Those ubiquitous cut polyhedra

CCCG 2010 Paul Erdös Lecture

This lecture is dedicated to Hazel Everett (1963-2010)

David Avis

Kyoto University and McGill University

August 9, 2010

Backdrop

L_1-embedding

Hypercubes

Correlations

Cut Polytope

Quantum correlations

Mining

August 3, 1975: Oakland Stadium

about:blank

Photo: Jim Marshall

1

Meanwhile at San Francisco airport...

Meanwhile at San Francisco airport...

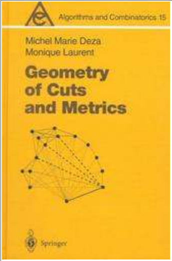
Photo: Adrian Bondy

Geometry of Cuts and Metrics

about:blank

Michel Deza **Monique Laurent**

and a book ......



Equidistant points in L_2

Equidistant points in L_2

- Easy question: How many L_2-equidistant points are there in R^n?

Equidistant points in L_2

- Easy question: How many L_2-equidistant points are there in R^n?
- Answer: $n + 1$

Equidistant points in L_2

- Easy question: How many L_2-equidistant points are there in R^n?
- Answer: $n + 1$
- Hard question: How many pairs of unit L_2-distances can a planar set of n points have?

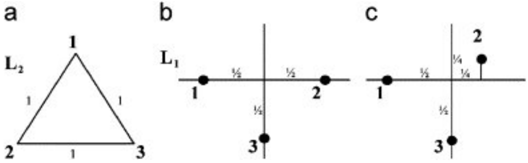
Equidistant points in L_2

- Easy question: How many L_2-equidistant points are there in R^n?
- Answer: $n+1$
- Hard question: How many pairs of unit L_2-distances can a planar set of n points have?
- Answer: Erdös knows now, but he is not telling ......

Equidistant points in L_2

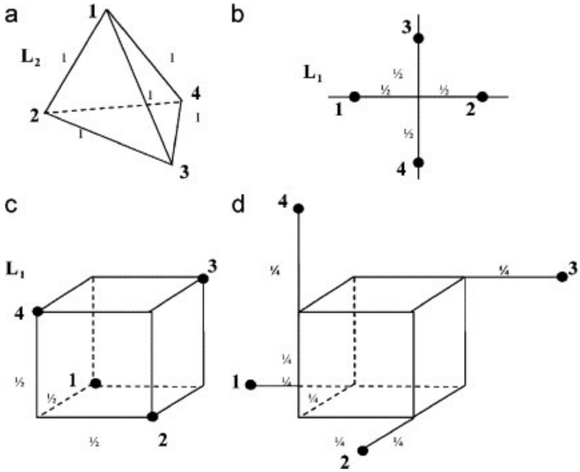
- Easy question: How many L_2-equidistant points are there in R^n?
- Answer: $n + 1$
- Hard question: How many pairs of unit L_2-distances can a planar set of n points have?
- Answer: Erdös knows now, but he is not telling ......
- How about L_1?

Three equidistant points: L_1 vs L_2



From: Fichet (DAM,2008)

Four equidistant points: L_1 vs L_2



From: Fichet (DAM,2008)

The equilateral problem in L_1

Let $e(L_1^n)$ = maximum number of L_1-equidistant points in R^n.

The equilateral problem in L_1

Let $e(L_1^n)$ = maximum number of L_1-equidistant points in R^n.

- Kusner's Conjecture(1983): $e(L_1^n) = 2n$

The equilateral problem in L_1

Let $e(L_1^n)$ = maximum number of L_1-equidistant points in R^n.

- Kusner's Conjecture(1983): $e(L_1^n) = 2n$
- H.-J. Bandelt, V. Chepoi and M. Laurent (1998): True for $d = 3$

The equilateral problem in L_1

Let $e(L_1^n)$ = maximum number of L_1-equidistant points in R^n.

- Kusner's Conjecture(1983): $e(L_1^n) = 2n$
- H.-J. Bandelt, V. Chepoi and M. Laurent (1998):
  True for $d = 3$
- J. Koolen, M. Laurent and A. Schrijver (2000):
  True for $d = 4$

The equilateral problem in L_1

Let $e(L_1^n)$ = maximum number of L_1-equidistant points in R^n.

- Kusner's Conjecture(1983): $e(L_1^n) = 2n$
- H.-J. Bandelt, V. Chepoi and M. Laurent (1998):
  True for $d = 3$
- J. Koolen, M. Laurent and A. Schrijver (2000):
  True for $d = 4$
- N. Alon and P. Pudlák (2003):
  $e(L_1^n) = O(nlogn)$

Embedding problems

- A distance is a non-negative vector $d = (d_{ij}), 1 \leq i < j \leq n$.

Embedding problems

- A distance is a non-negative vector $d = (d_{ij}), 1 \leq i < j \leq n$.
- d is L_p-embeddable if $\exists\ u^1, u^2, ..., u^n \in R^m$ s.t.

$$d_{ij} = \|u^i - u^j\|_p, \qquad 1 \leq i < j \leq n$$

Embedding problems

- A distance is a non-negative vector $d = (d_{ij}), 1 \leq i < j \leq n$.
- d is L_p-embeddable if $\exists\ u^1, u^2, ..., u^n \in R^m$ s.t.

$$d_{ij} = \|u^i - u^j\|_p, \qquad 1 \leq i < j \leq n$$

- We are concerned with exact embedding for $p = 1, 2$

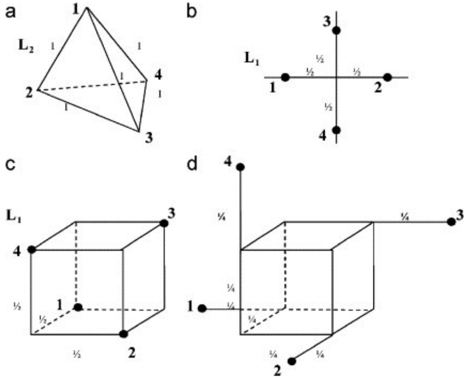
Embedding problems

- A distance is a non-negative vector $d = (d_{ij}), 1 \leq i < j \leq n$.
- d is L_p-embeddable if $\exists$ $u^1, u^2, ..., u^n \in R^m$ s.t.

$$d_{ij} = \|u^i - u^j\|_p, \qquad 1 \leq i < j \leq n$$

- We are concerned with exact embedding for $p = 1, 2$
- Which distances are L_P-embeddable?

Four equidistant points: L_1 vs L_2



(a) $u^1 = (0,0,0)$ $u^2 = (1,0,0)$ $u^3 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ $u^4 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{3}})$

(b) $u^1 = (-\frac{1}{2}, 0)$ $u^2 = (\frac{1}{2}, 0)$ $u^3 = (0, \frac{1}{2})$ $u^4 = (0, -\frac{1}{2})$

(c) $u^1 = (0,0,0)$ $u^2 = (\frac{1}{2}, \frac{1}{2}, 0)$ $u^3 = (\frac{1}{2}, 0, \frac{1}{2})$ $u^4 = (0, \frac{1}{2}, \frac{1}{2})$

(d) $u^1 = (-\frac{1}{4}, 0, 0)$ $u^2 = (\frac{1}{4}, -\frac{1}{2}, 0)$ $u^3 = (\frac{1}{2}, 0, \frac{1}{4})$ $u^4 = (0, -\frac{1}{4}, \frac{1}{2})$

Hypercube embedding

- d is *h-embeddable* if $\exists$ vertices $u^1, u^2, \ldots, u^n$ of a unit hypercube H_m s.t. d_{ij} is the Hamming distance from u^i to u^j.

Hypercube embedding

- d is *h*-embeddable if $\exists$ vertices $u^1, u^2, ..., u^n$ of a unit hypercube H_m s.t. d_{ij} is the Hamming distance from u^i to u^j.
- d is *h*-embeddable if $\exists$ $u^1, u^2, ..., u^n \in 0, 1^m$ s.t.

$$d_{ij} = \|u^i - u^j\|_1, \qquad 1 \leq i < j \leq n$$

Hypercube embedding

- d is *h-embeddable* if $\exists$ vertices $u^1, u^2, ..., u^n$ of a unit hypercube H_m s.t. d_{ij} is the Hamming distance from u^i to u^j.
- d is *h-embeddable* if $\exists$ $u^1, u^2, ..., u^n \in 0, 1^m$ s.t.

$$d_{ij} = \|u^i - u^j\|_1, \qquad 1 \leq i < j \leq n$$

- d is h-embeddable $\Rightarrow$ d is L_1-embeddable.

Hypercube embedding

- d is *h*-embeddable if $\exists$ vertices $u^1, u^2, ..., u^n$ of a unit hypercube H_m s.t. d_{ij} is the Hamming distance from u^i to u^j.
- d is *h*-embeddable if $\exists$ $u^1, u^2, ..., u^n \in 0, 1^m$ s.t.

$$d_{ij} = \|u^i - u^j\|_1, \qquad 1 \leq i < j \leq n$$

- d is h-embeddable $\Rightarrow$ d is L_1-embeddable.
- A rational d is L_1-embeddable $\Rightarrow$ kd is h-embeddable for some integer k.

Testing h-embeddability: complexity

- In P if d is a graph metric. [Dj73]

Testing h-embeddability: complexity

- In P if d is a graph metric. [Dj73]
- NP-complete if $d_{ij} \in \{2, 3, 4, 6\}$ [Ch80]

Testing h-embeddability: complexity

- In P if d is a graph metric. [Dj73]
- NP-complete if $d_{ij} \in \{2, 3, 4, 6\}$ [Ch80]
- In P if $d_{ij} \in \{1, 2, 3\}$ [Av90]

Testing h-embeddability: complexity

- In P if d is a graph metric. [Dj73]
- NP-complete if $d_{ij} \in \{2, 3, 4, 6\}$ [Ch80]
- In P if $d_{ij} \in \{1, 2, 3\}$ [Av90]
- In P if $d_{ij} \in \{x, y, x + y\}$ where only one of x, y is odd. [La93]

Testing h-embeddability: complexity

- In P if d is a graph metric. [Dj73]
- NP-complete if $d_{ij} \in \{2, 3, 4, 6\}$ [Ch80]
- In P if $d_{ij} \in \{1, 2, 3\}$ [Av90]
- In P if $d_{ij} \in \{x, y, x + y\}$ where only one of x, y is odd. [La93]
- (curve ball)

Testing h-embeddability: complexity

- In P if d is a graph metric. [Dj73]
- NP-complete if $d_{ij} \in \{2, 3, 4, 6\}$ [Ch80]
- In P if $d_{ij} \in \{1, 2, 3\}$ [Av90]
- In P if $d_{ij} \in \{x, y, x + y\}$ where only one of x, y is odd. [La93]
- (curve ball)
- Is testing h-embeddability in NP?

Hypercube embedding and symmetric differences

- The following two statements are equivalent for a distance d on n points:

Hypercube embedding and symmetric differences

- The following two statements are equivalent for a distance d on n points:
- d is h-embeddable to vertices $u^1, u^2, \ldots, u^n$ of H_m

Hypercube embedding and symmetric differences

- The following two statements are equivalent for a distance d on n points:
- d is h-embeddable to vertices $u^1, u^2, \ldots, u^n$ of H_m
- $\exists\ A_1, A_2, \ldots, A_n \subseteq \{1, 2, \ldots, m\}$ s.t.

$$d_{ij} = |A_i \triangle A_j|$$

where $\triangle$ is symmetric difference.

Hypercube embedding and symmetric differences

- The following two statements are equivalent for a distance d on n points:
- d is h-embeddable to vertices $u^1, u^2, \ldots, u^n$ of H_m
- $\exists\ A_1, A_2, \ldots, A_n \subseteq \{1, 2, \ldots, m\}$ s.t.

$$d_{ij} = |A_i \triangle A_j|$$

  where $\triangle$ is symmetric difference.
- Proof:

Hypercube embedding and symmetric differences

- The following two statements are equivalent for a distance d on n points:
- d is h-embeddable to vertices $u^1, u^2, ..., u^n$ of H_m
- $\exists\ A_1, A_2, ..., A_n \subseteq \{1, 2, ..., m\}$ s.t.

$$d_{ij} = |A_i \triangle A_j|$$

  where $\triangle$ is symmetric difference.
- Proof:
- $u^i_k = 1 \Leftrightarrow k \in A_i, \qquad k = 1, ..., m \quad i = 1, ..., n$

(anti)-Correlations

- Let $A, B, C, \ldots$ be events in a probability space.

(anti)-Correlations

- Let $A, B, C, \ldots$ be events in a probability space.
- $A \triangle B$ is the event that exactly one of A and B occurs.

(anti)-Correlations

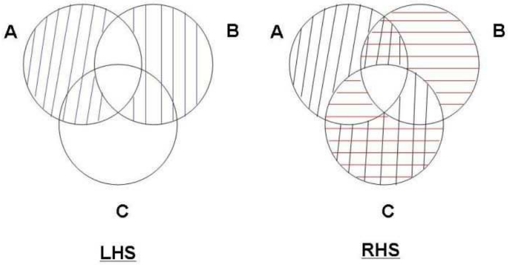
- Let $A, B, C, \dots$ be events in a probability space.
- $A \triangle B$ is the event that exactly one of A and B occurs.
- $Pr(A \triangle B) \leq Pr(A \triangle C) + Pr(B \triangle C)$

(anti)-Correlations

- Let $A, B, C, \ldots$ be events in a probability space.
- $A \triangle B$ is the event that exactly one of A and B occurs.
- $Pr(A \triangle B) \leq Pr(A \triangle C) + Pr(B \triangle C)$
- Proof:

(anti)-Correlations

- Let $A, B, C, \ldots$ be events in a probability space.
- $A \triangle B$ is the event that exactly one of A and B occurs.
- $Pr(A \triangle B) \leq Pr(A \triangle C) + Pr(B \triangle C)$
- Proof:



-

MenOnline.com

KKK Dating claims to supply the following men:

- 90% are either rich **or** university educated,
- 50% are either rich **or** tall,
- 35% are either tall **or** university educated,

MenOnline.com

KKK Dating claims to supply the following men:

- 90% are either rich **or** university educated,
- 50% are either rich **or** tall,
- 35% are either tall **or** university educated,

Alice doubts this, since if
A={rich}, B={univ. edu.}, C={tall}

$$Pr(A \triangle B) = .9 > .85 = Pr(A \triangle C) + Pr(B \triangle C)$$

MenOnline.com

- KKK Dating claims to supply men with these properties:

tall, handsome, rich, strong, intelligent

MenOnline.com

- KKK Dating claims to supply men with these properties:

  tall, handsome, rich, strong, intelligent

- They claim that for every pair of properties, at least 62.5% of the men have exactly one the properties.

MenOnline.com

- KKK Dating claims to supply men with these properties:

  tall, handsome, rich, strong, intelligent

- They claim that for every pair of properties, at least 62.5% of the men have exactly one the properties.
- For 4 properties tall, handsome, rich, strong it is possible:
  $B = \{$tall,strong$\}$, $D = \{$tall,handsome$\}$,
  $J = \{$strong,handsome$\}$

MenOnline.com

- KKK Dating claims to supply men with these properties:

  tall, handsome, rich, strong, intelligent

- They claim that for every pair of properties, at least 62.5% of the men have exactly one the properties.
- For 4 properties tall, handsome, rich, strong it is possible:
  $B = \{\text{tall,strong}\}$, $D = \{\text{tall,handsome}\}$,
  $J = \{\text{strong,handsome}\}$
- For each pair of properties, two of the three have exactly one of them.

MenOnline.com

- KKK Dating claims to supply men with these properties:

  tall, handsome, rich, strong, intelligent

- They claim that for every pair of properties, at least 62.5% of the men have exactly one the properties.
- For 4 properties tall, handsome, rich, strong it is possible: $B = \{\text{tall,strong}\}$, $D = \{\text{tall,handsome}\}$, $J = \{\text{strong,handsome}\}$
- For each pair of properties, two of the three have exactly one of them.
- For 5 properties Alice has doubts...

Correlations

- Let $A_1, \ldots, A_n$ be events in a joint probability space.

Correlations

- Let $A_1, \ldots, A_n$ be events in a joint probability space.
- Define $\quad x_{ij} = Pr(A_i \triangle A_j), \quad 1 \leq i < j \leq n$

Correlations

- Let $A_1, \ldots, A_n$ be events in a joint probability space.
- Define $x_{ij} = Pr(A_i \triangle A_j), \quad 1 \leq i < j \leq n$
- Problem: Characterize vectors $x = (x_{ij})$ for which there exists a joint probability space.

Correlations

- Let $A_1, \ldots, A_n$ be events in a joint probability space.
- Define $\quad x_{ij} = Pr(A_i \triangle A_j), \quad 1 \leq i < j \leq n$
- Problem: Characterize vectors $x = (x_{ij})$ for which there exists a joint probability space.
- Answer: Cut polytope (Av77)

Cut Polytope: definition

- Let $S \subseteq \{1, \ldots, n\}$ and $\oplus$ denote exclusive or.

Cut Polytope: definition

- Let $S \subseteq \{1, \ldots, n\}$ and $\oplus$ denote exclusive or.
- Define

$$x_{ij}^{S} = \begin{cases} 1 & i \oplus j \in S \\ 0 & otherwise \end{cases} \qquad 1 \leq i < j \leq n$$

Cut Polytope: definition

- Let $S \subseteq \{1, \ldots, n\}$ and $\oplus$ denote exclusive or.
- Define

$$x_{ij}^S = \begin{cases} 1 & i \oplus j \in S \\ 0 & otherwise \end{cases} \qquad 1 \le i < j \le n$$

- x^S is the edge-incidence vector of the cut $[S, V - S]$ in K_n.

Cut Polytope: definition

- Let $S \subseteq \{1, \ldots, n\}$ and $\oplus$ denote exclusive or.
- Define

$$x_{ij}^S = \begin{cases} 1 & i \oplus j \in S \\ 0 & otherwise \end{cases} \qquad 1 \leq i < j \leq n$$

- x^S is the edge-incidence vector of the cut $[S, V - S]$ in K_n.
- The cut polytope is

$$Cut_n = CH\{x^S : S \subseteq \{1, \ldots, n-1\}\}$$

Cut Polytope: definition

- Let $S \subseteq \{1, \ldots, n\}$ and $\oplus$ denote exclusive or.
- Define

$$x_{ij}^S = \begin{cases} 1 & i \oplus j \in S \\ 0 & otherwise \end{cases} \qquad 1 \le i < j \le n$$

- x^S is the edge-incidence vector of the cut $[S, V - S]$ in K_n.
- The cut polytope is

$$Cut_n = CH\{x^S : S \subseteq \{1, \ldots, n-1\}\}$$

- Membership test is NP-complete (Av & De 91)

Cut Polytope and correlations

- Let $S \subseteq \{A_1, \ldots, A_n\}$ be a deterministic outcome.

Cut Polytope and correlations

- Let $S \subseteq \{A_1, \ldots, A_n\}$ be a deterministic outcome.
- Define

$$x_{ij}^S \equiv Pr(A_i \triangle A_j) = \begin{cases} 1 & A_i \triangle A_j \in S \\ 0 & otherwise \end{cases} \qquad 1 \leq i < j \leq n$$

Cut Polytope and correlations

- Let $S \subseteq \{A_1, \ldots, A_n\}$ be a deterministic outcome.
- Define

$$x_{ij}^S \equiv Pr(A_i \triangle A_j) = \begin{cases} 1 & A_i \triangle A_j \in S \\ 0 & otherwise \end{cases} \qquad 1 \leq i < j \leq n$$

- A joint probability space is just a probabilistic mixture of deterministic outcomes.

Cut Polytope and correlations

- Let $S \subseteq \{A_1, \ldots, A_n\}$ be a deterministic outcome.
- Define

$$x_{ij}^S \equiv Pr(A_i \triangle A_j) = \begin{cases} 1 & A_i \triangle A_j \in S \\ 0 & otherwise \end{cases} \qquad 1 \leq i < j \leq n$$

- A joint probability space is just a probabilistic mixture of deterministic outcomes.
- Let p_S be the probability that S occurs.

Cut Polytope and correlations

- Let $S \subseteq \{A_1, \ldots, A_n\}$ be a deterministic outcome.
- Define

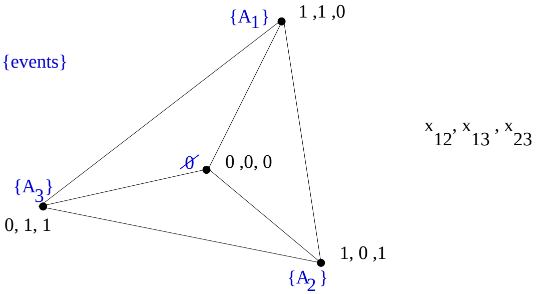
$$x_{ij}^S \equiv Pr(A_i \triangle A_j) = \begin{cases} 1 & A_i \triangle A_j \in S \\ 0 & otherwise \end{cases} \qquad 1 \leq i < j \leq n$$

- A joint probability space is just a probabilistic mixture of deterministic outcomes.
- Let p_S be the probability that S occurs.
- So the set of all probabilistic outcomes is

$$\left\{ \sum_S p_S x^S : \sum_S p_S = 1, p_S \geq 0 \right\} \;=\; CUT_n.$$

Cut_3 and correlations

	$x_{12} = Pr(A_1 \triangle A_2)$	$x_{13} = Pr(A_1 \triangle A_3)$	$x_{23} = Pr(A_2 \triangle A_3)$
Events $\downarrow \{S\}$			
$\emptyset$ *or* $\{A_1, A_2, A_3\}$	0	0	0
$\{A_1\}$ *or* $\{A_2, A_3\}$	1	1	0
$\{A_2\}$ *or* $\{A_1, A_3\}$	1	0	1
$\{A_3\}$ *or* $\{A_1, A_2\}$	0	1	1



$x \in CUT_3 \iff A_1, A_2, A_3$ have joint distribution x

Some facets of Cut_n

- Triangle inequalities:

$$x_{12} \leq x_{13} + x_{23} \qquad\qquad x_{12} + x_{13} + x_{23} \leq 2$$

Some facets of Cut_n

- Triangle inequalities:

$$x_{12} \leq x_{13} + x_{23} \qquad\qquad x_{12} + x_{13} + x_{23} \leq 2$$

- Pentagon inequality:

$$\sum_{1 \leq i < j \leq 5} x_{ij} \leq 6$$

Some facets of Cut_n

- Triangle inequalities:

$$x_{12} \le x_{13} + x_{23} \qquad\qquad x_{12} + x_{13} + x_{23} \le 2$$

- Pentagon inequality:

$$\sum_{1 \le i < j \le 5} x_{ij} \le 6$$

- Setting $x_{ij} = .625$ violates the pentagon inequality ...

Some facets of Cut_n

- Triangle inequalities:

$$x_{12} \leq x_{13} + x_{23} \qquad\qquad x_{12} + x_{13} + x_{23} \leq 2$$

- Pentagon inequality:

$$\sum_{1 \leq i < j \leq 5} x_{ij} \leq 6$$

- Setting $x_{ij} = .625$ violates the pentagon inequality ...
- ... so Alice was right!

L_1 embedding and the Cut cone CUT_n

L_1 embedding and the Cut cone CUT_n

- CUT_n is the cone defined by non-negative combinations of cut vectors

L_1 embedding and the Cut cone CUT_n

- CUT_n is the cone defined by non-negative combinations of cut vectors
- (Switching) The vertex figure at each vertex of Cut_n is identical to CUT_n

L_1 embedding and the Cut cone CUT_n

- CUT_n is the cone defined by non-negative combinations of cut vectors
- (Switching) The vertex figure at each vertex of Cut_n is identical to CUT_n
- $x \in CUT_n \Leftrightarrow x$ is L_1-embeddable (As77)

Quantum correlations

EPR, Bell & Aspect: The Original References

1. A. Einstein, B. Podolsky, N. Rosen:

"Can quantum-mechanical description of physical reality be considered complete?"

Physical Review 41, 777 (15 May 1935).

2. J.S. Bell:

"On the Einstein Podolsky Rosen paradox"

Physics 1 #3, 195 (1964).

3. A. Aspect, Dalibard, G. Roger:

"Experimental test of Bell's inequalities using time-varying analyzers"

Physical Review Letters 49 #25, 1804 (20 Dec 1982).

Quantum correlations

- Its another long story but.....

Quantum correlations

- Its another long story but.....
- ... the cut polytope models what correlations are possible in classical physics.

Quantum correlations

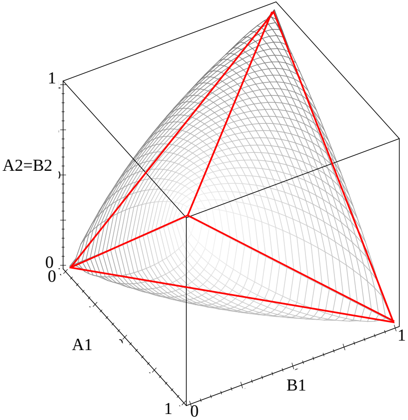
- Its another long story but.....
- ... the cut polytope models what correlations are possible in classical physics.
- In a quantum world you can stray outside the cut polytope ......

Quantum correlations

- Its another long story but.....
- ... the cut polytope models what correlations are possible in classical physics.
- In a quantum world you can stray outside the cut polytope ......
- ... there can be 62.5% of quantum men with exactly one of each pair of five properties!

The Classical and Quantum Regions

Plot of $x_{ij} = Pr(A_i \triangle B_j)$, $1 \le i, j \le 2$ with $A_2 = B_2$.



Quantum outcomes (elliptope) in grey.

Classical outcomes (cut polytope) in red.

Open pit mining

Open pit mining

- Yet another long story so ...

Open pit mining

- Yet another long story so ...
- ... please read Conor's thesis!

Open question

Open question

- A pit is convex if whenever blocks *a* and *b* are mined, all blocks directly between them are also mined.

Open question

- A pit is convex if whenever blocks a and b are mined, all blocks directly between them are also mined.
- What is the complexity of finding the maximum value convex pit?