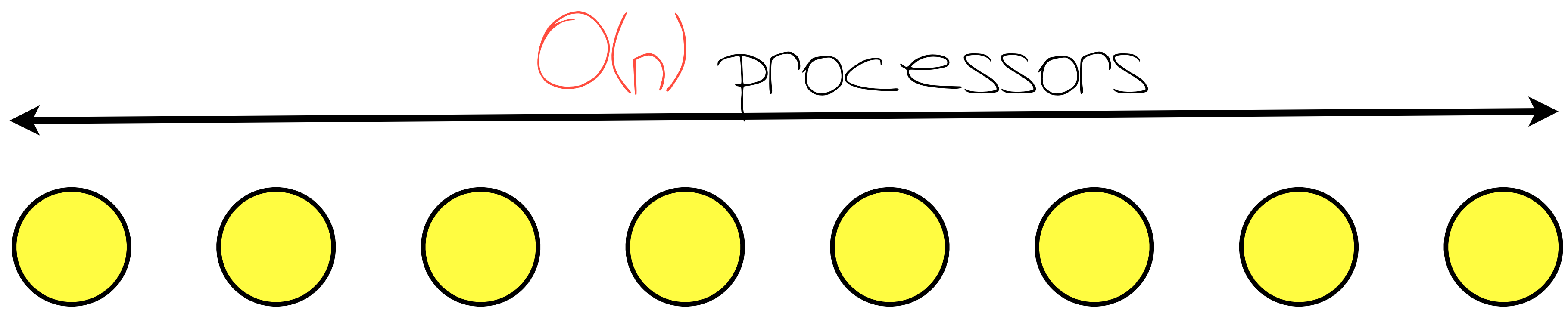
Can answer half of them with one ranking

Look at the first level



Each processor does one comparison =

is x_{ij} left of x^*?

Can answer half of them with one ranking

$O(\log n)$ rankings for one level

$O(\log n)$ levels -> $O(n \log^3 n)$

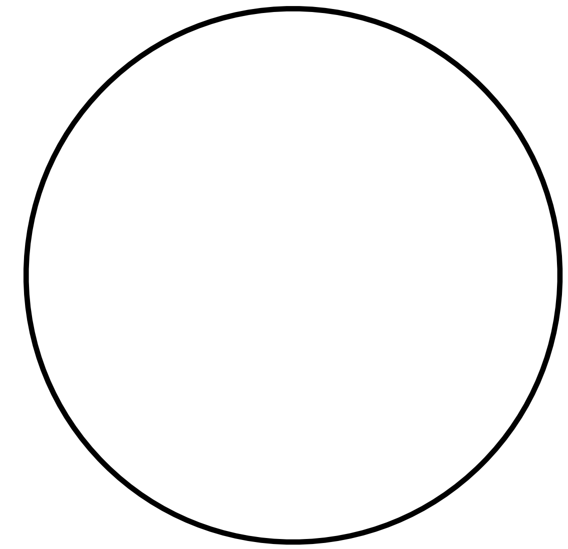
Can be improved to $O(n \log n)$

[Cole Salowe Steiger Szemeredi 1989]

Fast and easy?

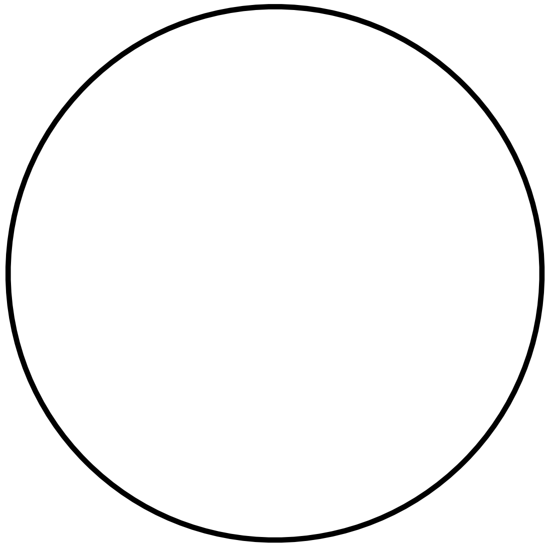
More Tools

ε- Nets



- Set S of n points
- Family R = all subsets of S that are contained in some disk
- N ⊆ S is an **ε**- net for (S,R) if every set in R that contains more than **ε** n points contains a witness in N

ε- Nets



- Set S of n points
- Family R = all subsets of S that are contained in some disk
- N ⊆ S is an **ε**- net for (S,R) if every set in R that contains more than **ε** n points contains a witness in N
- Thm: **ε**- nets of size $O((1/\varepsilon)\log(1/\varepsilon))$ always exist

ε- Nets

- Set S of n points
- Family R = all subsets of S that are contained in some halfplane
- N ⊆ S is an ε- net for (S,R) if every set in R that contains more than ε n points contains a witness in N
- Thm: ε- nets of size $O((1/\varepsilon)\log(1/\varepsilon))$ always exist

ε-Nets

- Set S of n lines in R^2
- Family R = all subsets of S that intersect some segment
- N ⊆ S is an ε-net for (S,R) if every set in R that contains more than ε n lines contains a witness in N
- Thm: ε-nets of size $O((1/\varepsilon)\log(1/\varepsilon))$ always exist

ε-Nets

- Set S of n hyperplanes in R^d
- Family R = all subsets of S that intersect some simplex
- $N \subseteq S$ is an ε-net for (S,R) if every set in R that contains more than ε n hyperplanes contains a witness in N
- Thm: ε-nets of size $O((d/\varepsilon)\log(d/\varepsilon))$ always exist

ε-Nets

- Set S of n elements
- Family R = subsets of S of finite VC-Dimension d
- $N \subseteq S$ is an ε-net for (S,R) if every set in R that contains more than εn elements contains a witness in N
- Thm: ε-nets of size $O((d/\varepsilon)\log(d/\varepsilon))$ always exist

Questions

- What is the best partition of n lines by 3 lines? 4? ...
- For red and blue sets of points, caracterize the (r,b) for which ther is a line with r red points and b blue points above.
- Ham-sandwich cuts in a plane with 2 speeds.
- How many measures can we split with a triangle, etc...