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Lecture 5

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1 Duality

The purpose of this lecture is to introduce duality, which is an important concept in linear programming. One of the main uses is to give a certificate of optimality.

1.1 Certificates of optimality

Recall our previous example.

$$\begin{aligned} max \quad z = 5x_1 + 6x_2 + 9x_3 \\ x_1 + 2x_2 + 3x_3 \le 5 \\ x_1 + x_2 + 2x_3 \le 3 \\ x_1, x_2, x_3 \ge 0 \end{aligned} \tag{1}$$

The problem has an intial dictionary:

$$\begin{aligned} x_4 &= 5 - x_1 - 2x_2 - 3x_3 \ge 0 \\ x_5 &= 3 - x_1 - x_2 - 2x_3 \ge 0 \\ z &= 5x_1 + 6x_2 + 9x_3 \end{aligned} \tag{2}$$

It has optimum solution $x_1^* = 1, x_2^* = 2, x_3^* = 0, z^* = 17$, obtained from the final dictionary:

$$\begin{aligned} x_1 &= 1 - x_3 + x_4 - 2x_5 \\ x_2 &= 2 - x_3 - x_4 + x_5 \\ z &= 17 - 2x_3 - x_4 - 4x_5 \end{aligned} \tag{3}$$

Our original proof of optimality came from the fact that all coefficients in the final row are non-positive. In general $\overline{c}_j \le 0$ as this is the stopping condition for the simplex method. Compare the objecitve row in (3) with that in the initial dictionary (2). Since they are equivalent, and since the slack variables in (2) only appear once, we can obtain the final objective function by adding the equation for x_4 and four times the equation for x_5 to the initial objective function.

Or we may work directly with the original inequalities in (1). If we add the first constraint to four times the second constraint, we get:

$$\begin{array}{rcl} x_1 + 2x_2 + 3x_3 & \leq & 5 \\ 4x_1 + 4x_2 + 8x_3 & \leq & 12 \\ ------------ & - & ---- \\ 5x_1 + 6x_2 + 11x_3 & \leq & 17 \end{array} \tag{4}$$

All of these inequalities are valid for every feasible x_1, x_2, x_3. This gives us a proof of optimality for the LP, since for every feasible x_1, x_2, x_3 we have

$$z = 5x_1 + 6x_2 + 9x_3 \leq 5x_1 + 6x_2 + 11x_3 \leq 17 \tag{5}$$

Let y_1 be the multiplier for the first constraint, and y_2 be the multiplier for the second. We used $y_1 = 1$ and $y_2 = 4$. Observe that using the $\bar{c}_j$ in the objective row of the final dictionary (3) we have

$$y_1 = -\bar{c}_4,\ y_2 = -\bar{c}_5.$$

1.2 Formulation of the dual

In the previous section we hinted that the certificate provided by the multipliers y_i can be obtained from the optimum dictionary. In this section we give an independent way of doing this. We may manipulate inequalities by one of the following two operations:

1. multiply by non-negative numbers
2. add inequalities together.

We will now formulate the properties that the y_i should satisfy. The idea is that they should give a way to combine the inequalities in the primal in order to give an upper bound on z. So we must have:

(i) $y_1, y_2 \geq 0$, since otherwise they are not valid multipliers.

(ii) After combining constraints, we must bound the objective z, so:

$$y_1 + y_2 \geq 5 \quad 2y_1 + y_2 \geq 6 \quad 3y_1 + 2y_2 \geq 9$$

Here are some multipliers satisfying (i) and (ii) and the upper bounds on z that they give:

$y_1 = 5$	$y_2 = 0$	$z \leq 25$
$y_1 = 0$	$y_2 = 6$	$z \leq 18$
$y_1 = 1$	$y_2 = 4$	$z \leq 17$

The bound on z is given by $5y_1 + 3y_2$. Since we want the lowest bound possible, we have one further condition.

(iii) Find

$$min\ w\ = 5y_1 + 3y_2$$

Putting all this together we have formulated the dual problem, which is also a linear program:

$$\begin{aligned} min \quad w = 5y_1 + 3y_2 \\ y_1 + y_2 \geq 5 \\ 2y_1 + y_2 \geq 6 \\ 3y_1 + 2y_2 \geq 9 \\ y_1, y_2 \geq 0 \end{aligned} \tag{6}$$

This has optimum solution: $y_1^* = 1, y_2^* = 4, w^* = 17$.

Using exactly the same logic, we may define a dual for every LP in standard form. The original LP is called the primal.

- **Primal**

$$max \qquad z = \sum_{j=1}^{n} c_j x_j \tag{7}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, ..., m) \tag{8}$$

$$x_j \geq 0 \qquad \text{(j=1,...n)}$$

- **Dual**

$$min \qquad w = \sum_{i=1}^{m} b_i y_i \tag{9}$$

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad (j = 1, ..., n) \tag{10}$$

$$y_i \geq 0 \qquad \text{(i=1,...m)}$$

If $x_1, .., x_n$ satisfy the constraints of the primal, they are called *primal feasible*. Similarly $y_1, ..., y_m$ are *dual feasible* if they satisfy the constraints of the dual.

1.3 Duality theorems

By construction, the purpose of the dual is to provide an upper bound on the value of the primal solution. This idea is formalized in the weak duality theorem.

Theorem 1. *(Weak Duality Theorem) Let $x_1, x_2, ...x_n$ be primal feasible and let $y_1, y_2, ...y_n$ be dual feasible then*

$$z \;=\; \sum_{j=1}^{n} c_j x_j \;\leq\; \sum_{i=1}^{m} b_i y_i \;=\; w \tag{11}$$

Proof. Given feasible $x_1 \cdots x_n$ and $y_1 \cdots y_n$,

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

for each $i = 1, .., m$, and since $y_i \geq 0$ we have

$$\sum_{j=1}^{n} a_{ij} x_j y_i \leq b_i y_i.$$

Summing over all i:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j y_i \leq \sum_{i=1}^{m} b_i y_i = w.$$

Similarly

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j$$

for each $j = 1, ...n$, and since $x_j \geq 0$ we have

$$\sum_{i=1}^{m} a_{ij} x_j y_i \geq c_j x_j.$$

Summing over j we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j y_i \geq \sum_{j=1}^{n} c_j x_j = z$$

Combining and noting that we can reverse the order of summation in any finite sum, we have

$$z = \sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j y_i \leq \sum_{i=1}^{m} b_i y_i = w. \tag{12}$$

□

It follows immediately that if equations hold throughout in (12), then the corresponding solutions are both optimal. The strong duality theorem states that this is always the case when an LP has an optimal solution.

Theorem 2. *(Strong Duality Theorem) If a linear programming problem has an optimal solution, so does its dual, and the respective objective functions are equal.* □

1.4 Outcomes for primal/dual problems

The weak and strong duality theorems give us information on what is the relationship between the primal and dual problems. This is summarized in the table below.

Primal \ Dual	opt	Infeasible	unbounded
opt	$\surd$	$\times$	$\times$
Infeasible	$\times$	$\surd$	$\surd$
unbounded	$\times$	$\surd$	$\times$

$\times$:can not happen
$\surd$: can happen

In particular we may conclude that:

primal unbounded $\Rightarrow$dual infeasible
primal infeasible $\Rightarrow$dual infeasible or unbounded

References

[1] V. Chvatal *Linear Programming* (W.H.FREEMAN AND COMPANY, New York, 1983)