1

Introduction

In this short chapter, we shall explain what is meant by linear programming and sketch a history of this subject.

A DIET PROBLEM

Polly wonders how much money she must spend on food in order to get all the energy (2,000 kcal), protein (55 g), and calcium (800 mg) that she needs every day. (For iron and vitamins, she will depend on pills. Nutritionists would disapprove, but the introductory example ought to be simple.) She chooses six foods that seem to be cheap sources of the nutrients; her data are collected in Table 1.1.

Table 1.1 Nutritive Value per Serving

Food	Serving size	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving (cents)
Oatmeal	28 g	110	4	2	3
Chicken	100 g	205	32	12	24
Eggs	2 large	160	13	54	13
Whole milk	237 cc	160	8	285	9
Cherry pie	170 g	420	4	22	20
Pork with beans	260 g	260	14	80	19

Then she begins to think about her menu. For example, 10 servings of pork with beans would take care of all her needs for only (?) $1.90 per day. On the other hand, 10 servings of pork with beans is a lot of pork with beans—she would not be able to stomach more than 2 servings a day. She decides to impose servings-per-day limits on all six foods:

Oatmeal	at most 4 servings per day
Chicken	at most 3 servings per day
Eggs	at most 2 servings per day
Milk	at most 8 servings per day
Cherry pie	at most 2 servings per day
Pork with beans	at most 2 servings per day.

Now, another look at the data shows Polly that 8 servings of milk and 2 servings of cherry pie every day will satisfy the requirements nicely and at a cost of only $1.12. In fact, she could cut down a little on the pie or the milk or perhaps try a different combination. But so many combinations seem promising that one could go on and on, looking for the best one. Trial and error is not particularly helpful here. To be systematic, we may speculate about some as yet unspecified menu consisting of x_1 servings of oatmeal, x_2 servings of chicken, x_3 servings of eggs, and so on. In order to stay below the upper limits, that menu must satisfy

$$\begin{aligned} &0 \le x_1 \le 4 \\ &0 \le x_2 \le 3 \\ &0 \le x_3 \le 2 \\ &0 \le x_4 \le 8 \\ &0 \le x_5 \le 2 \\ &0 \le x_6 \le 2. \end{aligned} \tag{1.1}$$

And, of course, there are the requirements for energy, protein, and calcium; they lead to the inequalities

$$\begin{aligned} 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 &\ge 2{,}000 \\ 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 &\ge 55 \\ 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 &\ge 800. \end{aligned} \tag{1.2}$$

If some numbers $x_1, x_2, \ldots, x_6$ satisfy inequalities (1.1) and (1.2), then they describe a satisfactory menu; such a menu will cost, in cents per day,

$$3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6. \tag{1.3}$$

In designing the most economical menu, Polly wants to find numbers $x_1, x_2, \ldots, x_6$ that satisfy (1.1) and (1.2), and make (1.3) as small as possible. As a mathematician would put it, she wants to

$$\begin{aligned} \text{minimize}\quad & 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\ \text{subject to}\quad & 0 \le x_1 \le 4 \\ & 0 \le x_2 \le 3 \\ & 0 \le x_3 \le 2 \\ & 0 \le x_4 \le 8 \\ & 0 \le x_5 \le 2 \\ & 0 \le x_6 \le 2 \\ & 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \ge 2000 \\ & 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \ge 55 \\ & 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \ge 800. \end{aligned} \tag{1.4}$$

Her problem is known as a *diet problem.*

LINEAR PROGRAMMING

Problems of this kind are called "linear programming problems," or "LP problems" for short; linear programming is the branch of applied mathematics concerned with these problems. Here are other examples:

$$\begin{aligned} \text{maximize}\quad & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to}\quad & 2x_1 + 3x_2 + x_3 \le 5 \\ & 4x_1 + x_2 + 2x_3 \le 11 \\ & 3x_1 + 4x_2 + 2x_3 \le 8 \\ & x_1, x_2, x_3 \ge 0 \end{aligned} \tag{1.5}$$

(with "$x_1, x_2, x_3 \ge 0$" used as shorthand for "$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$") or

$$\begin{aligned} \text{minimize}\quad & 3x_1 - x_2 \\ \text{subject to}\quad & -x_1 + 6x_2 - x_3 + x_4 \ge -3 \\ & 7x_2 + 2x_4 = 5 \\ & x_1 + x_2 + x_3 = 1 \\ & x_3 + x_4 \le 2 \\ & x_2, x_3 \ge 0. \end{aligned} \tag{1.6}$$

In general, if $c_1, c_2, \ldots, c_n$ are real numbers, then the function f of real variables $x_1, x_2, \ldots, x_n$ defined by

$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n = \sum_{j=1}^{n} c_jx_j$$

is called a *linear function.* If f is a linear function and if b is a real number, then the equation

$$f(x_1, x_2, \ldots, x_n) = b$$

is called a *linear equation* and the inequalities

$$f(x_1, x_2, \ldots, x_n) \leq b$$
$$f(x_1, x_2, \ldots, x_n) \geq b$$

are called *linear inequalities.* Linear equations and linear inequalities are both referred to as *linear constraints.* Finally, a *linear programming problem* is the problem of maximizing (or minimizing) a linear function subject to a finite number of linear constraints. We shall usually attach different subscripts i to different constraints and different subscripts j to different variables. For simplicity of exposition, we shall restrict ourselves in Chapters 1–7 to LP problems of the following form:

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & (i = 1, 2, \ldots, m) \\ & x_j \geq 0 & (j = 1, 2, \ldots, n). \end{array} \tag{1.7}$$

These problems will be referred to as LP problems in the *standard form.* (The reader should be warned that the terminology is far from unified; several authors prefer the terms *canonical* or *symmetric* form, and others reserve these adjectives for altogether different problems.) For example, (1.5) is a problem in the standard form (with $n = 3, m = 3, a_{11} = 2, a_{12} = 3$, and so on). What distinguishes the problems in the standard form from the rest? First, all of their constraints are linear *inequalities.* Secondly, the last n of the $m + n$ constraints in (1.7) are very special: they simply stipulate that none of the n variables may assume negative values. Such constraints are called *nonnegativity constraints.* (Note that problem (1.6) differs from the standard form on both counts: two of its constraints are linear equations and the variables x_1, x_4 may assume negative values.)

The linear function that is to be maximized or minimized in an LP problem is called the *objective function* of that problem. For example, the function z of variables $x_1, x_2, x_3, x_4, x_5, x_6$ defined by

$$z(x_1, x_2, \ldots, x_6) = 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6$$

is the objective function of Polly's diet problem (1.4). Numbers $x_1, x_2, \ldots, x_n$ that satisfy all the constraints of an LP problem are said to constitute a *feasible solution* of that problem. For instance, we have observed that

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 8, \quad x_5 = 2, \quad x_6 = 0$$

is a feasible solution of (1.4). Finally, a feasible solution that maximizes the objective function (or minimizes it, depending on the form of the problem) is called an *optimal solution*; the corresponding value of the objective function is called the *optimal value* of the problem. As it turns out, the unique optimal solution of (1.4) is

$$x_1 = 4, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 4.5, \quad x_5 = 2, \quad x_6 = 0$$

or simply (4, 0, 0, 4.5, 2, 0). Accordingly, the optimal value of (1.4) is 92.5. Not every LP problem has a unique optimal solution; some problems have many different optimal solutions and others have no optimal solutions at all. The latter may occur for one of two radically different reasons: either there are no feasible solutions at all or there are, in a sense, too many of them. The first case may be illustrated on the problem

$$\begin{array}{lrcl} \text{maximize} & 3x_1 - x_2 & & \\ \text{subject to} & x_1 + x_2 & \leq & 2 \\ & -2x_1 - 2x_2 & \leq & -10 \\ & x_1, x_2 & \geq & 0 \end{array} \tag{1.8}$$

which has no feasible solutions at all. Such problems are called *infeasible.* On the other hand, even though the problem

$$\begin{array}{lrcl} \text{maximize} & x_1 - x_2 & & \\ \text{subject to} & -2x_1 + x_2 & \leq & -1 \\ & -x_1 - 2x_2 & \leq & -2 \\ & x_1, x_2 & \geq & 0 \end{array} \tag{1.9}$$

does have feasible solutions, none of them is optimal: for every number M there is a feasible solution x_1, x_2 such that $x_1 - x_2 > M$. In a sense, (1.9) has such an abundance of feasible solutions that none of them can aspire to be the best. Problems with this property are called *unbounded.* As we shall prove later (Theorem 3.4), every linear programming problem belongs to one of the three categories noted here: it has an optimal solution, is infeasible, or is unbounded.

HISTORY OF LINEAR PROGRAMMING

As mathematical disciplines go, linear programming is quite young. It started in 1947 when G. B. Dantzig designed the "simplex method" for solving linear programming formulations of U.S. Air Force planning problems. What followed was an exciting period of rapid development in this new field. It soon became clear that a surprisingly wide range of apparently unrelated problems in production management could be stated in linear programming terms and, most importantly, solved by the simplex method. Such problems, if noticed at all, had traditionally been tackled by a hit-or-miss approach guided only by experience and intuition. The use of linear programming often brought about a considerable increase in the efficiency of the whole operation. (Until then, expansion of the efficiency frontier usually came from technological innovations. This new way to increase efficiency—*under existing technological conditions*—by improvements in organization and planning, made many managers appreciate the practical importance of mathematics. At least, it made them aware of the advantage of stating their decision problems in clear-cut and well-defined terms.) As the popularity of linear programming theory increased, applications in new areas occurred, many of them far from obvious. In turn, these applications stimulated further theoretical research by pointing out the need for solving problems that would have otherwise seemed uninteresting. In this fascinating interplay between theory and applications, a new branch of applied mathematics established itself.

As calculus developed from the seventeenth century's need to solve problems of mechanics, linear programming developed from the twentieth century's need to solve problems of management. Yet other profound influences stimulated the evolution of the new field from its very inception. Economics was one of them: as early as 1947, T. C. Koopmans began pointing out that linear programming provided an excellent framework for the analysis of classical economic theories, such as the renowned system proposed in 1874 by L. Walras. On the other hand, linear programming brought together previously known theorems of pure mathematics concerning such diverse topics as the geometry of convex sets, extremal problems of combinatorial nature, and the theory of two-person games. Finally, it was fortunate and perhaps even inevitable that linear programming developed concurrently with modern computer technology: without electronic computers, present-day large-scale linear programming would be unthinkable.

Scientific fields are rarely born overnight; with the advantage of hindsight, one can often track down the sources that paved the way for the decisive breakthrough. The field of linear programming is no exception. At the core of its mathematical theory is the study of systems of linear inequalities; such systems were investigated by Fourier as far back as 1826. Since then, quite a few other mathematicians have considered the subject, although none of them has devised an algorithm whose efficiency has come close to that of the simplex method. Nevertheless, some of them proved various special cases of a fundamental theorem that is now called the *duality theorem* of linear programming. On the applied side, L. V. Kantorovich pointed out the practical significance of a restricted class of LP problems, and proposed a rudimentary algorithm for their solution as early as 1939. Regrettably, this effort remained neglected in the U.S.S.R. and unknown elsewhere until long after linear programming became an elegant theory through the independent work of Dantzig and others.

In the 1970s, linear programming came twice to public attention. On October 14, 1975, the Royal Sweden Academy of Sciences awarded the Nobel Prize in economic science to L. V. Kantorovich and T. C. Koopmans "for their contributions to the theory of optimum allocation of resources." (As the reader may know, there is no Nobel Prize in mathematics. Apparently the Academy regarded the work of G. B. Dantzig, who is universally recognized as the father of linear programming, as being too mathematical.) The second event was even more dramatic. Ever since the invention of the simplex method, mathematicians had been looking for a *theoretically* satisfactory algorithm to solve LP problems. (A word of explanation is in order: theoretical criteria for judging the efficiency of algorithms are quite different from practical ones. Thus, an algorithm like the simplex method, which is eminently satisfactory in practical applications, may be found theoretically unsatisfactory. The converse is also true: theoretically satisfactory algorithms may be thoroughly useless in practice. We shall return to this distinction in

Chapter 4.) The breakthrough came in 1979 when L. G. Khachian published a description of such an algorithm (based on earlier works by Shor, and by Judin and Nemirovskii). Newspapers around the world published reports of this result, some of them full of hilarious misinterpretations. We shall present the algorithm in the appendix.

For a thorough survey of the history of linear programming, the reader is referred to Chapter 2 of Dantzig's monograph (1963). References to many applications of linear programming may be found in Riley and Gass (1958). Some of the more recent applications are referenced in Gass (1975). □

PROBLEMS

Answers to problems marked with the symbol △ are found at the back of the book.

1.1 Which of the problems below are in the standard form?

a. Maximize $3x_1 - 5x_2$

subject to
$$\begin{aligned} 4x_1 + 5x_2 &\geq 3 \\ 6x_1 - 6x_2 &= 7 \\ x_1 + 8x_2 &\leq 20 \\ x_1, x_2 &\geq 0. \end{aligned}$$

b. Minimize $3x_1 + x_2 + 4x_3 + x_4 + 5x_5$

subject to
$$\begin{aligned} 9x_1 + 2x_2 + 6x_3 + 5x_4 + 3x_5 &\leq 5 \\ 8x_1 + 9x_2 + 7x_3 + 9x_4 + 3x_5 &\leq 2 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

c. Maximize $8x_1 - 4x_2$

subject to
$$\begin{aligned} 3x_1 + x_2 &\leq 7 \\ 9x_1 + 5x_2 &\leq -2 \\ x_1, x_2 &\geq 0. \end{aligned}$$

1.2 State in the standard form:

minimize $-8x_1 + 9x_2 + 2x_3 - 6x_4 - 5x_5$

subject to
$$\begin{aligned} 6x_1 + 6x_2 - 10x_3 + 2x_4 - 8x_5 &\geq 3 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0. \end{aligned}$$

1.3 Prove that (1.8) is infeasible and (1.9) is unbounded.

△ **1.4** Find necessary and sufficient conditions for the numbers s and t to make the LP problem

maximize $x_1 + x_2$

subject to
$$\begin{aligned} sx_1 + tx_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

a. have an optimal solution,

b. be infeasible,

c. be unbounded.

△ 1.5 Prove or disprove: If problem (1.7) is unbounded, then there is a subscript k such that the problem

$$\begin{aligned} \text{maximize} \quad & x_k \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (i = 1, 2, \ldots, m) \\ & x_j \ge 0 \qquad (j = 1, 2, \ldots, n) \end{aligned}$$

is unbounded.

△ 1.6 [Adapted from Greene et al. (1959).] A meat packing plant produces 480 hams, 400 pork bellies, and 230 picnic hams every day; each of these products can be sold either fresh or smoked. The total number of hams, bellies, and picnics that can be smoked during a normal working day is 420; in addition, up to 250 products can be smoked on overtime at a higher cost. The *net* profits are as follows:

	Fresh	Smoked on regular time	Smoked on overtime
Hams	\$8	\$14	\$11
Bellies	\$4	\$12	\$7
Picnics	\$4	\$13	\$9

For example, the following schedule yields a total net profit of \$9,965:

	Fresh	Smoked	Smoked (overtime)
Hams	165	280	35
Bellies	295	70	35
Picnics	55	70	105

The objective is to find the schedule that maximizes the total net profit. Formulate as an LP problem in the standard form.

1.7 [Adapted from Charnes et al. (1952).] An oil refinery produces four types of raw gasoline: alkylate, catalytic-cracked, straight-run, and isopentane. Two important characteristics of each gasoline are its performance number PN (indicating antiknock properties) and its vapor pressure RVP (indicating volatility). These two characteristics, together with the production levels in barrels per day, are as follows:

	PN	RVP	Barrels produced
Alkylate	107	5	3,814
Catalytic-cracked	93	8	2,666
Straight-run	87	4	4,016
Isopentane	108	21	1,300

These gasolines can be sold either raw, at \$4.83 per barrel, or blended into aviation gasolines (Avgas A and/or Avgas B). Quality standards impose certain requirements on the aviation gasolines; these requirements, together with the selling prices, are as follows:

	PN	RVP	Price per barrel
Avgas A	at least 100	at most 7	\$6.45
Avgas B	at least 91	at most 7	\$5.91

The PN and RVP of each mixture are simply weighted averages of the PNs and RVPs of its constituents. For example, the refinery could adopt the following strategy:

- Blend 2,666 barrels of alkylate and 2,666 barrels of catalytic into 5,332 barrels of Avgas A with

$$\text{PN} = \frac{(2{,}666 \times 107) + (2{,}666 \times 93)}{5{,}332} = 100$$

$$\text{RVP} = \frac{(2{,}666 \times 5) + (2{,}666 \times 8)}{5{,}332} = 6.5.$$

- Blend 1,148 barrels of alkylate, 4,016 barrels of straight-run, and 1,024 barrels of isopentane into 6,188 barrels of Avgas B with

$$\text{PN} = \frac{(1{,}148 \times 107) + (4{,}016 \times 87) + (1{,}024 \times 108)}{6{,}188} \doteq 94.2$$

$$\text{RVP} = \frac{(1{,}148 \times 5) + (4{,}016 \times 4) + (1{,}024 \times 21)}{6{,}188} \doteq 7.$$

Sell 276 barrels of isopentane raw.

This sample plan yields a total profit of

$$(5{,}332 \times 6.45) + (6{,}188 \times 5.91) + (276 \times 4.83) \doteq \$72{,}296.$$

The refinery aims for the plan that yields the largest possible profit. Formulate as an LP problem in the standard form.

1.8 An electronics company has a contract to deliver 20,000 radios within the next four weeks. The client is willing to pay \$20 for each radio delivered by the end of the first week, \$18 for those delivered by the end of the second week, \$16 by the end of the third week, and \$14 by the end of the fourth week. Since each worker can assemble only 50 radios per week, the company cannot meet the order with its present labor force of 40; hence it must hire and train temporary help. Any of the experienced workers can be taken off the assembly line to instruct a class of three trainees; after one week of instruction, each of the trainees can either proceed to the assembly line or instruct additional new classes.

At present, the company has no other contracts; hence some workers may become idle once the delivery is completed. All of them, whether permanent or temporary, must be kept on the payroll till the end of the fourth week. The weekly wages of a worker, whether assembling, instructing, or being idle, are \$200; the weekly wages of a trainee are \$100. The production costs, excluding the worker's wages, are \$5 per radio.

For example, the company could adopt the following program.

First week: 10 assemblers, 30 instructors, 90 trainees
Workers' wages: \$8,000

Trainees' wages: \$9,000
Profit from 500 radios: \$7,500
Net loss: \$9,500

Second week: 120 assemblers, 10 instructors, 30 trainees
Workers' wages: \$26,000
Trainees' wages: \$3,000
Profit from 6,000 radios: \$78,000
Net profit: \$49,000

Third week: 160 assemblers
Workers' wages: \$32,000
Profit from 8,000 radios: \$88,000
Net profit: \$56,000

Fourth week: 110 assemblers, 50 idle
Workers' wages: \$32,000
Profit from 5,500 radios: \$49,500
Net profit: \$17,500

This program, leading to a total net profit of \$113,000, is one of many possible programs. The company's aim is to maximize the total net profit. Formulate as an LP problem (not necessarily in the standard form).

△ 1.9 [S. Masuda (1970); see also V. Chvátal (1983).] The *bicycle problem* involves n people who have to travel a distance of ten miles, and have one single-seat bicycle at their disposal. The data are specified by the walking speed w_j and the bicycling speed b_j of each person j ($j = 1, 2, \ldots, n$); the task is to minimize the arrival time of the last person. (Can you solve the case of $n = 3$ and $w_1 = 4$, $w_2 = w_3 = 2$, $b_1 = 16$, $b_2 = b_3 = 12$?) Show that the optimal value of the LP problem

$$\begin{array}{lrll} \text{minimize} & t & & \\ \text{subject to} & t - x_j - x'_j - y_j - y'_j \geq & 0 & (j = 1, 2, \ldots, n) \\ & t \quad - \sum_{j=1}^{n} y_j - \sum_{j=1}^{n} y'_j \geq & 0 & \\ & w_j x_j - w_j x'_j + b_j y_j - b_j y'_j = & 10 & (j = 1, 2, \ldots, n) \\ & \sum_{j=1}^{n} b_j y_j - \sum_{j=1}^{n} b_j y'_j \leq & 10 & \\ & x_j, x'_j, y_j, y'_j \geq & 0 & (j = 1, 2, \ldots, n) \end{array}$$

provides a lower bound on the optimal value of the bicycle problem.

2

How the Simplex Method Works

In this chapter, we shall learn to solve LP problems in the standard form by the simplex method. A rigorous analysis of the details will be deferred to Chapter 3.

FIRST EXAMPLE

We shall illustrate the simplex method on the following example:

$$\begin{array}{lrl} \text{maximize} & 5x_1 + 4x_2 + 3x_3 & \\ \text{subject to} & 2x_1 + 3x_2 + x_3 \leq & 5 \\ & 4x_1 + x_2 + 2x_3 \leq & 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq & 8 \\ & x_1, x_2, x_3 \geq & 0. \end{array} \qquad (2.1)$$

A preliminary step of the method consists of introducing so-called slack variables.

In order to motivate this concept, let us consider the first of our constraints,

$$2x_1 + 3x_2 + x_3 \leq 5. \tag{2.2}$$

For every feasible solution x_1, x_2, x_3, the value of the left-hand side of (2.1) is at most the value of the right-hand side; often, there may be a slack between the two values. We shall denote the slack by x_4. That is, we shall *define* $x_4 = 5 - 2x_1 - 3x_2 - x_3$; with this notation, inequality (2.2) may now be written as $x_4 \geq 0$. In an analogous way, the next two constraints give rise to variables x_5 and x_6. Finally, following a time-honored convention, we shall denote the objective function $5x_1 + 4x_2 + 3x_3$ by z. To summarize: for every *choice* of numbers x_1, x_2, and x_3, we shall *define* numbers x_4, x_5, x_6, and z by the formulas

$$\begin{aligned} x_4 &= 5 - 2x_1 - 3x_2 - x_3 \\ x_5 &= 11 - 4x_1 - x_2 - 2x_3 \\ x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \\ z &= 5x_1 + 4x_2 + 3x_3. \end{aligned} \tag{2.3}$$

With this notation, our problem may be restated as

$$\text{maximize} \quad z \quad \text{subject to} \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \tag{2.4}$$

The new variables x_4, x_5, x_6 defined by (2.3) are called *slack variables*; the old variables x_1, x_2, x_3 are usually referred to as the *decision variables*. It is crucial to note that the equations in (2.3) spell out an equivalence between (2.1) and (2.4). More precisely:

- Every feasible solution x_1, x_2, x_3 of (2.1) can be extended, in the unique way determined by (2.3), into a feasible solution $x_1, x_2, \ldots, x_6$ of (2.4).
- Every feasible solution $x_1, x_2, \ldots, x_6$ of (2.4) can be restricted, simply by deleting the slack variables, into a feasible solution x_1, x_2, x_3 of (2.1).
- This correspondence between feasible solutions of (2.1) and feasible solutions of (2.4) carries optimal solutions of (2.1) onto optimal solutions of (2.4), and vice versa.

The grand strategy of the simplex method is that of *successive improvements*: having found some feasible solution $x_1, x_2, \ldots, x_6$ of (2.4), we shall try to proceed to another feasible solution $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_6$, which is better in the sense that

$$5\bar{x}_1 + 4\bar{x}_2 + 3\bar{x}_3 > 5x_1 + 4x_2 + 3x_3.$$

Repeating this process a finite number of times, we shall eventually arrive at an optimal solution.

To begin with, we need some feasible solution $x_1, x_2, \ldots, x_6$. Finding one in our example presents no difficulty: setting the decision variables x_1, x_2, x_3 at zero, we evaluate the slack variables x_4, x_5, x_6 from (2.3). Hence our initial solution,

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 5, \quad x_5 = 11, \quad x_6 = 8 \tag{2.5}$$

yields $z = 0$.

In the spirit of the grand strategy sketched above, we should now look for a feasible solution that yields a higher value of z. Finding such a solution is not difficult. For example, if we keep $x_2 = x_3 = 0$ and increase the value of x_1, we obtain $z = 5x_1 > 0$. Thus, if we keep $x_2 = x_3 = 0$ and set $x_1 = 1$, we obtain $z = 5$ (and $x_4 = 3, x_5 = 7, x_6 = 5$). Better yet, if we keep $x_2 = x_3 = 0$ and set $x_1 = 2$, we obtain $z = 10$ (and $x_4 = 1$, $x_5 = 3$, $x_6 = 2$). However, if we keep $x_2 = x_3 = 0$ and set $x_1 = 3$, we obtain $z = 15$ and $x_4 = x_5 = x_6 = -1$; this won't do, since feasibility requires $x_i \geq 0$ for every i. The moral is that we cannot increase x_1 too much. The question is: *Just how much can we increase x_1 (keeping $x_2 = x_3 = 0$ at the same time) and still maintain feasibility* ($x_4, x_5, x_6 \geq 0$)?

The condition $x_4 = 5 - 2x_1 - 3x_2 - x_3 \geq 0$ implies $x_1 \leq \frac{5}{2}$; similarly, $x_5 \geq 0$ implies $x_1 \leq \frac{11}{4}$ and $x_6 \geq 0$ implies $x_1 \leq \frac{8}{3}$. Of these three bounds, the first is the most stringent. Increasing x_1 up to that bound we obtain our next solution,

$$x_1 = \frac{5}{2}, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0, \quad x_5 = 1, \quad x_6 = \frac{1}{2}. \tag{2.6}$$

Note that this solution yields $z = \frac{25}{2}$, which is indeed an improvement over $z = 0$.

Next, we should look for a feasible solution that is even better than (2.6). However, this task seems a little more difficult. What made the first iteration so easy? We had at our disposal not only the feasible solution (2.5), but also the system of linear equations (2.3), which guided us in our quest for an improved feasible solution. If we wish to continue in a similar way, we should manufacture a new system of linear equations that relates to (2.6) much as system (2.3) relates to (2.5).

What properties should the new system have? Note that (2.3) expresses the variables that assume positive values in (2.5) in terms of the variables that assume zero values in (2.5). Similarly, the new system should express those variables that assume positive values in (2.6) in terms of the variables that assume zero values in (2.6): in short, it should express x_1, x_5, x_6 (as well as z) in terms of x_2, x_3, and x_4. In particular, the variable x_1, which just changed its value from zero to positive should change its position from the right-hand side to the left-hand side of the system of equations. Similarly, the variable x_4, which just changed its value from positive to zero, should move from the left-hand side to the right-hand side.

To construct the new system, we shall begin with the newcomer to the left-hand side, namely, the variable x_1. The desired formula for x_1 in terms of x_2, x_3, x_4 is obtained easily from the first equation in (2.3):

$$x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4. \tag{2.7}$$

Next, in order to express x_5, x_6, and z in terms of x_2, x_3, x_4, we simply substitute from (2.7) into the corresponding rows of (2.3):

$$\begin{aligned} x_5 &= 11 - 4\left(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4\right) - x_2 - 2x_3 \\ &= 1 + 5x_2 + 2x_4, \\ x_6 &= 8 - 3\left(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4\right) - 4x_2 - 2x_3 \\ &= \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4, \\ z &= 5\left(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4\right) + 4x_2 + 3x_3 \\ &= \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4. \end{aligned}$$

Hence our new system reads

$$\begin{aligned} x_1 &= \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_5 &= 1 + 5x_2 + 2x_4 \\ x_6 &= \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\ z &= \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4. \end{aligned} \tag{2.8}$$

As we did in the first iteration, we shall now try to increase the value of z by increasing the value of a suitably chosen right-hand side variable, while at the same time keeping the remaining right-hand side variables fixed at zero. Note that increases in the values of x_2 or x_4 would bring about *decreases* in the value of z, which is very much against our intentions. Thus, we have no choice: the right-hand side variable to increase its value is necessarily x_3. How much can we increase x_3? The answer can be read directly from system (2.8): with $x_2 = x_4 = 0$, the constraint $x_1 \geq 0$ implies $x_3 \leq 5$, the constraint $x_5 \geq 0$ imposes no restriction at all, and the constraint $x_6 \geq 0$ implies $x_3 \leq 1$. Hence, $x_3 = 1$ is the best we can do; our new solution is

$$x_1 = 2, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 0, \quad x_5 = 1, \quad x_6 = 0. \tag{2.9}$$

(Note that the value of z just increased from 12.5 to 13.)

As we have learned, getting just the improved solution isn't good enough; we also want a system of linear equations to go with (2.9). In this system, the positive-valued variables x_1, x_3, x_5 will appear on the left, whereas the zero-valued variables x_2, x_4, x_6 will appear on the right. To construct the system, we begin again with the newcomer to the left-hand side, namely, the variable x_3. From the third equation in (2.8), we have $x_3 = 1 + x_2 + 3x_4 - 2x_6$; substituting for x_3 into the remaining equations in (2.8), we obtain

$$\begin{aligned} x_3 &= 1 + x_2 + 3x_4 - 2x_6 \\ x_1 &= 2 - 2x_2 - 2x_4 + x_6 \\ x_5 &= 1 + 5x_2 + 2x_4 \\ z &= 13 - 3x_2 - x_4 - x_6. \end{aligned} \tag{2.10}$$

Now it's time for the third iteration. First of all, from the right-hand side of (2.10) we have to choose a variable whose increase brings about an increase of the objective function. However, there is no such variable: indeed, if we increase any of the right-hand side variables x_2, x_4, x_6, we will make the value of z *decrease*. Thus, it seems that we have come to a standstill. In fact, the very presence of this standstill indicates that we are done; we have solved our problem; the solution described by the last table is optimal. Why? The answer lies hidden in the last row of (2.10):

$$z = 13 - 3x_2 - x_4 - x_6. \tag{2.11}$$

Our last solution (2.9) yields $z = 13$; proving that this solution is optimal amounts to proving that every feasible solution satisfies the inequality $z \leq 13$. Since every feasible solution $x_1, x_2, \ldots, x_6$ satisfies, among other relations, the inequalities $x_2 \geq 0$, $x_4 \geq 0$, and $x_6 \geq 0$, the desired inequality $z \leq 13$ follows directly from (2.11).

DICTIONARIES

In general, given a problem

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^{n} c_j x_j \\ &\text{subject to} && \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m) \\ &&& x_j \geq 0 \qquad (j = 1, 2, \ldots, n) \end{aligned} \tag{2.12}$$

we first introduce the slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ and denote the objective function by z. That is, we define

$$\begin{aligned} x_{n+i} &= b_i - \sum_{j=1}^{n} a_{ij} x_j \qquad (i = 1, 2, \ldots, m) \\ z &= \sum_{j=1}^{n} c_j x_j. \end{aligned} \tag{2.13}$$

In the framework of the simplex method, each feasible solution $x_1, x_2, \ldots, x_n$ of (2.12) is represented by $n + m$ *nonnegative* numbers $x_1, x_2, \ldots, x_{n+m}$, with $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ defined by (2.13). In each iteration, the simplex method moves from some feasible solution $x_1, x_2, \ldots, x_{n+m}$ to another feasible solution $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+m}$, which is better than the previous one in the sense that

$$\sum_{j=1}^{n} c_j\bar{x}_j > \sum_{j=1}^{n} c_j x_j.$$

(Actually, the last statement is not quite correct: the inequality is not always strict. This point and other subtleties will be discussed in Chapter 3.)

As we have seen, it is convenient to associate a system of linear equations with each of the feasible solutions: such systems make it easier to find the improved feasible solutions. They do so by translating any choice of values of the right-hand side variables into the corresponding values of the left-hand side variables and of the objective function. Following J. E. Strum (1972), we shall refer to these systems as *dictionaries*. Thus, every dictionary associated with (2.12) will be a system of linear equations in the variables $x_1, x_2, \ldots, x_{n+m}$ and z. However, not every system of linear equations in these variables constitutes a dictionary. To begin with, we have defined $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ and z in terms of $x_1, x_2, \ldots, x_n$, and so the $n + m + 1$ variables are heavily interdependent. This interdependence must be captured by every dictionary associated with (2.12): the translations must be correct. More precisely, we shall insist that:

> Every solution of the set of equations comprising a dictionary must be also a solution of (2.13), and vice versa. (2.14)

For example, for every choice of numbers $x_1, x_2, \ldots, x_6$ and z, the following three statements are equivalent:

- $x_1, x_2, \ldots, x_6, z$ constitute a solution of (2.3),
- $x_1, x_2, \ldots, x_6, z$ constitute a solution of (2.8),
- $x_1, x_2, \ldots, x_6, z$ constitute a solution of (2.10).

In that sense, the three dictionaries (2.3), (2.8), and (2.10) contain the same information concerning the interdependence among the seven variables. Nevertheless, each of the three dictionaries presents this information in its very own way. The form of (2.3) suggests that we are free to choose the numerical values of x_1, x_2, and x_3 at will, whereupon the values of x_4, x_5, x_6, and z are determined: in this dictionary, the decision variables x_1, x_2, x_3 act as independent variables, while z and the slack variables x_4, x_5, x_6 are dependent on them. Dictionary (2.8) presents x_2, x_3, x_4 as independent and x_1, x_5, x_6, z as dependent. In dictionary (2.10), the independent variables are x_2, x_4, x_6 and the dependent ones are x_3, x_1, x_5, z. In general:

> The equations of every dictionary must express m of the variables $x_1, x_2, \ldots, x_{n+m}$ and the objective function z in terms of the remaining n variables. (2.15)

The properties (2.14) and (2.15) are the defining properties of dictionaries.

In addition to these two properties, dictionaries (2.3), (2.8), and (2.10) have the following property:

> Setting the right-hand side variables at zero and evaluating the left-hand side variables, we arrive at a *feasible* solution.

Dictionaries with this additional property will be called *feasible dictionaries*. Hence, every feasible dictionary describes a feasible solution. However, not every feasible solution is described by a feasible dictionary; for instance, no dictionary describes the feasible solution $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 2, x_5 = 5, x_6 = 3$ of (2.1). Feasible solutions that can be described by dictionaries are called *basic*. The characteristic feature of the simplex method is the fact that it works exclusively with basic feasible solutions and ignores all other feasible solutions.

SECOND EXAMPLE

We shall complete our preview of the simplex method by applying it to another LP problem:

$$\begin{array}{ll} \text{maximize} & 5x_1 + 5x_2 + 3x_3 \\ \text{subject to} & x_1 + 3x_2 + x_3 \le 3 \\ & -x_1 + 3x_3 \le 2 \\ & 2x_1 - x_2 + 2x_3 \le 4 \\ & 2x_1 + 3x_2 - x_3 \le 2 \\ & x_1, x_2, x_3 \ge 0. \end{array}$$

In this case, the initial feasible dictionary reads

$$\begin{array}{l} x_4 = 3 - x_1 - 3x_2 - x_3 \\ x_5 = 2 + x_1 - 3x_3 \\ x_6 = 4 - 2x_1 + x_2 - 2x_3 \\ x_7 = 2 - 2x_1 - 3x_2 + x_3 \\ \hline z = 5x_1 + 5x_2 + 3x_3. \end{array} \qquad (2.16)$$

(Even though the order of the equations in a dictionary is quite irrelevant, we shall make a habit of writing the formula for z last and separating it from the rest of the table by a solid line. Of course, that does *not* mean that the last equation is the sum of the previous ones.) This feasible dictionary describes the feasible solution

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 3, \quad x_5 = 2, \quad x_6 = 4, \quad x_7 = 2.$$

However, there is no need to write this solution down, as we just did: the solution is implicit in the dictionary.

In the first iteration, we shall attempt to increase the value of z by making one of the right-hand side variables positive. At this moment, any of the three variables x_1, x_2, x_3 would do. In small examples, it is common practice to choose the variable that, in the formula for z, has the largest coefficient: the increase in that variable will make z increase at the fastest rate (but not necessarily to the highest level). In our case, this rule leaves us a choice between x_1 and x_2; choosing arbitrarily, we decide to make x_1 positive. As the value of x_1 increases, so does the value of x_5. However, the values of x_4, x_6, and x_7 decrease, and none of them is allowed to become negative. Of the three constraints $x_4 \geq 0, x_6 \geq 0, x_7 \geq 0$ that impose upper bounds on the increment of x_1, the last constraint $x_7 \geq 0$ is the most stringent: it implies $x_1 \leq 1$. In the improved feasible solution, we shall have $x_1 = 1$ and $x_7 = 0$. Without writing the new solution down, we shall now construct the new dictionary. All we need to know is that x_1 just made its way from the right-hand side to the left, whereas x_7 went in the opposite direction. From the fourth equation in (2.16), we have

$$x_1 = 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7. \tag{2.17}$$

Substituting from (2.17) into the remaining equations of (2.16), we arrive at the desired dictionary

$$\begin{aligned}
x_1 &= 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7 \\
x_4 &= 2 - \frac{3}{2}x_2 - \frac{3}{2}x_3 + \frac{1}{2}x_7 \\
x_5 &= 3 - \frac{3}{2}x_2 - \frac{5}{2}x_3 - \frac{1}{2}x_7 \\
x_6 &= 2 + 4x_2 - 3x_3 + x_7 \\
\hline
z &= 5 - \frac{5}{2}x_2 + \frac{11}{2}x_3 - \frac{5}{2}x_7.
\end{aligned} \tag{2.18}$$

The construction of (2.18) completes the first iteration of the simplex method.

Digression on Terminology

The variables x_j that appear on the left-hand side of a dictionary are called *basic*; the variables x_j that appear on the right-hand side are *nonbasic*. The basic variables are said to constitute a *basis*. Of course, the basis changes with each iteration: for example, in the first iteration, x_1 entered the basis whereas x_7 left it. In each iteration, we first choose the nonbasic variable that is to enter the basis and then we find out which basic variable must leave the basis. The choice of the *entering* variable is motivated by our desire to increase the value of z; the determination of the *leaving* variable is based on the requirement that all variables must assume nonnegative values. The leaving variable is that basic variable whose nonnegativity imposes the most stringent upper bound on the increment of the entering variable. The formula for the leaving variable appears in the *pivot row* of the dictionary; the computational process of constructing the new dictionary is referred to as *pivoting*.

Back to the Second Example

In our example, the variable to enter the basis during the second iteration is quite unequivocally x_3. This is the only nonbasic variable in (2.18) whose coefficient in the last row is positive. Of the four basic variables, x_6 imposes the most stringent upper bound on the increase of x_3, and, therefore, has to leave the basis. Pivoting, we arrive at our third dictionary,

$$\begin{aligned}
x_3 &= \frac{2}{3} + \frac{4}{3}x_2 + \frac{1}{3}x_7 - \frac{1}{3}x_6 \\
x_1 &= \frac{4}{3} - \frac{5}{6}x_2 - \frac{1}{3}x_7 - \frac{1}{6}x_6 \\
x_4 &= 1 - \frac{7}{2}x_2 \qquad\qquad + \frac{1}{2}x_6 \\
x_5 &= \frac{4}{3} - \frac{29}{6}x_2 - \frac{4}{3}x_7 + \frac{5}{6}x_6 \\
\hline
z &= \frac{26}{3} + \frac{29}{6}x_2 - \frac{2}{3}x_7 - \frac{11}{6}x_6.
\end{aligned} \tag{2.19}$$

In the third iteration, the entering variable is x_2 and the leaving variable is x_5. Pivoting yields the dictionary

$$\begin{aligned}
x_2 &= \frac{8}{29} - \frac{8}{29}x_7 + \frac{5}{29}x_6 - \frac{6}{29}x_5 \\
x_3 &= \frac{30}{29} - \frac{1}{29}x_7 - \frac{3}{29}x_6 - \frac{8}{29}x_5 \\
x_1 &= \frac{32}{29} - \frac{3}{29}x_7 - \frac{9}{29}x_6 + \frac{5}{29}x_5 \\
x_4 &= \frac{1}{29} + \frac{28}{29}x_7 - \frac{3}{29}x_6 + \frac{21}{29}x_5 \\
\hline
z &= 10 - 2x_7 - x_6 - x_5.
\end{aligned} \tag{2.20}$$

At this point, no nonbasic variable can enter the basis without making the value of z decrease. Hence, the last dictionary describes an optimal solution of our example. That solution is

$$x_1 = \frac{32}{29}, \quad x_2 = \frac{8}{29}, \quad x_3 = \frac{30}{29}$$

and it yields $z = 10$.

FURTHER REMARKS

The reader may have noticed that, having first carefully laid down the definition of a dictionary, we then proceeded to refer to (2.18), (2.19), and (2.20) as dictionaries, without bothering to verify that they do indeed have property (2.14). Such carelessness can be easily justified. Take, for example, system (2.18). Since (2.18) arises from (2.16) by arithmetical operations (namely, pivoting with x_1 entering and x_7 leaving), every solution of (2.16) must be also a solution of (2.18). The converse is also true, since (2.16) can be obtained from (2.18) by pivoting with x_7 entering and x_1 leaving. Hence, *every solution of* (2.18) *is a solution of* (2.16), *and vice versa*. Similar arguments show that *every solution of* (2.19) *is a solution of* (2.18), *and vice versa*; and that *every solution of* (2.20) *is a solution of* (2.19), *and vice versa.*

Another point of concern is the question of the *uniqueness*, as opposed to the *existence*, of optimal solutions. This question will be of no great interest to us; nevertheless, it is easy to deal with and so we will get it out of the way now. Note that in each of our two examples, we not only found an optimal solution, but we also collected the evidence to prove that there is only one optimal solution. For instance, the final dictionary for our first problem reads

$$\begin{array}{l} x_3 = 1 + x_2 + 3x_4 - 2x_6 \\ x_1 = 2 - 2x_2 - 2x_4 + x_6 \\ x_5 = 1 + 5x_2 + 2x_4 \\ \hline z = 13 - 3x_2 - x_4 - x_6. \end{array}$$

The last row shows that every feasible solution with $z = 13$ satisfies $x_2 = x_4 = x_6 = 0$; the rest of the dictionary shows that every such solution satisfies $x_3 = 1, x_1 = 2, x_5 = 1$; therefore, there is just one optimal solution. A similar argument applies to the second problem.

Of course, there are LP problems with more than just one optimal solution; having solved

such problems by the simplex method, we can effectively describe all the optimal solutions. For example, consider the following dictionary:

$$\begin{array}{l} x_4 = 3 + x_2 - 2x_5 + 7x_3 \\ x_1 = 1 - 5x_2 + 6x_5 - 8x_3 \\ x_6 = 4 + 9x_2 + 2x_5 - x_3 \\ \hline z = 8 \qquad\qquad\qquad\; - x_3. \end{array}$$

The last row shows that every optimal solution satisfies $x_3 = 0$ (but not necessarily $x_2 = 0$ or $x_5 = 0$). For such solutions, the rest of the dictionary implies

$$\begin{array}{l} x_4 = 3 + x_2 - 2x_5 \\ x_1 = 1 - 5x_2 + 6x_5 \\ x_6 = 4 + 9x_2 + 2x_5. \end{array} \qquad (2.21)$$

We conclude that every optimal solution arises by the substitution formulas (2.21) from some x_2 and x_5 such that

$$\begin{array}{r} -x_2 + 2x_5 \le 3 \\ 5x_2 - 6x_5 \le 1 \\ -9x_2 - 2x_5 \le 4 \\ x_2, x_5 \ge 0. \end{array}$$

(In fact, the inequality $-9x_2 - 2x_5 \le 4$ is clearly redundant; its validity is forced by $x_2 \ge 0$ and $x_5 \ge 0$.)

There are a few other rough spots we deliberately failed to point out in our overview of the simplex method. We shall discuss them in Chapter 3.

TABLEAU FORMAT

The simplex method is often introduced in a format differing from ours. To outline the more popular *tableau format*, we shall return to the first example of this chapter. To begin, let us write down the equations of the first dictionary in a slightly modified form:

$$\begin{array}{rcl} 2x_1 + 3x_2 + x_3 + x_4 & & = 5 \\ 4x_1 + x_2 + 2x_3 \qquad + x_5 & & = 11 \\ 3x_1 + 4x_2 + 2x_3 \qquad\qquad + x_6 & & = 8 \\ \hline -z + 5x_1 + 4x_2 + 3x_3 & & = 0. \end{array}$$

Recording just the coefficients at the x_j's, together with the right-hand sides, we obtain our first *tableau*:

2	3	1	1	0	0	5
4	1	2	0	1	0	11
3	4	2	0	0	1	8
5	4	3	0	0	0	0.

In a similar way, the equations of the second dictionary,

$$\begin{array}{lllll} x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 & & & = & \frac{5}{2} \\ - 5x_2 & - 2x_4 + x_5 & & = & 1 \\ - \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{3}{2}x_4 & & + x_6 & = & \frac{1}{2} \\ \hline -z \quad - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 & & & = & -\frac{25}{2} \end{array}$$

give rise to a second tableau:

$$\begin{array}{ccccccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{5}{2} \\ 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & 0 & 1 & \frac{1}{2} \\ \hline 0 & -\frac{7}{2} & \frac{1}{2} & -\frac{5}{2} & 0 & 0 & -\frac{25}{2}. \end{array}$$

It is a routine matter to translate the pivoting rules, previously derived in terms of dictionaries, into the language of tableaus. The following steps describe the procedure; the reader should have no trouble verifying its correctness. (At any rate, the procedure is not important for our exposition since we do not use the tableau format.)

Step 1. Examine all numbers in the last row (except the one farthest right, which equals the current value of $-z$). If all of them are negative or zero, stop: the tableau describes an optimal solution. Otherwise find the largest of these numbers; the column in which it appears is called the *pivot column* and corresponds to the entering variable.

For example, the pivot column in our first tableau is the first one:

$$\begin{array}{|c|cccccc} 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ \hline 5 & 4 & 3 & 0 & 0 & 0 & 0 \end{array}$$

Step 2. For each row whose entry r in the pivot column is positive, look up the entry s in the rightmost column. The row with the smallest ratio $\frac{s}{r}$ is called the *pivot row* and corresponds to the leaving variable. (If all the entries in the pivot column are negative or zero, then the problem is unbounded; more on that in Chapter 3.)

In our example, the pivot row is the first row $\left(\text{with } \frac{s}{r} = \frac{5}{2}\right)$:

$$\begin{array}{|c|cccccc|} \hline 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ \hline 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ \hline 5 & 4 & 3 & 0 & 0 & 0 & 0. \end{array}$$

Step 3. Divide every entry in the pivot row by the *pivot number*, found in the intersection of the pivot row with the pivot column:

$$\begin{array}{ccccccc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{5}{2} \\ 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ \hline 5 & 4 & 3 & 0 & 0 & 0 & 0. \end{array}$$

Step 4. From every remaining row, subtract a suitable multiple of the new pivot row. This operation is designed to make every entry in the pivot column (except for the pivot number) become zero; hence, the "suitable multiple" results when the new pivot row is multiplied by the entry appearing in the pivot column and in the row in question. (In our example, step 4 results in the second tableau.)

A tableau is nothing but a cryptic recording of a dictionary with all the variables collected on the left-hand side and the symbols for these variables omitted. We shall continue to use dictionaries instead, since they are more explicit. (Of course, nothing prevents the reader tired of writing the same symbols $x_1, x_2, \ldots$ over and over again from using the tableau shorthand.) □

A WARNING

There is often more than one way of describing a particular algorithm; descriptions aimed at clarifying underlying concepts are often quite different from those that suggest efficient computer implementations. The simplex method is no exception. Dictionaries may provide a convenient tool for explaining its basic principles. However, in implementing the method for computer solutions of large problems, considerations of computational efficiency and numerical accuracy overshadow such didactic niceties. We shall begin to study efficient implementations of the simplex method in Chapters 7 and 8.

PROBLEMS

△ **2.1** Solve the following problems by the simplex method:

a. maximize $3x_1 + 2x_2 + 4x_3$

subject to
$$\begin{aligned} x_1 + x_2 + 2x_3 &\le 4 \\ 2x_1 + 3x_3 &\le 5 \\ 2x_1 + x_2 + 3x_3 &\le 7 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

b. maximize $5x_1 + 6x_2 + 9x_3 + 8x_4$

subject to
$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &\le 5 \\ x_1 + x_2 + 2x_3 + 3x_4 &\le 3 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

c. maximize $2x_1 + x_2$

subject to
$$\begin{aligned} 2x_1 + 3x_2 &\le 3 \\ x_1 + 5x_2 &\le 1 \\ 2x_1 + x_2 &\le 4 \\ 4x_1 + x_2 &\le 5 \\ x_1, x_2 &\ge 0. \end{aligned}$$

2.2 Use the simplex method to describe *all* the optimal solutions of the following problem:

maximize $2x_1 + 3x_2 + 5x_3 + 4x_4$

subject to
$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &\le 5 \\ x_1 + x_2 + 2x_3 + 3x_4 &\le 3 \\ x_1, x_2, x_3, x_4 &\ge 0. \end{aligned}$$

3

Pitfalls and How to Avoid Them

The examples illustrating the simplex method in the preceding chapter were purposely smooth. They did not point out the dangers that can occur. The purpose of the present chapter, therefore, is to rigorously analyze the method by scrutinizing its every step.

THREE KINDS OF PITFALLS

Three kinds of pitfalls can occur in the simplex method.

(i) INITIALIZATION. We might not be able to start: How do we get hold of a feasible dictionary?

(ii) ITERATION. We might get stuck in some iteration: Can we always choose an entering variable, find the leaving variable, and construct the next feasible dictionary by pivoting?

(iii) TERMINATION. We might not be able to finish: Can the simplex method construct an endless sequence of dictionaries without ever reaching an optimal solution?